

Incomplete Markets as the Outcome of Bilateral Bargaining*

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Abstract

To complement the theory of incomplete markets under perfect competition and anonymity, this paper examines the theory of incomplete markets under strategic bargaining. Households bargain over bilateral nominal contracts that specify transfers for all states of uncertainty. The lone institutional feature is the limit on the number of contracts that a household can agree to. These contract limits for all households determine whether or not the equilibrium allocations are first-best. Specifically, if all households can be linked via a series of contracts to all other households, then the equilibrium allocations will be first-best; if not, then the allocations will generically not even be second-best.

Keywords incomplete markets – bargaining – strategic foundations – constrained suboptimality – regularity – asymmetric information

JEL Classification D52 · D82 · E21 · G11

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1 Introduction

One of the significant achievements by economists in the past 30 years is a broad understanding of the outcomes of decision-making in competitive markets with an uncertain future. As research in the field of finance trends toward a greater use of heterogeneous agent models and macroeconomists switch to general equilibrium models, variants of the GEI financial model will continue to be widely used. The equilibrium properties when financial contracts are competitively traded are well-known (see Magill and Quinzii 1996), but very little is known if competitive trade is replaced with strategic bargaining. This paper demonstrates that when households bargain over bilateral contracts the normative results are in line with those attained for competitive markets. In particular, if the institutional primitives allow all households to be linked via a series of contracts, then the equilibrium allocations will be Pareto optimal.

Economists often view competitive markets as idealized mechanisms and have sought to justify their use by relaxing the price-taking assumption. Price taking is a feature of the institutions buttressing the model. At a more fundamental level lie the opportunities and choices of households. This paper considers a dynamic setting in which households make strategic decisions about the contracts they use to transfer wealth across time and uncertainty.

In place of institutions specifying asset payouts and asset prices, the model only requires a limit on the number of contracts that a household can agree to. The contracts are reached through a simple bargaining game: (i) each pair of households meets only once and (ii) bargaining within a pair occurs via a "take it or leave it" offer. Earlier work in static pure-exchange models was able to determine conditions sufficient so that the strategic outcome approaches the competitive outcome (see Feldman 1973, Goldman and Starr 1982, Gale 2000, Yildiz 2003, Dávila and Eeckhout 2008, and Penta 2011).

Other than the strategic foundations of competitive static equilibria, a handful of broad attempts have been made to blend a competitive environment with strategic decision-making. Examples include club theory (see Ellickson et al. 1999 and Ellickson et al. 2001) and strategic market games (see Shapley and Shubik 1969 and Giraud 2003). In all cases, economists incorporate the broad notion that a household's trading opportunities can be impacted by available actions or information lying outside the competitive paradigm. These fields and others (on the topic of incomplete markets, see Yildiz 2002 and Oksendal and Sulem 2008) share with the current paper the question of whether the strategic outcomes are commensurate with the competitive outcomes.

To model time and uncertainty, this paper considers two time periods with a finite number

of states of uncertainty in the second period. Importantly, wealth transfers are no longer made according to an exogenous asset structure. Instead, households bargain over bilateral nominal contracts specifying transfers for all states of uncertainty. The choices of households are made simultaneously in the first period, where a choice contains three components: the vector of contract offers to propose, the conditions under which contracts are accepted, and the contingent consumption vector for the competitive commodity markets in all states of uncertainty.

With nominal contracts and a fixed asset structure, it is known that GEI equilibria are generically indeterminate (see Cass 1992). Yet, when households exchange nominal contracts and are not constrained by a fixed asset structure, I show that equilibria are generically determinate.

In a sense, each household is selecting, subject to its contract limit, individual assets to trade. This brings into the discussion the literature on financial innovation, where an outside agent (typically called the innovator) sets the asset structure in order to maximize asset trade. The literature is split into two approaches. The first approach is Allen and Gale (1988) who show that the equilibrium allocations that arise in their model are constrained optimal, but not Pareto optimal.¹

The second approach is Pesendorfer (1995) whose model presumes a set of standard securities with trading costs. Innovators then create new securities from the standard ones and seek to market them (at a cost) to investors in competitive markets. With only one innovator, as the trading and marketing costs approach zero, the equilibria become determinate and their respective allocations become Pareto optimal. In my model, the equilibria are determinate over a generic subset of parameters and the contract limits determine in a precise way whether the equilibrium allocations are Pareto optimal or constrained Pareto suboptimal (generically).

Though not a model of financial innovation, the Kiyotaki and Wright model (1991) shows that households may choose to trade intrinsically worthless fiat money because it can reduce the cost of search frictions. The Kiyotaki and Wright (1991) search frictions are the reason why trade may occur via a monetary transaction, where a monetary transaction is simply a bilateral nominal contract. In my paper, I utilize bilateral nominal contracts in a different bargaining environment not as an explanation for money, but rather as an explanation for incomplete risk-sharing opportunities. Instead of search frictions, the frictions in my model are the contract limits.

The paper contains two orthogonal normative results. Consider those economies in which

¹The follow-up paper Allen and Gale (1991) shows that with imperfect competition, as is required to prove existence without imposing short sale constraints, allocations are not even constrained optimal.

multiple commodities are traded in each state. The first normative result states that if all households can be linked via a series of contracts to all other households, then the equilibrium allocations are Pareto optimal.

A weaker concept of Pareto optimality is constrained Pareto optimality, which states that the equilibrium allocation cannot be Pareto improved using the same financial institutions that are utilized in the equilibrium. For a competitive model, this requires using the same asset structure to implement wealth transfers. For this strategic model, this requires using the same networks of households to implement wealth transfers. The second normative result states that if the households cannot all be linked via a series of contracts (that is, more than one network is present), then the equilibrium allocations are constrained Pareto suboptimal, where the result holds generically over the set of household parameters (utility functions and endowments). In particular, a planner can implement transfers just between households in the same network and make all households better off.

This paper is organized into five remaining sections and an Appendix. In Section 2, I pair the contracting environment with a standard two-period general equilibrium model. In Section 3, I consider the conditions for the Pareto optimality property. In Section 4, I consider the conditions for the constrained Pareto suboptimality property. In Section 5, I examine the normative effects of asymmetric information. Section 6 concludes and Appendix A contains the proofs of the main results.

2 The Model

Consider a two-period general equilibrium model with s^* states of uncertainty in the second period. Denoting the first period as the $s = 0$ state, I number the states as $s \in S = \{0, \dots, s^*\}$. In each state, a finite number of households $h \in H = \{1, \dots, h^*\}$ trade and consume a finite number of physical commodities $l \in L = \{1, \dots, l^*\}$. The commodities are denoted by the variable x . Define the total number of goods as $g^* = l^*(s^* + 1)$. Concerning notation, the vector $x^h \in \mathbb{R}_+^{g^*}$ contains the entire consumption by household h , the vector $x^h(s) \in \mathbb{R}_+^{l^*}$ contains the consumption by household h in state s (of all commodities), and the scalar $x_l^h(s) \in \mathbb{R}_+$ is the consumption by household h of the good (s, l) , or the l^{th} physical commodity in state s .

Households are endowed with commodities in all states. These endowments are denoted by $e = (e^h)_{h \in H} \in \mathbb{R}_+^{h^* g^*}$. I assume that all households have strictly positive endowments:

Assumption 1 $e^h \gg 0 \quad \forall h \in H.$ ²

²The vector notation $y \gg 0$ means that $y_i > 0 \quad \forall i$, whereas $y \geq 0$ means $y_i \geq 0 \quad \forall i$ and $y > 0$ means

Define the set of endowments satisfying Assumption 1 as $\mathcal{E} = \{(e^h)_{h \in H} : e^h \gg 0\}$. The consumption set X^h is assumed to be the interior of $\mathbb{R}_+^{g^*}$:

Assumption 2 $X^h = \mathbb{R}_{++}^{g^*} \forall h \in H$.

The utility function $u^h : X^h \rightarrow \mathbb{R}$ is subject to the following assumptions:

Assumption 3 u^h is C^3 , differentiably strictly increasing (i.e., $Du^h(x^h) \gg 0 \forall x^h \in X^h$), differentiably strictly concave (i.e., $D^2u^h(x^h)$ is negative definite $\forall x^h \in X^h$), and satisfies the boundary condition $(clU^h(x^h) \subset X^h$ where $U^h(x^h) = \{x' \in X^h : u^h(x') \geq u^h(x^h)\}) \forall h \in H$.

Define the set of utility functions satisfying Assumption 3 as $\mathcal{U} = \{(u^h)_{h \in H} : u^h \text{ satisfies Assumption 3}\}$.

The commodity markets are Walrasian, so I introduce the commodity prices $p \in \mathbb{R}^{g^*} \setminus \{0\}$.³ Under Assumption 3, the prices satisfy $p \in \mathbb{R}_{++}^{g^*}$.

2.1 Contracting Environment

The commodities are perishable, so households require a means by which to transfer wealth between states. In this paper, that means is provided through bilateral nominal contracts. The terms of the contracts are reached using "take it or leave it" bargaining offers.

In the initial period, prior to the realization of uncertainty in the final period, households simultaneously perform the following tasks: (i) bargaining, (ii) consumption choices for the competitive commodity market $s = 0$ and (iii) contingent consumption choices for the competitive commodity markets in states $s > 0$.⁴ The remainder of this section breaks down the manner in which bargaining is modeled.

Bilateral bargaining requires two households, and in this model (unlike Gale 2000), a pair has only a single opportunity for bargaining. In that single opportunity, one household makes a "take it or leave it" offer and the other household decides to either accept or reject that offer. As all decisions are made simultaneously, the latter household is actually selecting the set of all contracts that will be accepted.

The key feature governing the contract offers and acceptance decisions is the "contract limit" of a household. For each household $h \in H$, χ^h is the "contract limit," that is, the

$y \geq 0$ and $y \neq 0$.

³These Walrasian markets can equivalently be viewed as the outcome of a bilateral bargaining process under certain conditions (see Gale 2000).

⁴Assumption 3 can be strengthened to guarantee that the contingent consumption choices are time consistent, but the approach in this paper will simply assume that the contingent consumption choices are fully committed to.

total number of contracts that h can enter into. I define an economy as the parameters $(e^h, u^h, \chi^h)_{h \in H}$. The following assumption specifies that all households have the opportunity to use contracts to transfer wealth:

Assumption 4 $\chi^h \geq 1 \forall h \in H$.

Consider a contracting pair (h, h') . Recall that any pair has only the single opportunity for bargaining, so one household makes the "take it or leave it" offer and the other household either accepts or rejects the offer. The following assumption imposes transitivity:

Assumption 5 If h makes the "take it or leave it" offer to h' and h' makes the "take it or leave it" offer to h'' , then h will make the "take it or leave it" offer to h'' .

To impose structure on the model and avoid circular bargaining networks, a clear ordering of bargaining power is required. To justify Assumption 5, consider that the institutions on the periphery of the economic model that permit household h to make a contract offer to h' and household h' to make a contract offer to h'' are also likely to permit household h to make a contract offer to h'' . Without loss of generality, the labeling of households is such that Assumption 5 is equivalent to the following statement.

For all $h, h' \in H$, the household $h < h'$ is the one in the pair (h, h') to make the "take it over leave it" offer.

Assumption 5 specifies that household $h = 1$ makes "take it or leave it" offers in all pairs $(1, h')$ for $h' > 1$, household $h = 2$ makes "take it or leave it" offers in all pairs $(2, h'')$ for $h'' > 2$, and so forth. The labeling of households is common knowledge.

