Final Exam
Thursday, October 12, 2017

Directions

- This is a 90-minute exam worth 100 points.
- This exam contains 1 question.
- You may use a calculator for this exam. You are not permitted to access user-generated files in the calculator.
- Justify your answers by showing all work.
- If you feel that additional assumptions are required to answer the question, state these assumptions clearly.

1. **Neoclassical Growth Model with Public Goods**

   Consider a deterministic model in which the government uses tax revenue in order to provide a public good (instead of rebating it lump-sum back to the household).

   Assume that there is a set of homogeneous consumers in an economy with an infinite time horizon. Each consumer has one unit of time that is split between labor and leisure. Households consume private goods and public goods. The public goods are provided by the governmnent and the amount of public good consumption is taken as given by the households. In period t, household consumption of the private good is denoted c_t, consumption of the public good is denoted G_t, and labor supply is denoted n_t.

   The households derive utility from public goods G_t in proportion to the amount of time devoted to leisure. The preferences are

$$\sum_{t=0}^{\infty} \beta^t u\left(c_t, G_t, n_t\right),$$

   where the utility function is assumed to be of the Cobb-Douglas form

$$u\left(c_t, G_t, n_t\right) = \left(c_t\right)^{\epsilon} \left(G_t\left(1-n_t\right)\right)^{1-\epsilon} \text{ for } \epsilon \in (0,1).$$

   The government levies a labor income tax in order to raise tax revenue. The tax rate is $\tau \in (0,1)$ and is taken as given by the households. Households choose investment i_t, have current capital stock k_t, and take as given the factor prices R_t and w_t. The household budget constraint is given by:

$$c_t + i_t \leq R_t k_t + w_t n_t \left(1-\tau\right).$$

   The law of motion for capital is:

$$k_{t+1} = \left(1-\delta\right) k_t + z_t i_t,$$

   where $\delta \in (0,1)$ is the depreciation rate and z_t is the deterministic investment return. Assume that $z_t = \left(\lambda_z\right)^t$ for $\lambda_z > 1$.

   Firms produce output Y_t using capital K_t and labor N_t as inputs. The production function is a standard Cobb-Douglas production function:

$$Y_t = A_t \left(K_t\right)^{\alpha} \left(N_t\right)^{1-\alpha} \text{ for } \alpha \in (0,1),$$

   A_t is the deterministic total factor productivity. Assume that $A_t = \left(\lambda_A\right)^t$ for $\lambda_A > 1$.

   The tax revenue raised by the government in period t is equal to $\tau w_t n_t$. The government produces public good G_t using the following government technology function:

$$G_t = \Omega_t \tau w_t n_t.$$

   Ω_t is the deterministic public good productivity. Assume that $\Omega_t = \left(\lambda_\Omega\right)^t$ for $\lambda_\Omega > 1$.

(a) State the Contraction Mapping Theorem.

(b) A balanced growth path is an equilibrium in which the labor supply n_t is constant and the remaining variables grow at constant rates. Denote the growth rates as

$$g_c = \frac{c_{t+1}}{c_t} \quad g_G = \frac{G_{t+1}}{G_t} \quad g_k = \frac{k_{t+1}}{k_t} \quad g_I = \frac{i_{t+1}}{i_t} \quad g_Y = \frac{Y_{t+1}}{Y_t} \ .$$

Determine the growth rates $(g_c, g_G, g_k, g_I, g_Y)$ as a function of parameters, including $(\lambda_z, \lambda_A, \lambda_\Omega)$.

(c) Define scaled variables as follows:

$$\hat{c}_t = \frac{c_t}{(g_c)^t} \quad \hat{G}_t = \frac{G_t}{(g_G)^t} \quad \hat{k}_t = \frac{k_t}{(g_k)^t} \quad \hat{\imath}_t = \frac{i_t}{(g_I)^t} \quad \hat{Y}_t = \frac{Y_t}{(g_Y)^t} \ .$$

What assumptions are required on $(\lambda_z, \lambda_A, \lambda_\Omega)$ to ensure that the Bellman equation, written in terms of these scaled variables, is well-defined?

(d) State Berge's Maximum Theorem.

(e) The space $CB(\mathbf{A})$ is the set of continuous and bounded functions on compact domain $\mathbf{A} = [0, \bar{k}]$ and it is a complete metric space. The upper bound on capital is $\bar{k} = \left(\frac{1-\tau(1-\alpha)}{g_k-(1-\delta)}\right)^{\frac{1}{1-\alpha}}$. Define the appropriate mapping $T : CB(\mathbf{A}) \to CB(\mathbf{A})$ and the appropriate budget correspondence Γ required for Berge's Maximum Theorem and the Contraction Mapping Theorem.

(f) State and use Blackwell's sufficient conditions to verify that T is a contraction.

(g) Define a recursive competitive equilibrium (RCE) in terms of the scaled variables. Make sure that you clearly distinguish between aggregate state variables and individual state variables.

(h) For the recursive consumer's problem specified in part (g), write down the first order condition with respect to labor supply and the Euler equation for capital investment.