The notational convention is that for any pair (h, h') with $h < h'$, the contract proposal from h to h' is $\gamma_{h'}^h \in \mathbb{R}^{s^*+1}$. The proposed contract specifies that household h receives the nominal transfer of $+\gamma_{h'}^h(s) \forall s \in S$, provided the proposal is accepted. The agreed-upon contract between the proposer h and the receiver h' is $\tilde{\gamma}_{h'}^h$. If h' accepts the proposal $\gamma_{h'}^h$, then $\tilde{\gamma}_{h'}^h = \gamma_{h'}^h$. If h' rejects the proposal, then $\tilde{\gamma}_{h'}^h = 0$. For consistency (zero net transfers), I require that the accepted contract $\tilde{\gamma}_{h'}^h$ specifies that h' will receive the nominal transfer of $-\tilde{\gamma}_{h'}^h(s) \forall s \in S$.

Recall that household h proposes contracts in all pairs (h, h') with $h < h'$. Define $\gamma^h = (\gamma_{h'}^h)_{h' > h}$ as the contract proposals by household h . Define $\tilde{\gamma}_{h'} = (\tilde{\gamma}_{h'}^h)_{h' > h}$ as the accept/reject decisions by household h' .

The contract $\tilde{\gamma}_{h'}^h = 0$ is the trivial contract. When determining whether or not a household violates its contract limit, I am only concerned with nontrivial contracts. The "contract

limit" constraint (CL) for household h is:

$$\sum_{h' > h} 1 \{ \tilde{\gamma}_{h'}^h \neq 0 \} + \sum_{h' < h} 1 \{ \tilde{\gamma}_h^{h'} \neq 0 \} \leq \chi^h. \quad (\text{CL})$$

In this environment, all contracts are fully committed to. In all states $s \in S$, the commodity markets are competitive Walrasian markets.

2.2 Unconstrained Financial Equilibrium

To define the initial equilibrium concept, I first define the household problem. For simplicity,

any contract $\gamma_{h'}^h \in \mathbb{R}^{s^*+1}$ is a column vector and $P = \begin{bmatrix} p(0) & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & p(s^*) \end{bmatrix}$ is the $(s^* + 1) \times g^*$

price matrix. For household h , taking as given the variables p , $(\gamma^{h'})_{h' \neq h}$, and $(\tilde{\gamma}_{h'})_{h' \neq h}$, the unconstrained household problem (UHP) is given by:

$$\begin{aligned} & \max_{x^h \in X^h, \gamma^h \in \mathbb{R}^{(h^* - h)(s^* + 1)}, \tilde{\gamma}_h \in \mathbb{R}^{(h-1)(s^* + 1)}} u^h(x^h) \\ & \text{subj. to} \quad \begin{aligned} & 1. \quad P(e^h - x^h) + \sum_{h' > h} \tilde{\gamma}(h, h') - \sum_{h' < h} \tilde{\gamma}(h', h) \geq 0. \\ & 2. \quad \text{Constraint (CL) holds.}^6 \end{aligned} \end{aligned} \quad (\text{UHP})$$

Given the household problem (UHP), I define an *unconstrained financial equilibrium*.

Definition 1 $\left((x^h, \gamma^h, \tilde{\gamma}^h)_{h \in H}, p \right)$ is an *unconstrained financial equilibrium* if

1. $\forall h \in H$, given $\left(p, (\gamma^{h'})_{h' \neq h}, (\tilde{\gamma}_{h'})_{h' \neq h} \right)$, $(x^h, \gamma^h, \tilde{\gamma}_h)$ is an optimal solution to the unconstrained household problem (UHP),
2. for any pair (h, h') with $h < h'$, $\tilde{\gamma}_{h'}^h = \gamma_{h'}^h$ if h' accepts the proposal; $\tilde{\gamma}_{h'}^h = 0$ otherwise,
3. markets clear:

$$\sum_{h \in H} (x_l^h(s) - e_l^h(s)) = 0 \quad \forall (l, s) \in L \times S.$$

⁵The indicator function $1 \{p\}$ takes the value 1 when the statement p is true and 0 otherwise.

⁶The budget constraint and the constraint (CL) are functions of the decision made by other households. The choice of household h is invalid if either constraint is violated, where an invalid choice results in an infinitely large utility penalty.

2.3 Acceptance Condition

In order for the contract proposal $\gamma_{h'}^h$ (a choice variable of h) to be included in the budget constraint of h , the household must take into account the *acceptance condition* of household h' . The following paragraphs introduce the notation required to specify the *acceptance condition*, which states that household h' will accept the contract proposal $\gamma_{h'}^h$.

Define a household's total nominal payoff from contracts in state $s \in S$ as $\tau^h(s) = \sum_{h' > h} \gamma_{h'}^h(s) - \sum_{h' < h} \gamma_{h'}^h(s)$ with $\tau^h = (\tau^h(s))_{s \in S}$. These payoffs are contingent on all contracts being accepted. For h , define the best response

$$b^h : \mathbb{R}^{\left(\frac{(h^* - 1) \cdot h^*}{2} - (h^* - h)\right)(s^* + 1)} \rightarrow \mathbb{R}^{(s^* + 1)}$$

$$(\gamma^k)_{k \neq h} \mapsto \sum_{h' > h} \gamma_{h'}^h(s)$$

as the sum of the $(h^* - h)$ contract proposals that are made by household h .

Define $\sigma^h : \mathbb{R}^{(s^* + 1)} \rightarrow X^h$ as the consumption demand function given the transfers $\tau^h \in \mathbb{R}^{(s^* + 1)} : x^h = \sigma^h(\tau^h)$. The function also depends on the commodity prices p and the parameters of the model, notably endowments e^h , but these are omitted for simplicity.

For the pair (h, h') with $h < h'$, define $\gamma_{\setminus h'}^h = \left((\gamma^k)_{k \notin \{h, h'\}}, (\gamma_k^h)_{k \neq h'} \right)$ as all contract proposals $(\gamma^k)_{k \neq h'}$, except $\gamma_{h'}^h$. By definition, $\tilde{\gamma}_{h'}^h = \gamma_{h'}^h$ iff

$$u^{h'} \left\{ \sigma^{h'} \left[b^{h'} \left(\gamma_{\setminus h'}^h, \gamma_{h'}^h \right) - \sum_{k < h' : k \neq h} \gamma_{h'}^k - \gamma_{h'}^h \right] \right\} \geq u^{h'} \left\{ \sigma^{h'} \left[b^{h'} \left(\gamma_{\setminus h'}^h, 0 \right) - \sum_{k < h' : k \neq h} \gamma_{h'}^k \right] \right\}, \quad (AC)$$

where the inequality is referred to as the *acceptance condition* (AC).

As written, h' expects that it will accept all contract proposals and that each of its contract proposals will be accepted. This is because all households make their contract proposals subject to (AC) above. First, I will define an equilibrium with this restriction in the household problem (HP). Then, I will show that the decision by households to make contract proposals subject to (AC) is not a restriction, but an equilibrium choice.

2.4 Financial Equilibrium

The main equilibrium concept of this paper is a *financial equilibrium*. For household h , taking as given the variables p and $(\gamma^{h'})_{h' \neq h}$, the household problem (HP) is given by:

$$\begin{aligned}
& \max_{x^h \in X^h, \gamma^h \in \mathbb{R}^{(h^* - h)(s^* + 1)}} u^h(x^h) \\
& \text{subj. to} \quad \begin{aligned}
& 1. \quad P(e^h - x^h) + \sum_{h' > h} \gamma(h, h') - \sum_{h' < h} \gamma(h', h) \geq 0. \\
& 2. \quad \text{Constraint (CL) holds } (\gamma_{h'}^h \text{ replaces } \tilde{\gamma}_{h'}^h). \\
& 3. \quad \text{Constraint (AC) holds } \forall h' > h.
\end{aligned} \tag{HP}
\end{aligned}$$

Definition 2 $\left((x^h, \gamma^h)_{h \in H}, p \right)$ is a financial equilibrium if

1. $\forall h \in H$, given $\left(p, (\gamma^{h'})_{h' \neq h} \right)$,
 (x^h, γ^h) is an optimal solution to the household problem (HP),

2. markets clear:

$$\sum_{h \in H} (x_l^h(s) - e_l^h(s)) = 0 \quad \forall (l, s) \in L \times S.$$

2.5 Relation Between Equilibrium Concepts

I assume for simplicity that trivial contracts, and any other contracts that a household is indifferent toward, are always accepted. This does not affect the results in any way, only the way that I describe equilibrium choices. Suppose that a household h makes a nontrivial contract proposals that is rejected. As contracts decisions are made simultaneously, this household h is indifferent between proposing this nontrivial contract and proposing the trivial contract. Thus, the set of *financial equilibria* is a subset of the set of *unconstrained financial equilibria*. Households only proposing contracts that are accepted do not have an incentive to deviate once this restriction is removed.

What I do in the sequel is show that the set of *financial equilibria* is allocation-equivalent to the set of *unconstrained financial equilibria*. To accomplish this, I first characterize the set of *financial equilibria*. I then show that there does not exist an *unconstrained financial equilibrium* allocation that lies outside this set (see Theorem 4 in Subsection 3.3). This justifies my earlier claim that the decision by households to make contract proposals subject to (AC) is an equilibrium choice.

3 "Complete" Markets

The following version of the "First Basic Welfare Theorem" motivates the analysis in this section as a connection is made between complete markets in the GEI financial model and households' contract limits.

Theorem 1 *Under Assumptions 1-5, if $\chi^h = h^* - 1 \forall h \in H$, then any financial equilibrium allocation is Pareto optimal.*

Proof. Suppose otherwise, that is, a *financial equilibrium* $\left((x^h, \gamma^h)_{h \in H}, p \right)$ exists with a suboptimal allocation. Using Assumption 3, the allocation is pairwise Pareto suboptimal for some pair (h, h') , where $h < h'$ without loss of generality.⁷ As the commodity markets are Walrasian, there must exist $\alpha \in \mathbb{R}^{s^*+1} \setminus \{0\}$ such that the alternative allocation for the two households $(\hat{x}^h, \hat{x}^{h'})$, reached using nominal contracts $(\tau^h + \alpha, \tau^{h'} - \alpha)$, strictly dominates the original equilibrium allocation: $(u^h(\hat{x}^h), u^{h'}(\hat{x}^{h'})) \gg (u^h(x^h), u^{h'}(x^{h'}))$. This contradicts that the contract $\gamma_{h'}^h$ is an optimal solution to the household problem (HP) for household h . This is because the households can never violate (CL) and the alternative contract $\gamma_{h'}^h + \alpha$ would be accepted by h' and makes h strictly better off. ■

Motivated by the above result, I examine the implications of bilateral bargaining for the Pareto optimality of the equilibrium allocations. I define a *branch* as the set of households that are linked through nontrivial contract offers (a formal definition is provided in the following subsection). The main result of this paper states that if the contract limits allow all households to belong to one *branch*, then any *financial equilibrium* allocation is Pareto optimal. This is a statement of the "First Basic Welfare Theorem" for this model of bilateral bargaining.

Theorem 2 *Under Assumptions 1-5, if all households can be linked through nontrivial contract offers, then any financial equilibrium allocation is Pareto optimal.*

3.1 Branches

In the household problem (HP), let $\lambda^h \in \mathbb{R}^{s^*+1}$ be the Lagrange multiplier for the budget constraints of household h and let $\mu_{h'}^h \in \mathbb{R}$ be the Lagrange multiplier for (AC) corresponding to $\gamma_{h'}^h$. Recalling (AC) from Subsection 2.3, define

$$V^{h'} \left(\gamma_{h'}^h, \gamma_{h'}^h \right) = \max_{\gamma_{h'}^h} u^{h'} \left\{ \sigma^{h'} \left[\sum_{k>h'} \gamma_k^{h'} - \sum_{k<h': k \neq h} \gamma_{h'}^k - \gamma_{h'}^h \right] \right\},$$

⁷See Gale (2000), citing previous works by Feldman (1973) and Goldman and Starr (1982), for the specific conditions under which Pareto optimality is equivalent to pairwise Pareto optimality.

with maximizer $\sum_{k>h'} \gamma_k^{*h'} = b^{h'} \left(\gamma_{\setminus h'}^h, \gamma_{h'}^h \right)$. From the Envelope Theorem, the derivatives of (AC) with respect to $\gamma_{h'}^h$ are given by:

$$\frac{\partial V^{h'} \left(\gamma_{\setminus h'}^h, \gamma_{h'}^h \right)}{\partial \gamma_{h'}^h} = \frac{\partial u^{h'} \left(\sigma^{h'} \left(\tau^{h'} \right) \right)}{\partial \gamma_{h'}^h} \Big|_{\gamma^{h'} = \gamma^{*h'}}$$

where $\tau^{h'} = \sum_{k>h'} \gamma_k^{*h'} - \sum_{k<h':k \neq h} \gamma_{h'}^k - \gamma_{h'}^h$.

The first order conditions for household h with respect to $\gamma_{h'}^h \in \mathbb{R}^{s^*+1} \setminus \{0\}$ are:

$$\lambda^h + \mu_{h'}^h \cdot \frac{\partial u^{h'} \left(\sigma^{h'} \left(\tau^{h'} \right) \right)}{\partial \gamma_{h'}^h} = 0. \quad (1)$$

The static household problem (SHP), in which the financial payoffs are held fixed at $\tau^{h'}$, is given by:

$$\begin{aligned} & \max_{x^{h'} \in X^{h'}} u^{h'}(x^{h'}) \\ & \text{subj. to } P(e^{h'} - x^{h'}) + \tau^{h'} \geq 0 \end{aligned} \quad (\text{SHP})$$

The function $x^{h'} = \sigma^{h'}(\tau^{h'})$ is an implicit function of the system of equations characterizing the solutions of (SHP) :

$$F^{SHP} \left(x^{h'}, \lambda^{h'}; \tau^{h'} \right) = \begin{pmatrix} \left(Du^{h'}(x^{h'}) - \lambda^{h'} P \right)^T \\ P(e^{h'} - x^{h'}) + \tau^{h'} \end{pmatrix} = 0.$$

Using the Implicit Function Theorem:

$$\begin{aligned} \frac{\partial \sigma^{h'}(\tau^{h'})}{\partial \tau^{h'}} &= - \begin{bmatrix} I_{g^*} & | & 0_{g^* \times s^*+1} \end{bmatrix} \cdot \left[D_{x,\lambda} F^{SHP}(\cdot) \right]^{-1} \cdot \left[D_{\tau} F^{SHP}(\cdot) \right] \\ &= - \begin{bmatrix} I_{g^*} & | & 0_{g^* \times s^*+1} \end{bmatrix} \cdot \begin{bmatrix} D^2 u^{h'}(x^{h'}) & -P^T \\ -P & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0_{g^* \times s^*+1} \\ I_{s^*+1} \end{bmatrix}. \end{aligned}$$

Using the properties of blockwise matrix inversion, the product reduces to:

$$\frac{\partial \sigma^{h'}(\tau^{h'})}{\partial \tau^{h'}} = - \left(D^2 u^{h'}(x^{h'}) \right)^{-1} P^T \left[-P \left(D^2 u^{h'}(x^{h'}) \right)^{-1} P^T \right]^{-1}.$$

By definition, $\frac{\partial u^{h'}(\sigma^{h'}(\tau^{h'}))}{\partial \gamma_{h'}^h} = Du^{h'}(x^{h'}) \cdot \frac{\partial \sigma^{h'}(\tau^{h'})}{\partial \tau^{h'}} \cdot \frac{\partial \tau^{h'}}{\partial \gamma_{h'}^h}$. From the first order conditions with

respect to $x^{h'}$, $Du^{h'}(x^{h'}) = \lambda^{h'} P$. Combining the previous two sentences, (1) reduces to:

$$\lambda^h + \mu_{h'}^h \cdot \lambda^{h'} \cdot \left[-P \left(D^2 u^{h'}(x^{h'}) \right)^{-1} P^T \right] \left[-P \left(D^2 u^{h'}(x^{h'}) \right)^{-1} P^T \right]^{-1} \cdot \frac{\partial \tau^{h'}}{\partial \gamma_{h'}^h} = 0$$

or simply

$$\lambda^h - \mu_{h'}^h \cdot \lambda^{h'} = 0, \quad (2)$$

as $\frac{\partial \tau^{h'}}{\partial \gamma_{h'}^h} = -I_{s^*+1}$.

From F^{SP} , $\lambda^h \gg 0 \quad \forall h \in H$, so (2) implies $\mu_{h'}^h > 0$ for any household h making a nontrivial contract offer to h' .

Suppose that household h offers a nontrivial contract to household h' , but not h'' . If household h' offers a nontrivial contract to h'' , the pair (h, h'') are still connected using two iterations of (2):

$$\lambda^h = \mu_{h'}^h \cdot \lambda^{h'} = \mu_{h'}^h \cdot \left(\mu_{h''}^{h'} \cdot \lambda^{h''} \right). \quad (3)$$

Suppose instead that both h and h'' offer a nontrivial contract to household h' , but not to each other. The pair (h, h'') are still connected using two iterations of (2):

$$\lambda^h = \mu_{h'}^h \cdot \lambda^{h'} = \mu_{h'}^h \cdot \left(\frac{\lambda^{h''}}{\mu_{h'}^{h''}} \right). \quad (4)$$

If all households $h' > 1$ have a nontrivial contract connection with $h = 1$, or with any household that has a nontrivial contract connection with $h = 1$, or with any household that has a nontrivial contract connection with a household that has a nontrivial contract connection with $h = 1$, ..., and so forth, then (3) and (4) imply that

$$\forall h' > 1 : \quad \lambda^1 = \kappa^{h'} \cdot \lambda^{h'} \quad \text{for some } \kappa^{h'} \in \mathbb{R}_{++}.$$

Therefore, the resulting *financial equilibrium* allocation is Pareto optimal.

Define the *branch* of households originating with household $h = 1$ and connecting all households $h' : \lambda^1 = \kappa^{h'} \lambda^{h'}$ for some $\kappa^{h'} > 0$ as the set $H_1 \subseteq H$. These households are usually connected through an iterated linking of nontrivial contracts, though it is possible that $\lambda^1 = \kappa^{h'} \lambda^{h'}$ for some $\kappa^{h'} > 0$ even without such a contract connection.

Suppose that household $h \notin H_1$. This requires that $\forall \kappa^h > 0$, $\lambda^1 \neq \kappa^h \lambda^h$. This implies that h does not have a contract connection with any household $h' \in H_1$. Thus, a second *branch* can be defined where household $h_2 = \min \{H \setminus H_1\}$ is the household that can make contract offers to all of the remaining households $h \in H \setminus H_1$. Define the *branch* of households originating with household h_2 and connecting all households $h : \lambda^{h_2} = \kappa^h \lambda^h$ for some $\kappa^h > 0$ as the set

$H_2 \subseteq H \setminus H_1$. By induction, define *branches* $i \in I = \{1, \dots, i^*\}$ so that $H = H_1 \cup \dots \cup H_{i^*}$. For simplicity, define $h_1 = 1$ and $h_i = \min \{\cap_{k < i} (H \setminus H_k)\}$ for $i = 2, \dots, i^*$. By definition, $\forall h \in H_i, \lambda^{h_i} = \kappa^h \lambda^h$ for some $\kappa^h > 0$.

3.2 Structure of Branches

Lemma 1 verifies that if it's possible for a *branch* to contain all households $h \in \cap_{k < i} (H \setminus H_k)$ that do not belong to the first $(i - 1)$ branches H_1, \dots, H_{i-1} , then it is also optimal for household $h_i = \min \cap_{k < i} (H \setminus H_k)$ to offer contracts so that $H_i = \cap_{k < i} (H \setminus H_k)$.

Definition 3 *The subset of households $A \subseteq H$ is achievable if $\#A - 1 \leq \frac{1}{2} \sum_{h \in A} \chi^h$, where $\#A$ indicates the number of elements in the set A .*

This definition accounts for the fact that contracts satisfying $(CL) \forall h \in A$ can be written that connect all households $h \in A$. Consider that all households $h \in A$ can be connected by as few as $\#A - 1$ contracts. The term $\frac{1}{2} \sum_{h \in A} \chi^h$ is the maximum number of bilateral contracts permitted for this subset according to (CL) .

Lemma 1 *For any i , if a branch $A = \cap_{k < i} (H \setminus H_k)$ is achievable, then household $h_i = \min A$ finds it optimal to offer contracts such that $H_i = A$.*

Proof. In the process of proving existence (see Section A.1), I will show that if $h^* - 1 \leq \frac{1}{2} \sum_{h \in H} \chi^h$ (meaning that H is achievable), then household $h = 1$ finds it optimal to offer contracts so that $H_1 = H$. This confirms Lemma 1 for the case $i = 1$ and $A = H$. The exact same argument (from Section A.1) can be used to show that Lemma 1 holds for any i and any $A = \cap_{k < i} (H \setminus H_k)$. ■

Using this result, we can determine the total number of branches, i^* , as a function of the number of households, h^* , and the total number of contract limits, $\sum_{h \in H} \chi^h$.

$$\begin{aligned} i^* = 1 & \quad \frac{1}{2} \sum_{h \in H} \chi^h \geq h^* - 1 \\ i^* = 2 & \quad \frac{1}{2} \sum_{h \in H} \chi^h \in [h^* - 2, h^* - 1) \\ & \quad \vdots \\ i^* = k & \quad \frac{1}{2} \sum_{h \in H} \chi^h \in [h^* - k, h^* - k + 1) \end{aligned}$$

Under Assumption 4, $\chi^h \geq 1 \forall h \in H$. Thus, $\frac{1}{2} \sum_{h \in H} \chi^h \geq \frac{h^*}{2}$, meaning that the number of *branches* is bounded: $i^* \leq \frac{h^*}{2}$.

Given Lemma 1 and the analysis in Subsection 3.1, I can restate Theorem 2, the main result of this paper, as a Corollary.

Corollary 1 *Under Assumptions 1-5, if $h^* - 1 \leq \frac{1}{2} \sum_{h \in H} \chi^h$, then any financial equilibrium allocation is Pareto optimal.*

The following theorem shows that under the same conditions used for Corollary 1, the existence of a *financial equilibrium* is guaranteed. The proof proceeds by construction, beginning with a Pareto optimal allocation.

Theorem 3 *Under Assumptions 1-5, if $h^* - 1 \leq \frac{1}{2} \sum_{h \in H} \chi^h$, then a financial equilibrium exists.*

Proof. See Section A.1. ■

3.3 Equilibria Equivalence

The following result verifies the equivalence (in terms of equilibrium allocation) between the set of *unconstrained financial equilibria* and the set of *financial equilibria*. The result is valid even if the conditions for the First Basic Welfare Theorem are not met. Due to this result, I make all future statements only in terms of the set of *financial equilibria*.

Theorem 4 *The set of unconstrained financial equilibria is allocation-equivalent to the set of financial equilibria.*

Proof. See Section A.2. ■

4 "Incomplete" Markets

For this section, I hold fixed the partition of H into *branches* $(H_i)_{i \in I}$. The results that I obtain for any one partition will hold for any of the other possible partitions.

I need to introduce some additional notation and the mathematical tools necessary to prove the generic regularity of *financial equilibria*. As a corollary of the regularity result, I prove that if more than one *branch* exists, $h^* - \frac{1}{2} \sum_{h \in H} \chi^h > 1$, then any *financial equilibrium* allocation is generically Pareto suboptimal (Subsection 4.1). Moreover, if $1 < h^* - \frac{1}{2} \sum_{h \in H} \chi^h \leq s^* + 1$ and $l^* > 1$, then any *financial equilibrium* allocation is generically constrained Pareto suboptimal (Subsection 4.2).

4.1 Regularity

I begin by normalizing the commodity prices $p_{l^*}(s) = 1 \forall s \in S$.

Recalling (AC), in equilibrium, all proposals made to household h are such that the utility for h is equal to the utility it would receive by rejecting all contract proposals: $u^h \left\{ \sigma^h \left[b^h \left((\gamma^k)_{k \neq h} : (\gamma_h^{h'})_{h' < h} = 0 \right) \right] \right\}$. By rejecting all contract proposals, h then has the freedom to make contract proposals to households $h' > h$ (as specified by the best response function b^h). This is of course only an off-the-equilibrium path consideration as all contract proposals are accepted in equilibrium (as shown in Theorem 4). Define this outside utility value as

$$U^h = u^h \left\{ \sigma^h \left[b^h \left((\gamma^k)_{k \neq h} : (\gamma_h^{h'})_{h' < h} = 0 \right) \right] \right\} \quad \forall h > 1.$$

In equilibrium, these values $(U^h)_{h > 1}$ are fixed.

Using Lemma 1 and the fact that the subsets $(H_i)_{i \in I}$ are held fixed, any *financial equilibrium* of the true economy can be expressed as the unique equilibrium of a hypothetical economy with two features. The first feature requires that the only contract proposals are made by households $h \in \{h_1, \dots, h_{i^*}\}$. The second feature requires that these proposals must provide the households $h' \notin \{h_1, \dots, h_{i^*}\}$ with utility value at least equal to $(U^{h'})_{h' \notin \{h_1, \dots, h_{i^*}\}}$, which are the fixed outside utility values obtained in an equilibrium of the true economy. In this hypothetical economy, the contract proposal from $h \in \{h_1, \dots, h_{i^*}\}$ to a particular $h' \notin \{h_1, \dots, h_{i^*}\}$ is given by $-\delta^{h'} \in \mathbb{R}^{s^*+1}$. This means that this one contract has financial payoff $-\delta^{h'}$ for household h and $\delta^{h'}$ for household h' . The contract proposals $(\delta^{h'})_{h' \notin \{h_1, \dots, h_{i^*}\}}$ from households $h \in \{h_1, \dots, h_{i^*}\}$ must satisfy

$$u^{h'} \left\{ \sigma^{h'} \left[\delta^{h'} \right] \right\} \geq U^{h'} \quad \forall h' \notin \{h_1, \dots, h_{i^*}\}, \quad (5)$$

where the parameters $(U^{h'})_{h' > 1}$ are the equilibrium outside utility values in the true economy.

For any $i \in I$, the household problems for $h \in H_i$ are updated as:

$\begin{array}{ll} \max_{x^h, (\delta^{h'})_{h' \in H_i \setminus \{h\}}} & u^h(x^h) \\ \text{subj. to} & \begin{array}{l} 1. \quad P(e^h - x^h) - \sum_{h' \in H_i \setminus \{h\}} \delta^{h'} \geq 0. \\ 2. \quad u^{h'} \left\{ \sigma^{h'} \left[\delta^{h'} \right] \right\} \geq U^{h'} \quad \forall h' \in H_i \setminus \{h\}. \end{array} \end{array}$	(6)
$\begin{array}{ll} \max_{x^h} & u^h(x^h) \\ \text{subj. to} & P(e^h - x^h) + \delta^h \geq 0. \end{array}$	if $h \neq h_i$

Let $(\lambda^h)_{h \in H}$ be the Lagrange multipliers that correspond to the budget constraints in

(6). From the first order conditions, $\lambda^h \in \mathbb{R}_{++}^{s^*+1} \forall h \in H$. Let $(\mu^{h'})_{h' \notin \{h_1, \dots, h_{i^*}\}}$ be the Lagrange multipliers that correspond to the *acceptance conditions* in (6). From (2), $\mu^{h'} > 0 \forall h' \notin \{h_1, \dots, h_{i^*}\}$. The equilibrium set of variables is defined as:

$$\xi = \left((x^h, \lambda^h)_{h \in H}, (\delta^{h'}, \mu^{h'})_{h' \notin \{h_1, \dots, h_{i^*}\}}, p \right).$$

The variables $\xi \in \Xi = \prod_{h \in H} (X^h \times \mathbb{R}_{++}^{s^*+1}) \times \mathbb{R}^{(h^* - i^*) \cdot (s^* + 1)} \times \mathbb{R}_{++}^{h^* - i^*} \times \mathbb{R}_{++}^{g^* - (s^* + 1)}$, an open set. Define the set of parameters as $\theta = (e^h, u^h)_{h \in H}$. The parameters belong to the set $\Theta = \mathcal{E} \times \mathcal{U}$, also an open set.

The equilibrium manifold $\Phi : \Xi \times \Theta \rightarrow \mathbb{R}^n$ is defined by n equations, where $n = h^*(g^* + s^* + 1) + (h^* - i^*)(s^* + 2) + g^* - (s^* + 1)$, and is defined so that $\xi \in \Xi$ is a *financial equilibrium* iff $\Phi(\xi, \theta) = 0$ for

$$\Phi = \left(\begin{array}{c|c} \left((Du^h(x^h) - \lambda^h P)^T \right)_{h \in H} & FOCx \\ \hline \left(\begin{array}{c} P(e^h - x^h) - \sum_{h' \in H_i \setminus \{h\}} \delta^{h'} \quad \text{if } h = h_i \\ P(e^h - x^h) + \delta^h \quad \text{if } h \neq h_i \end{array} \right)_{i \in I} & BC \\ \hline \left((\mu^h \lambda^h - \lambda^{h_i})_{h \in H_i \setminus \{h_i\}} \right)_{i \in I} & FOC\delta \\ \hline \left((u^h \{ \sigma^h [\delta^h] \} - U^h)_{h \in H_i \setminus \{h_i\}} \right)_{i \in I} & AC \\ \hline \left(\sum_{h \in H} e_{\setminus l^*}^h(s) - x_{\setminus l^*}^h(s) \right)_{s \in S} & MCx \end{array} \right),$$

where the notation $x_{\setminus l^*}^h(s)$ denotes the column vector containing the elements $l \neq l^* : (x_1^h(s), \dots, x_{l^*-1}^h(s))^T$.

Define the equilibrium projection $\pi : \Xi \times \Theta \rightarrow \Theta$ so that $\theta \in \pi(\xi, \theta)$ iff $\Phi(\xi, \theta) = 0$. The mapping π is *proper* iff π is C^0 and for compact $\Theta' \subset \Theta$, $\pi^{-1}(\Theta')$ is also compact.

Theorem 5 *Under Assumptions 1-5, the set of regular values of π is a generic subset of Θ . Specifically, all financial equilibria $\pi^{-1}(\theta)$ are regular for all $(u^h)_{h \in H} \in \mathcal{U}$ and for all $(e^h)_{h \in H}$ in a generic subset of \mathcal{E} .*

Proof. To prove Theorem 5, it suffices to prove that (a) π is proper and (b) $D_{\xi, e} \Phi(\xi, \theta)$ has full row rank n whenever $\Phi(\xi, \theta) = 0$. The proof of part (a) is standard and the details can be found in Villanacci et al. (2002). The proof of part (b) is contained in Section A.3. ■

The following corollary addresses the Pareto optimality of the equilibrium allocations.

Corollary 2 *Under Assumptions 1-5, over a generic subset of household endowments \mathcal{E} and provided $h^* - \frac{1}{2} \sum_{h \in H} \chi^h > 1$, then any financial equilibrium allocation is Pareto suboptimal.*

Proof. See Section A.4. ■

4.2 Generic Constrained Pareto Suboptimality

The previous subsection demonstrated that any *financial equilibrium* allocation is generically Pareto suboptimal. The comparison between an equilibrium allocation and a Pareto optimal allocation seems unfair as a Pareto optimal allocation implicitly allows for transfers between any pair of households. Given the contract limits, an equilibrium allocation can only be achieved through a limited number of bilateral contracts. I define the property of *constrained Pareto optimality* to incorporate this restriction.

For any allocation, the definition of a *branch* can be used to construct the partition $(H_i)_{i \in I}$. By the definition of a *branch*, a nominal transfer can be made between any two households in a *branch* as these households are linked through a series of contracts.

Definition 4 *The vector of nominal transfers $(\tau^h)_{h \in H} \in \mathbb{R}^{h^*(s^*+1)}$ is feasible with respect to the partition $(H_i)_{i \in I}$ if $\sum_{h \in H_i} \tau^h = 0 \forall i \in I$.*

Definition 5 *An allocation $(x^h)_{h \in H}$ is constrained Pareto optimal if there does not exist a vector of transfers $(\tau^h)_{h \in H} \in \mathbb{R}^{h^*(s^*+1)}$ that are feasible with respect to the partition $(H_i)_{i \in I}$ corresponding to $(x^h)_{h \in H}$ satisfying the two properties:*

- (i) $(\hat{x}^h)_{h \in H}$ are equilibrium consumption choices of the static household problem (SHP) given $(\tau^h)_{h \in H}$ and
- (ii) $(\hat{x}^h)_{h \in H}$ Pareto dominates $(x^h)_{h \in H}$.

The following result proves that the bargaining framework is not sufficient to ensure that the *financial equilibrium* allocations are constrained Pareto optimal. That is, given a *financial equilibrium*, a planner can intervene with nominal transfers only between households in the same *branch* and still make all households strictly better off. The result requires that multiple physical commodities are traded in each state.

Assumption 6 $l^* > 1$.

Theorem 6 *Under Assumptions 1-6, over a generic subset of household endowments and utility functions $\mathcal{E} \times \mathcal{U}$, if $1 < h^* - \frac{1}{2} \sum_{h \in H} \chi^h \leq s^* + 1$, then any financial equilibrium allocation is constrained Pareto suboptimal.*

Proof. See Section A.5. ■

5 Asymmetric Information

This section analyzes the effects of asymmetric information. The theory of asymmetric information is discussed in Subsection 5.1. Basically, a household making contract proposals cannot distinguish among the households that it is proposing contracts to. Thus, it is possible that a contract intended for a certain household is actually accepted by a different household. An example of this nature is considered in Subsection 5.2. The key question is whether the incentive compatibility constraints are satisfied for the *financial equilibrium* with complete information. If they are not satisfied, then the equilibrium contracts with asymmetric information will likely differ. This difference has normative implications.

5.1 Updating the Model

As in Section 2, the parameters $(\chi^h)_{h \in H}$ and household labeling are common knowledge. Now, however, household h cannot observe the household parameters $(e^{h'}, u^{h'})$ for $h' > h$. The household h does observe the distribution $\{(e^{h'}, u^{h'})_{h' > h}\}$, but not which parameters belong with which household.

As a consequence, the contract proposals $\gamma^h = (\gamma_{h'}^h)_{h' > h}$ are not household specific, whereby $\gamma_{h'}^h$ is for household h' and only h' , $\gamma_{h''}^h$ is for household h'' and only h'' , and so forth. Rather, the vector γ^h is a menu of proposals, and each can be accepted by any household $h' > h$.

The definition of an equilibrium with asymmetric information requires the following adjustment of the definition of an *unconstrained financial equilibrium*: $\tilde{\gamma}_{h'}^h$ is chosen by household h' from the set $\{(\gamma_k^h)_{k > h}, 0\}$ to maximize utility.

5.2 Example 1

Consider an economy with three households ($h^* = 3$), only one state of uncertainty in the final period ($s^* = 1$), and two commodities traded in each state ($l^* = 2$). Household $h = 1$ has a contract limit of $\chi^1 = 2$, while households $h = 2, 3$ have contract limits of $\chi^2 = \chi^3 = 1$. As $\frac{1}{2} \sum_{h \in H} \chi^h \geq h^* - 1$, the *financial equilibrium* allocations (complete information) are Pareto optimal. The utility functions for all households $h \in H$ are of the Cobb-Douglas form:

$$u^h(x^h) = \alpha^h(0) \log(x_1^h(0)) + (1 - \alpha^h(0)) \log(x_2^h(0)) + \beta^h \{ \alpha^h(1) \log(x_1^h(1)) + (1 - \alpha^h(1)) \log(x_2^h(1)) \}$$

The utility parameters are given in Table I.

$h = 1$	$\beta^1 = 3.5$	$\alpha^1(0) = 0.228$	$\alpha^1(1) = 0.567$
$h = 2$	$\beta^2 = 0.49$	$\alpha^2(0) = 0.25$	$\alpha^2(1) = 0.824$
$h = 3$	$\beta^3 = 0.31$	$\alpha^3(0) = 0.444$	$\alpha^3(1) = 0.5$

Table I: Utility parameters

The household endowments are given in Table II.

$h = 1$	$e^1 = (5, 6, 3, 7)$
$h = 2$	$e^2 = (4, 6, 7, 3)$
$h = 3$	$e^3 = (8, 5, 3, 6)$

Table II: Endowments

The unique *financial equilibrium* (complete information) can be calculated and is given in Table III.⁸

$p_1(0) = \frac{1}{3}$	$p_2(0) = \frac{2}{3}$	$p_1(1) = \frac{2}{3}$	$p_2(1) = \frac{1}{3}$
$\gamma_2^1 = (-1.569, 2.322)$			
$\gamma_3^1 = (-1.569, 2.122)$			
$x^1 = (1.732, 2.928, 7.459, 11.412)$		$u^1(x^1) = 8.637$	
$x^2 = (5.176, 7.765, 4.132, 1.771)$		$u^2(x^2) = 2.569$	
$x^3 = (10.092, 6.307, 1.409, 2.818)$		$u^3(x^3) = 2.262$	

Table III: Financial equilibrium

With complete information, household $h = 1$ can make the offer γ_2^1 to only household $h = 2$, who in turn accepts the offer. Likewise, $h = 1$ can make the offer γ_3^1 to only household $h = 3$, who in turn accepts the offer. With asymmetric information, this is no longer possible. Consider what would happen if $h = 1$ were to make the same two offers as in Table III. Both households $h = 2, 3$ would elect to accept contract γ_3^1 as it dominates γ_2^1 (recall that payoffs for $h = 2, 3$ are $-\gamma_h^1$).

There are two options available to household $h = 1$ in the model with asymmetric information, where the contract proposals are given by $(\hat{\gamma}_2^1, \hat{\gamma}_3^1) \in \mathbb{R}^2$:

1. Offer a pooling contract $\hat{\gamma}_2^1 = \hat{\gamma}_3^1 = \hat{\gamma}^*$ that is accepted by both households (this could

⁸Uniqueness is guaranteed in this particular example because in equilibrium both (i) $u^h(\sigma^h(0)) = u^h(e^h)$ for $h = 2, 3$, and (ii) $U^h = u^h(e^h)$ for $h = 2, 3$, where U^h is the outside utility that can be achieved only through a contract $\gamma(2, 3)$. By construction, the initial endowments (e^2, e^3) are Pareto optimal for the subeconomy consisting of only $h = 2, 3$. This allows $h = 1$ to make offers that only provide the utility of endowment $(u^2(x^2), u^3(x^3)) = (u^2(e^2), u^3(e^3))$ to $h = 2, 3$.

include a trivial contract).

2. Offer a menu of separating contracts $\hat{\gamma}_2^1 \neq \hat{\gamma}_3^1$ such that only $h = 2$ accepts the contract $\hat{\gamma}_2^1$ and only $h = 3$ accepts the contract $\hat{\gamma}_3^1$. The contract $\hat{\gamma}_3^1 = \gamma_3^1$ can continue to be offered, but a new contract $\hat{\gamma}_2^1$ must be written so that both (i) $h = 2$ accepts $\hat{\gamma}_2^1$ rather than $\hat{\gamma}_3^1$ and (ii) $h = 3$ continues to accept $\hat{\gamma}_3^1$.

Pooling equilibrium If household $h = 1$ elects to offer a pooling contract, the optimal pooling contract $\hat{\gamma}^*$ is determined from the following maximization problem:

$$\begin{aligned} & \max_{\hat{\gamma}^* \in \mathbb{R}^2} && u^1(x^1) \\ \text{subj. to} & && \begin{aligned} & 1. \quad u^2(x^2) \geq u^2(e^2) \\ & 2. \quad u^3(x^3) \geq u^3(e^3) \end{aligned} \quad (AC) \end{aligned} \quad (7)$$

where the household consumption choices are given by

$$\begin{aligned} x^1(s) &= \left(\alpha^1(s) \frac{p(s)e^1(s) + 2\hat{\gamma}^*(s)}{p_1(s)}, (1 - \alpha^1(s)) \frac{p(s)e^1(s) + 2\hat{\gamma}^*(s)}{p_2(s)} \right) & s \in S \\ x^h(s) &= \left(\alpha^h(s) \frac{p(s)e^h(s) - \hat{\gamma}^*(s)}{p_1(s)}, (1 - \alpha^h(s)) \frac{p(s)e^h(s) - \hat{\gamma}^*(s)}{p_2(s)} \right) & h > 1, s \in S \end{aligned}$$

Note that the conditions (AC) are correct as written given the facts from the footnote preceding Table III, namely that the outside utility from rejecting a proposal from $h = 1$ equals the utility of endowment for both $h = 2, 3$.

It is not possible for the contract $\hat{\gamma}^*$ to be such that (AC) binds for both households. If the complete information contract offers are made, household $h = 2$ does not accept the offer γ_2^1 , electing instead to accept γ_3^1 . Thus, the optimal pooling contract will be such that (AC) does not bind for $h = 2$, but does bind for $h = 3$. Use μ^3 as the Lagrange multiplier for the household $h = 3$ constraint (AC) in (7). The first order conditions of the maximization problem in (7) are then given by:

$$\begin{aligned} \frac{2}{p(0)e^1(0) + 2\hat{\gamma}^*(0)} - \frac{\mu^3}{p(0)e^3(0) - \hat{\gamma}^*(0)} &= 0. \\ \frac{2}{p(1)e^1(1) + 2\hat{\gamma}^*(1)} - \frac{\mu^3}{p(1)e^3(1) - \hat{\gamma}^*(1)} &= 0. \end{aligned}$$

Solving the system of equations, the equilibrium with the pooling contract can be calculated

and is given in Table IV.

$p_1(0) = 0.332$	$p_2(0) = 0.668$	$p_1(1) = 0.672$	$p_2(1) = 0.328$
$\hat{\gamma}^* = (-2.175, 6.132)$			
$x^1 = (1.900, 3.188, 7.026, 11.014)$		$u^1(x^1) = 8.551$	
$x^2 = (5.118, 7.620, 4.506, 1.979)$		$u^2(x^2) = 2.597$	
$x^3 = (9.981, 6.192, 1.468, 3.008)$		$u^3(x^3) = 2.262$	

Table IV: Equilibrium under pooling contract

Separating equilibrium At the commodity prices found in Table IV, what would an optimal menu of separating contracts look like? The contract $\hat{\gamma}_3^1 = \gamma_3^1$ can remain unchanged, but the contract $\hat{\gamma}_2^1$ needs to be adjusted so that both (i) $h = 2$ accepts $\hat{\gamma}_2^1$ rather than $\hat{\gamma}_3^1$ and (ii) $h = 3$ continues to accept $\hat{\gamma}_3^1$. The optimal offer $\hat{\gamma}_2^1$ is determined as the solution to the following maximization problem:

$$\begin{aligned} & \max_{\hat{\gamma}_2^1 \in \mathbb{R}^2} && u^1(x^1) \\ \text{subj. to} & \begin{aligned} & 1. \quad u^2[\sigma^2(\hat{\gamma}_2^1)] \geq u^2[\sigma^2(\hat{\gamma}_3^1)] \\ & 2. \quad u^3[\sigma^3(\hat{\gamma}_3^1)] = u^3(e^3) \geq u^3[\sigma^3(\hat{\gamma}_2^1)] \end{aligned} && (IC) \end{aligned} \quad (8)$$

where the household demand functions are given by

$$\sigma^h(s, \hat{\gamma}_k^1) = \left(\alpha^h(s) \frac{p(s)e^h(s) - \hat{\gamma}_k^1(s)}{p_1(s)}, (1 - \alpha^h(s)) \frac{p(s)e^h(s) - \hat{\gamma}_k^1(s)}{p_2(s)} \right) \quad \begin{aligned} & h, k \in (H \setminus \{1\})^2, \\ & s \in S. \end{aligned}$$

and the "incentive compatibility conditions" (*IC*) specify that the correct household is accepting each contract.

It is not possible for the contract $\hat{\gamma}_2^1$ to be such that (*IC*) binds for both households. The optimal separating contracts are such that (*IC*) only binds for $h = 2$. Use μ^2 as the Lagrange multiplier for the household $h = 2$ constraint (*IC*) in (8). The first order conditions of the maximization problem in (8) are then given by:

$$\begin{aligned} & \frac{1}{p(0)e^1(0) + \hat{\gamma}_2^1(0) + \hat{\gamma}_3^1(0)} - \frac{\mu^2}{p(0)e^2(0) - \hat{\gamma}_2^1(0)} = 0. \\ & \frac{1}{p(1)e^1(1) + \hat{\gamma}_2^1(1) + \hat{\gamma}_3^1(1)} - \frac{\mu^2}{p(1)e^2(1) - \hat{\gamma}_2^1(1)} = 0. \end{aligned}$$

Solving the system of equations that characterize an optimal solution to (8), the optimal separating contracts and $h = 1$ consumption, for the commodity prices given in Table IV,

are:

$p_1(0) = 0.332$	$p_2(0) = 0.668$	$p_1(1) = 0.672$	$p_2(1) = 0.328$
$\hat{\gamma}_2^1 = (1.361, -14.831)$			
$\hat{\gamma}_3^1 = (-1.569, 2.122)$			
$x^1 = (3.806, 6.385, 0.121, 0.190)$			$u^1(x^1) = -4.971$

The utility u^1 would be lower than what can be achieved through a pooling contract, so $h = 1$ chooses not to offer separating contracts.

Comments Recall that the commodity markets are Walrasian, so the household $h = 1$ takes the commodity prices as fixed. If the household $h = 1$ were to offer a pooling contract, then the prices required to satisfy market clearing must be $p^* = (0.332, 0.668, 0.672, 0.328)$ as in Table IV. At these prices p^* , it is not optimal for the household $h = 1$ to offer separating contracts. However, if the household $h = 1$ were to offer separating contracts, the prices required to satisfy market clearing would be different; call them \bar{p} . At these different prices \bar{p} , it may be the case that it is optimal to offer separating contracts and not a pooling contract. This remains to be seen, but certainly the presence of two equilibria, one involving a pooling contract at prices p^* and the second involving separating contracts at prices \bar{p} , is a possibility.

What is certainly true is that Table IV characterizes one equilibrium in this model with asymmetric information, and in this equilibrium a pooling contract offer is used. Comparing the equilibrium with complete information in Table III and the equilibrium with asymmetric information in Table IV, asymmetric information reduces the utility of household $h = 1$ by 1%, which is equivalent to a 1.9% decrease in consumption. Additionally, the equilibrium allocation with asymmetric information is no longer Pareto optimal.

6 Concluding Remarks

Within the canonical model of dynamic uncertainty, this paper has considered the implications of allowing strategic bargaining over the contracts transferring wealth across states of uncertainty. In this bargaining environment, contract limits replace the fixed asset structure present in the perfectly competitive GEI model. If the contract limits permit all households to be linked via a series of contracts, then the equilibrium allocations are Pareto optimal.

Consider that the introduction of strategic interaction has not introduced inefficiency into the competitive model. The inefficiency that occurs is due to both the contract limits and the fact that households fail to internalize the impact that financial decisions have on relative commodity prices. Generalizing the model to include asymmetric information or a different bargaining process can introduce inefficiency, the former being shown in this

paper. Regarding the bargaining process, future research will analyze the effects of sequential contract proposals.

A Proofs of Main Results

A.1 Proof of Theorem 3

The proof proceeds in two steps to construct a *financial equilibrium*. **Step 1**, using the fact that any *financial equilibrium* allocation is Pareto optimal, finds the real variables: household consumption $(x^h)_{h \in H}$ and commodity prices p . **Step 2** finds the financial variables: contract proposals $(\gamma^h)_{h \in H}$. We begin with the easier step, **Step 2**.

Step 2: There may be multiple sets of contracts that can support the *financial equilibrium* allocation $(x^h)_{h \in H}$ and commodity prices p . Select one vector of nontrivial contract proposals that satisfies $(CL) \forall h \in H$ and connects all households. Define the nontrivial contract proposals so that the budget constraints are satisfied:

$$P(e^h - x^h) + \sum_{h' > h} \gamma_{h'}^h - \sum_{h' < h} \gamma_h^{h'} = 0 \quad \forall h \in H.$$

The remaining contracts in $(\gamma^h)_{h \in H}$ can be defined as trivial contracts.

Step 1: The allocation and commodity prices of a *financial equilibrium* are constructed as a fixed point of the mapping Ψ . Define the price set as $p = (p(s))_{s \in S} \in (\Delta^{l^*-1})^{s^*+1}$, a compact and convex set. For all $h > 1$, define the set of outside utility values $\Lambda_h = [u^h(\sigma^h(0)), u^h(\sum_{h \in H} e^h)]$ for $h > 1$. All together, $\Lambda = \times_{h > 1} \Lambda_h$ is a compact and convex set. I will define the self map $\Psi : (\Delta^{l^*-1})^{s^*+1} \times \Lambda \rightarrow (\Delta^{l^*-1})^{s^*+1} \times \Lambda$.

The mapping Ψ is composed of two functions: Ψ_p (**Step 1(a)**) and Ψ_u (**Step 1(b)**).

Step 1(a): The first function $\Psi_p : \Lambda \rightarrow (\Delta^{l^*-1})^{s^*+1}$ works in two stages and is supported by one key fact. The two stages are: (i) given $(U^h)_{h > 1}$, the unique Pareto optimal allocation $(x^h)_{h \in H}$ of the economy is determined and (ii) if $(x^h)_{h \in H}$ is the equilibrium allocation, then the commodity prices $p = (p(s))_{s \in S}$ supporting the allocation $(x^h)_{h \in H}$ can be found. The key fact is that given $\frac{1}{2} \sum_{h \in H} \chi^h \geq h^* - 1$, household $h = 1$ finds it optimal to offer contracts so that the equilibrium allocation is equivalent to the Pareto optimal allocation $(x^h)_{h \in H}$. Stage (ii) is defined by the continuous functions (think Second Basic Welfare Theorem):

$$p_l(s) = \frac{D_{(l,s)} u^1(x^1)}{\sum_{l'} D_{(l',s)} u^1(x^1)} \quad \forall (l, s) \in L \times S.$$

Stage (i) is defined as the solution of the system $F^{PO} \left((x^h)_{h \in H}, (\mu^h)_{h > 1}; (U^h)_{h > 1} \right) = 0$ for

$$F^{PO} = \left\{ \begin{array}{l} \left((\mu^h D u^h(x^h) - D u^1(x^1))^T \right)_{h > 1} \\ \left(\sum_{h \in H} (e^h(s) - x^h(s)) \right)_{s \in S} \\ (u^h(x^h) - U^h)_{h > 1} \end{array} \right\}.$$

Lemma 2 *Holding fixed $(U^h)_{h > 1}$, the derivative matrix $D_{x, \mu} F^{PO} \left((x^h)_{h \in H}, (\mu^h)_{h > 1}; (U^h)_{h > 1} \right)$ is invertible.*

Proof. See Subsection A.1.1. ■

From Lemma 2, we can apply the Implicit Function Theorem. The implicit function $(x^h)_{h \in H} = f^{PO} \left((U^h)_{h > 1} \right)$ is C^1 .

I now verify the key fact.⁹ Denote the family of subsets

$$\mathcal{A} = \left\{ A \subseteq H : 1 \in A \text{ and } \#A - 1 \leq \frac{1}{2} \sum_{h \in A} \chi^h \right\}.$$

The subset $H \in \mathcal{A}$, since $\frac{1}{2} \sum_{h \in H} \chi^h \geq h^* - 1$. The household $h = 1$ is choosing $A \in \mathcal{A}$, the subset of households that it will be connected to. In doing this, $h = 1$ recognizes that any households in the set $H \setminus A$ must satisfy (1) $\left(p(s) \sum_{h \in H \setminus A} (e^h(s) - \hat{x}_A^h(s)) \right) = 0 \quad \forall s \in S$ and (2) $u^h(\hat{x}_A^h) \geq U^h \quad \forall h \in H \setminus A$. The objective of $h = 1$ is to solve:

$$\max_{A \in \mathcal{A}} u^1 \left(\hat{x}_A^1; (U^h)_{h > 1} \right),$$

where $(U^h)_{h > 1}$ are held fixed and $u^1 \left(\hat{x}_A^1; (U^h)_{h > 1} \right)$ is the value associated with optimal solutions of the following problem.

⁹This is the point at which I verify that Lemma 1 holds for the case of $i = 1$ and $A = H$.

$$\begin{aligned}
& \max_{(\hat{x}_A^h)_{h \in H}} && u^1(\hat{x}_A^1) \\
& \text{1.} && \left(p(s) \sum_{h \in A} (e^h(s) - \hat{x}_A^h(s)) \right) = 0 \quad \forall s \in S. \\
& \text{2.} && u^h(\hat{x}_A^h) \geq U^h \quad \forall h \in A \setminus \{1\}. \\
& \text{subj. to} && \\
& \text{3.} && (\hat{x}_A^h)_{h \in H \setminus A} : \left(p(s) \sum_{h \in H \setminus A} (e^h(s) - \hat{x}_A^h(s)) \right) = 0 \quad \forall s \in S. \\
& && u^h(\hat{x}_A^h) \geq U^h \quad \forall h \in H \setminus A.
\end{aligned} \tag{9}$$

When $A = H$, the solution $(\hat{x}_H^h)_{h \in H}$ is a Pareto optimal allocation (that is, it solves $F^{PO} \left((\hat{x}_H^h)_{h \in H}, (\mu^h)_{h>1}; (U^h)_{h>1} \right) = 0$ for some $(\mu^h)_{h>1}$). For any other $A \in \mathcal{A} \setminus \{H\}$, the solution to (9) either provides strictly lower utility for $h = 1$ or violates $u^h(\hat{x}_A^h) \geq U^h$ for some $h > 1$ (using the definition of Pareto optimality and Assumption 3). Therefore, $H = \arg \max_{A \in \mathcal{A}} u^1(\hat{x}_A^1; (U^h)_{h>1})$, finishing the argument.

Thus, the equilibrium allocation is Pareto optimal. The mapping Ψ_p is the composition of (i) the mapping from $(U^h)_{h>1}$ to Pareto optimal allocations $(x^h)_{h \in H} = f^{PO} \left((U^h)_{h>1} \right)$ and (ii) the mapping from $(x^h)_{h \in H}$ to p . As both mappings are C^1 , then Ψ_p is C^1 .

Step 1(b): The second function $\Psi_u : (\Delta^{l^*-1})^{s^*+1} \rightarrow \Lambda$ is recursively defined. Consider the problem faced by any household $h > 1$.

By construction, the outside utility value U^h is defined when h rejects all contract proposals that it receives, and chooses which households to propose contracts to. In fact, h decides more than just which households to propose contracts to, but decides the set of households that will be connected to h through contracts. That is, h chooses a set

$$A \in \mathcal{A}_h = \left\{ A \subseteq \{h, \dots, h^*\} : h \in A \text{ and } \frac{1}{2} \sum_{h \in A} \chi^h \geq \#A - 1 \right\}.$$

For each $A \in \mathcal{A}_h$, the variables that solve the household problem (10) are denoted $(\hat{x}_A^{h'})_{h' \in A}$. The household problem (10) is introduced below. The outside utility values $(U^{h'})_{h' \in A \setminus \{h\}}$ are variables in the household problem (10). Consider that the outside options for any household $h' \in A \setminus \{h\}$ are governed by the contract proposals made to $h'' > h'$, and h affects these proposals if $h'' \in A$. With slight abuse of notation, define the function $U^{h'} \left((U^{h''})_{h''>h'} \right)$ to indicate that the outside utility value for h' depends upon the outside utility values $(U^{h''})_{h''>h'}$.

For each $A \in \mathcal{A}_h$, the household problem (10) is given such that $U^h \left((U^{h'})_{h'>h} \right) =$

$\max_{A \in \mathcal{A}_h} U_A^h \left((U^{h'})_{h' > h} \right)$, where

$$\begin{aligned}
U_A^h \left((U^{h'})_{h' > h} \right) &= \max_{(\hat{x}_A^{h'})_{h' \in A}, (U^{h'})_{h' \in A \setminus \{h\}}} u^h(\hat{x}_A^h) \\
\text{subj. to} \quad & 1. \quad \left(p(s) \sum_{h' \in A} (e^{h'}(s) - \hat{x}_A^{h'}(s)) \right) = 0 \quad \forall s \in S. \\
& 2. \quad u^{h'}(\hat{x}_A^{h'}) \geq U^{h'} \left((U^{h''})_{h'' > h'} \right) \quad \forall h' \in A \setminus \{h\}.
\end{aligned} \tag{10}$$

By Assumption 3, the constraint $u^{h'}(\hat{x}_A^{h'}) - U^{h'} \left((U^{h''})_{h'' > h'} \right) \geq 0$ is strictly increasing and strictly convex in the variables $(U^{h''})_{h'' \in A: h'' > h'}$. This implies that a subset of the solutions $(U^{h'})_{h' \in A \setminus \{h\}}$ to (10) occur on the boundary, meaning that $U^{h'} = u^{h'}(\sigma^{h'}(0))$ for some h' . Example 2 in Section A.1.2 demonstrates that solutions can occur both on the boundary and in the interior, giving rise to a continuum of equilibria. The comments following the example (Section A.1.3) suggest that with multiple commodities per state ($l^* > 1$) economies with a continuum of equilibria are exceptional.

Thus, we construct only those equilibria in which the solutions $(U^{h'})_{h' \in A \setminus \{h\}}$ to (10) occur on the boundary. For any h , define the set of possible permutations of the order of households $\{h+1, \dots, h^*\}$ as $\Omega(h)$, with typical element ω . The set $\Omega(h)$ contains $(h-1)!$ elements. The permutation itself is $\phi_\omega : \{h+1, \dots, h^*\} \rightarrow \{h+1, \dots, h^*\}$ so that for permutation $\omega \in \Omega(h)$, the new order of households is given by $\phi_\omega^{-1}(h+1, \dots, h^*)$.

The intuition for the equivalent household problem in (13) is as follows. Consider the permutation $\omega \in \Omega(h)$. The household $k^* = \phi_\omega(h^*)$ has outside utility value

$$U_\omega^{k^*} = u^{k^*}(\sigma_\omega^{k^*}(0)). \tag{11}$$

Select any $\tilde{h} \in \{h+1, \dots, h^*-1\}$. The analysis employs the logic of backward induction. The values $\left(U_\omega^{\phi_\omega(h')} \right)_{h' > \tilde{h}}$ are held fixed. The outside utility of household $\tilde{k} = \phi_\omega(\tilde{h})$ is determined as $U_\omega^{\tilde{k}} = \max_{A \in \mathcal{A}_{\tilde{k}}} U_{\omega, A}^{\tilde{k}}$, where $U_{\omega, A}^{\tilde{k}}$ is the solution of the following optimization problem.

$$\begin{aligned}
U_{\omega, A}^{\tilde{k}} &= \max_{(\hat{x}_{\omega, A}^{h'})_{h' \in A}} u^{\tilde{k}}(\hat{x}_{\omega, A}^{\tilde{k}}) \\
\text{subj. to} \quad & 1. \quad \left(p(s) \sum_{h' \in A} (e^{h'}(s) - \hat{x}_{\omega, A}^{h'}(s)) \right) = 0 \quad \forall s \in S. \\
& 2. \quad u^{\phi_\omega(h')}(\hat{x}_{\omega, A}^{\phi_\omega(h')}) \geq U_\omega^{\phi_\omega(h')} \quad \forall h' > \tilde{h}.
\end{aligned} \tag{12}$$

For any $\omega \in \Omega(h)$, the values $(U_\omega^{h'})_{h' > h}$ are found by backward induction using (11) and

(12). The household problem for h that is equivalent to (10) is:

$$\begin{aligned} & \max_{\omega \in \Omega(h)} u^h(x_\omega^h) \\ & \text{subj. to } (U_\omega^h)_{h' > h} \text{ are found from (11) and (12).} \end{aligned} \quad (13)$$

This recursive process continues until $h = 1$ and the set $\Omega(1)$ of possible permutations of the order of households $\{2, 3, \dots, h^*\}$. The utility values of (11) and (12) are the building blocks for eventually obtaining the optimal $\omega \in \Omega(1)$ and the corresponding outside utility values $(U^h)_{h > 1}$. Applying the Maximum Theorem to (12) implies that all utility values, and importantly $(U^h)_{h > 1}$, are continuous functions of p .

In conclusion, the function Ψ_u mapping $p \mapsto (U^h)_{h > 1}$ is C^0 .

Define the mapping $\Psi = (\Psi_p, \Psi_u) : (\Delta^{l^*-1})^{s^*+1} \times \Lambda \rightarrow (\Delta^{l^*-1})^{s^*+1} \times \Lambda$. By definition, if $(p, (U^h)_{h > 1}) = \Psi(p, (U^h)_{h > 1})$, then the corresponding allocation $(x^h)_{h \in H} = f^{PO}((U^h)_{h > 1})$ is the equilibrium allocation. The mapping Ψ is a continuous self-map over a convex and compact set, so Brouwer's Fixed Point Theorem guarantees the existence of a fixed point.

A.1.1 Proof of Lemma 2

To show full rank, I set the equation

$$\left((\Delta x_h^T)_{h > 1}, \Delta p^T, (\Delta \mu_h)_{h > 1} \right) D_{x,\mu} F^{PO}(\cdot) = 0, \quad (14)$$

and must verify that $\left((\Delta x_h^T)_{h > 1}, \Delta p^T, (\Delta \mu_h)_{h > 1} \right) = 0$. (14) is given by:

$$\begin{aligned} & - \left(\sum_{h > 1} \Delta x_h^T \right) D^2 u^1(x^1) - \Delta p^T = 0 & (15.a) \\ & \mu^h \cdot \Delta x_h^T D^2 u^h(x^h) + \Delta \mu_h D u^h(x^h) - \Delta p^T = 0 \quad \forall h > 1 & (15.b) \\ & \Delta x_h^T (D u^h(x^h))^T = 0 \quad \forall h > 1 & (15.c) \end{aligned} \quad (15)$$

Post-multiply (15.b) by Δx_h and use (15.c) to obtain:

$$\mu^h \cdot \Delta x_h^T D^2 u^h(x^h) \Delta x_h - \Delta p^T \Delta x_h = 0 \quad \forall h > 1. \quad (16)$$

Summing (16) over all households $h > 1$, and using (15.a) to replace the term Δp^T , I obtain:

$$\sum_{h>1} \mu^h \cdot \Delta x_h^T D^2 u^h(x^h) \Delta x_h + \left(\sum_{h>1} \Delta x_h^T \right) D^2 u^1(x^1) \left(\sum_{h>1} \Delta x_h \right) = 0. \quad (17)$$

Using (17), the fact $(\mu^h)_{h>1} \gg 0$, and Assumption 3 ($D^2 u^h(x^h)$ negative definite), I obtain $(\Delta x_h^T)_{h>1} = 0$. From (15.a) and (15.b), $(\Delta p^T, (\Delta \mu_h)_{h>1}) = 0$. This completes the argument.

A.1.2 Example 2

Consider an economy with three households ($h^* = 3$), two states of uncertainty in the final period ($s^* = 2$), and one commodity traded in each state ($l^* = 1$). The contract limits are $(\chi^1, \chi^2, \chi^3) = (2, 1, 1)$. As $\frac{1}{2} \sum_{h \in H} \chi^h \geq h^* - 1$, then Corollary 1 implies that the resulting equilibrium allocation is Pareto optimal.

The utility functions are given by $u^1(x^1) = \frac{2}{3} \log(x^1(0)) + \frac{1}{6} \log(x^1(1)) + \frac{1}{6} \log(x^1(2))$ and $u^h(x^h) = \frac{1}{3} \log(x^h(0)) + \frac{1}{3} \log(x^h(1)) + \frac{1}{3} \log(x^h(2))$ for $h = 2, 3$, where the commodity subscripts have been omitted for simplicity. The endowments are given by $e^1 = (2, 2, 2)$, $e^2 = (2, 3, 1)$, and $e^3 = (2, 1, 3)$.

The contracts offered by $h = 1$ are set so that they are accepted by both households $h = 2, 3$, and they are optimal when they remove any incentive for a nontrivial contract γ_3^2 . This optimality for $h = 1$, in terms of the household problem (10), is achieved by extracting all possible surplus.

For this example, there are a continuum of *financial equilibria*. To illustrate this, select any value $U^* \in [u^3(e^3), U^2(u^3(e^3))]$. Set $U^3 = U^*$ and $U^2 = U^2(U^*)$.

Consider the logic. When household $h = 1$ offers contracts γ_2^1 and γ_3^1 such that $u^2(x^2) = U^2$, $u^3(x^3) = U^3$, and $\frac{(\lambda^2(1), \lambda^2(2))}{\lambda^2(0)} = \frac{(\lambda^3(1), \lambda^3(2))}{\lambda^3(0)}$, there is no incentive for $h = 2$ to offer anything but a nontrivial contract to $h = 3$. With $\frac{(\lambda^2(1), \lambda^2(2))}{\lambda^2(0)} = \frac{(\lambda^3(1), \lambda^3(2))}{\lambda^3(0)}$, the allocation (x^2, x^3) is optimal for the pair $h = 2, 3$. Any contract $\gamma_3^2 \neq 0$ would then provide lower utility for one of these households, so the only accepted contract is the trivial one.

Given $U^2 = U^2(U^*)$ and $U^3 = U^*$, the *financial equilibrium* is given by:

$x^1 = (3.125, 1.282, 1.282)$, $\gamma_2^1 = e^2 - x^2$, $\gamma_3^1 = e^3 - x^3$	$u^1(x^1) = 0.842$
$x^2 = \left(\exp\left(U^2(U^*) - \frac{1}{3}\right), \exp\left(U^2(U^*) + \frac{1}{6}\right), \exp\left(U^2(U^*) + \frac{1}{6}\right) \right)$	$u^2(x^2) = U^2(U^*)$.
$x^3 = \left(\exp\left(U^* - \frac{1}{3}\right), \exp\left(U^* + \frac{1}{6}\right), \exp\left(U^* + \frac{1}{6}\right) \right)$	$u^3(x^3) = U^*$

A.1.3 Comments on Example 2

Recall that $x^h = \sigma^h(\tau^h)$ is the implicit function of the system of equations characterizing the solutions to the static household problem (*SHP*), where $\tau^h = \sum_{h' > h} \gamma_{h'}^h - \sum_{h' < h} \gamma_h^{h'}$ are the financial payoffs. A continuum of *financial equilibria* is only possible if $D\sigma^h(\tau^h) = D\sigma^{h'}(\tau^{h'})$ for any h, h' receiving contract proposals from the same household $k < \min\{h, h'\}$. The transfers $(\tau^h, \tau^{h'})$ are those derived from equilibrium contract proposals. Recalling the result in Section 3.1, $D\sigma^h(\tau^h) = (D^2u^h(x^h))^{-1} P^T \left[-P (D^2u^h(x^h))^{-1} P^T \right]^{-1}$, where the right-hand side is evaluated at equilibrium consumption and prices. For $l^* = 1$, the term $D\sigma^h(\tau^h)$ is trivially equal for any two households. However, with $l^* > 1$, the term $D\sigma^h(\tau^h)$ is only equal for a pair of households in exceptional economies.

This suggests that with $l^* > 1$ a typical economy does not give rise to a continuum of equilibria.

A.2 Proof of Theorem 4

The definition of a *branch* remains valid for *unconstrained financial equilibria*. Suppose, for contradiction, that an *unconstrained financial equilibrium* exists in which household h makes a contract proposal $\gamma_{h'}^h \neq 0$ that is rejected by household h' . We know that h is indifferent between making this proposal and making the trivial contract proposal $\gamma_{h'}^h = 0$. What we have to verify is that the choice to propose $\gamma_{h'}^h \neq 0$ cannot support additional equilibria.

I claim that upon viewing the contract proposal $\gamma_{h'}^h \neq 0$, all households know that the proposal will be rejected, so its presence has no effect on any of the other proposed contracts. To verify the claim, there are two cases to consider.

Case I: $h, h' \in H_i$ for some i , even if the contract $\gamma_{h'}^h$ is rejected.

From the analysis in Subsection 3.1, $h, h' \in H_i$ implies $\lambda^h = \kappa^{h'} \lambda^{h'}$ for some $\kappa^{h'} > 0$. If the contract proposal $\gamma_{h'}^h \neq 0$ is accepted by household h' , then this must necessarily make h strictly worse off. And if the proposal $\gamma_{h'}^h \neq 0$ were to make h better off, then it would not be accepted by h' (as it would make h' strictly worse off). Then all households ignore the contract proposal as they know it will not be accepted.

Case II: If $\gamma_{h'}^h$ is rejected, $\nexists i$ such that $h, h' \in H_i$.

If the acceptance of this contract would cause either h or h' to violate its contract limit (*CL*), with the resulting infinitely large utility loss (see the footnote in the unconstrained household problem (*UHP*)), then all households ignore the contract proposal as they know it will not be accepted. Yet, if the contract can be accepted and both households h and h' continue to satisfy (*CL*), then a larger branch A such that $h, h' \in A$ is achievable. Citing

Lemma 1, either the contract is accepted or both households belong to the same branch (or both). The first outcome is a contradiction of the initial supposition that the contract $\gamma_{h'}^h$ would be rejected, while the second outcome is a contradiction of Case II. This completes the argument.

A.3 Proof of Theorem 5

The matrix $M = D_\xi \Phi(\xi, \theta)|_{\Phi(\xi, \theta)=0}$ is given below, where the rows correspond to equations of Φ and the columns correspond to variables $\xi = \left((x^h, \lambda^h)_{h \in H}, (\delta^h, \mu^h)_{h \notin \{h_1, \dots, h_{i^*}\}}, p \right)$, in that order. To conserve on space, I employ the following conventions:

$$c(A^h) = \begin{pmatrix} A^1 \\ \vdots \\ A^{h^*} \end{pmatrix}, \quad r(A^h) = \left(A^1 \quad \dots \quad A^{h^*} \right), \quad d(A^h) = \begin{pmatrix} A^1 & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & A^{h^*} \end{pmatrix},$$

where (c, r, d) stand for column, row, and diagonal, respectively.

$$M = \begin{pmatrix} d(D^2 u^h) & d(-P^T) & 0 & 0 & c(\Lambda_2^h) \\ d(-P) & 0 & \Lambda_3 & 0 & c(Z^h) \\ 0 & \Lambda_3^* & 0 & \Lambda_4 & 0 \\ 0 & 0 & \Lambda_4^* & 0 & 0 \\ r(-\Lambda) & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The $(s^* + 1) \times g^*$ price matrix P was previously defined in Section 2.2. The submatrices $(\Lambda, \Lambda_2^h, Z^h)$ are defined in the table below:

Matrix	Dimensions	Definition
Λ	$(g^* - s^* - 1) \times g^*$	$\begin{bmatrix} \left(I_{l^*-1} & 0 \right) & 0 & 0 \\ 0 & \dots & 0 & \\ 0 & 0 & \left(I_{l^*-1} & 0 \right) \end{bmatrix}$
Λ_2^h	$g^* \times (g^* - s^* - 1)$	$\begin{bmatrix} \left(\lambda^h(0) I_{l^*-1} \right) & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & \left(\lambda^h(s^*) I_{l^*-1} \right) \\ & & 0 \end{bmatrix}$
Z^h	$(s^* + 1) \times (g^* - s^* - 1)$	$\begin{bmatrix} \left(e_{\setminus l^*}^h(0) - x_{\setminus l^*}^h(0) \right)^T & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & \left(e_{\setminus l^*}^h(s^*) - x_{\setminus l^*}^h(s^*) \right)^T \end{bmatrix}$

The remaining submatrices $(\Lambda_3, \Lambda_3^*, \Lambda_4, \Lambda_4^*)$ are block diagonal matrices where each block corresponds to $i \in I$. Denote each of these blocks as $(\Lambda_3(i), \Lambda_3^*(i), \Lambda_4(i), \Lambda_4^*(i))$, respectively, and let $h \in H_i \setminus \{h_i\}$. The blocks $(\Lambda_3(i), \Lambda_3^*(i), \Lambda_4(i), \Lambda_4^*(i))$ are defined in the table below:

Matrix	Dimensions	Definition
$\Lambda_3(i)$	$\#H_i(s^* + 1) \times (\#H_i - 1)(s^* + 1)$	$\begin{bmatrix} -I_{s^*+1} & \dots & -I_{s^*+1} \\ I_{s^*+1} & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & I_{s^*+1} \end{bmatrix}$
$\Lambda_3^*(i)$	$(\#H_i - 1)(s^* + 1) \times \#H_i(s^* + 1)$	$\begin{bmatrix} -I_{s^*+1} & \dots & 0 & 0 \\ \vdots & 0 & \mu^h \cdot I_{s^*+1} & 0 \\ -I_{s^*+1} & 0 & 0 & \dots \end{bmatrix}$ ¹⁰
$\Lambda_4(i)$	$(\#H_i - 1)(s^* + 1) \times (\#H_i - 1)$	$\begin{bmatrix} \dots & 0 & 0 \\ 0 & (\lambda^h)^T & 0 \\ 0 & 0 & \dots \end{bmatrix}$
$\Lambda_4^*(i)$	$(\#H_i - 1) \times (\#H_i - 1)(s^* + 1)$	$\begin{bmatrix} \dots & 0 & 0 \\ 0 & \lambda^h & 0 \\ 0 & 0 & \dots \end{bmatrix}$

¹⁰The notation $\#H_i$ refers to the number of elements in the set H_i .

The final derivative matrix follows from the analysis in Section 3.1.

To show that $D_{\xi,e}\Phi(\xi,\theta)$ has full row rank, set $\nu^T D_{\xi,e}\Phi(\xi,\theta) = 0$ where

$$\nu^T = (\Delta x^T, \Delta \lambda^T, \Delta \delta^T, \Delta \mu^T, \Delta p^T) \in \mathbb{R}^n$$

is partitioned so that each subvector corresponds with the following equations of Φ :

$$\Delta x^T \iff FOCx$$

$$\Delta \lambda^T \iff BC$$

$$\Delta \delta^T \iff FOC\delta$$

$$\Delta \mu^T \iff AC$$

$$\Delta p^T \iff MCx.$$

The theorem is proved upon showing that $\nu^T = 0$.

First, for the columns corresponding to derivatives with respect to $(x^h)_{h \in H}$ and $(e^h)_{h \in H}$:

$$(\Delta x^h)^T D^2 u^h(x^h) - (\Delta \lambda^h)^T P - \Delta p^T \Lambda = 0 \quad h \in H \quad (18.a)$$

$$(\Delta \lambda^h)^T P + \Delta p^T \Lambda = 0 \quad h \in \{h_1, \dots, h_{i^*}\} \quad (18.b) \quad . \quad (18)$$

$$(\Delta \lambda^h)^T P + \Delta p^T \Lambda + \Delta \mu^h \lambda^h P = 0 \quad h \notin \{h_1, \dots, h_{i^*}\} \quad (18.c)$$

Regarding (18.c), the equation $u^h \{ \sigma^h [\delta^h] \}$ is an implicit function of e^h with derivative equal to $\lambda^h P$. This derivative is obtained as

$$\frac{\partial u^h \{ \sigma^h [\delta^h] \}}{\partial e^h} = Du^h(x^h) \cdot \frac{\partial \sigma^h [\delta^h]}{\partial e^h},$$

Φ contains the equality $Du^h(x^h) = \lambda^h P$, and the Implicit Function Theorem applied to the system $F^{SP}(x^h, \lambda^h; e^h) = 0$ (combined with some matrix algebra as in Section 3.1) yields

$$\frac{\partial \sigma^h [\delta^h]}{\partial e^h} = \left[\left(D^2 u^{h'}(x^{h'}) \right)^{-1} P^T \right] \left[-P \left(D^2 u^{h'}(x^{h'}) \right)^{-1} P^T \right]^{-1} P.$$

From the definition of Λ , (18.b) implies $(\Delta \lambda^h)^T = 0 \quad \forall h \in \{h_1, \dots, h_{i^*}\}$ and $\Delta p^T = 0$. Additionally, (18.a) and (18.b) imply $(\Delta x^h)^T D^2 u^h(x^h) \Delta x^h = 0 \quad \forall h \in \{h_1, \dots, h_{i^*}\}$. From Assumption 3, $(\Delta x^h)^T = 0 \quad \forall h \in \{h_1, \dots, h_{i^*}\}$.

Second, for households $h \notin \{h_1, \dots, h_{i^*}\}$, (18.a) and (18.c) imply $(\Delta x^h)^T D^2 u^h(x^h) =$

$-\Delta\mu^h\lambda^h P$. After postmultiplying by Δx^h :

$$(\Delta x^h)^T D^2 u^h(x^h)\Delta x^h = -\Delta\mu^h\lambda^h P\Delta x^h. \quad (19)$$

For the columns corresponding to derivatives with respect to $(\lambda^h)_{h \notin \{h_1, \dots, h_{i^*}\}}$, using the definition of Λ_3^* :

$$-(\Delta x^h)^T P^T + \mu^h (\Delta \delta^h)^T = 0. \quad (20)$$

For the columns corresponding to the derivatives with respect to $(\mu^h)_{h \notin \{h_1, \dots, h_{i^*}\}}$, using the definition of Λ_4 :

$$(\Delta \delta^h)^T (\lambda^h)^T = 0. \quad (21)$$

Combining (19), (20), and (21) yields $(\Delta x^h)^T D^2 u^h(x^h)\Delta x^h = 0 \forall h \notin \{h_1, \dots, h_{i^*}\}$. From Assumption 3, $(\Delta x^h)^T = 0 \forall h \notin \{h_1, \dots, h_{i^*}\}$.

Third, from (20), $(\Delta \delta^h)^T = 0 \forall h \notin \{h_1, \dots, h_{i^*}\}$. From (18.a) and (18.c), $(\Delta \lambda^h)^T = 0$ and $\Delta \mu^h = 0 \forall h \notin \{h_1, \dots, h_{i^*}\}$.

In conclusion, $\nu^T = 0$ and the proof of Theorem 5 is complete.

A.4 Proof of Corollary 2

A necessary condition for Pareto optimality is that $\frac{\lambda^1(1)}{\lambda^1(0)} = \frac{\lambda^{h_2}(1)}{\lambda^{h_2}(0)}$. I define the system of equations $\Phi^* : \Xi \times \Theta \rightarrow \mathbb{R}^{n+1}$ as $\Phi^* = \begin{pmatrix} \Phi \\ \lambda^1(1)\lambda^{h_2}(0) - \lambda^1(0)\lambda^{h_2}(1) \end{pmatrix}$. Define the equilibrium projection $\pi^* : \Xi \times \Theta \rightarrow \Theta$ so that $\theta \in \pi^*(\xi, \theta)$ iff $\Phi^*(\xi, \theta) = 0$.

Lemma 3 *The set of regular values of π^* is a generic subset of Θ .*

Given Lemma 3 and Φ^* with a greater number of equations than unknowns, for all $(e^h)_{h \in H}$ in a generic subset of \mathcal{E} , the equations $\Phi(\xi, \theta) = 0$ and $\lambda^1(1)\lambda^{h_2}(0) = \lambda^1(0)\lambda^{h_2}(1)$ cannot both hold. This implies that for all $(e^h)_{h \in H}$ in a generic subset of \mathcal{E} , $\lambda^1(1)\lambda^{h_2}(0) \neq \lambda^1(0)\lambda^{h_2}(1)$ and the allocation is Pareto suboptimal.

A.4.1 Proof of Lemma 3

It suffices to show that (a) π^* is proper and (b) $D_{\xi, e}\Phi^*(\xi, \theta)$ has full row rank $n+1$ whenever $\Phi^*(\xi, \theta) = 0$. The proof of part (a) is standard and the details can be found in Villanacci et al. (2002). For part (b), I apply the exact same three steps as in the proof of theorem 5. Notice that none of these steps considered the columns corresponding to the derivatives with respect to $(\lambda^1, \lambda^{h_2})$. After obtaining the linear independence of the first n rows of

$D_{\xi,e}\Phi^*(\xi, \theta)$, the final row is linearly independent by considering the columns corresponding to the derivatives with respect to $(\lambda^1, \lambda^{h_2})$. This completes the argument.

A.5 Proof of Theorem 6

The proof of this theorem will follow the framework of Citanna et al. (1998). The following two lemmas are crucial for the argument.

Lemma 4 *Over a generic subset of household endowments \mathcal{E} , $\sum_{h \in H'} (x_1^h(s) - e_1^h(s)) \neq 0 \forall s \in S$ and for any subset $H' \subset H$ with $H' \neq H$.*

Proof. See Section A.5.3. ■

Lemma 5 *Over a generic subset of household endowments \mathcal{E} and provided $i^* \leq s^* + 1$, the matrix*

$$Z = \begin{pmatrix} \lambda^1(0) \sum_{h \in H_1} (e_1^h(0) - x_1^h(0)) & \dots & \lambda^{h_{i^*}}(0) \sum_{h \in H_{i^*}} (e_1^h(0) - x_1^h(0)) \\ \vdots & & \vdots \\ \lambda^1(s^*) \sum_{h \in H_1} (e_1^h(s^*) - x_1^h(s^*)) & \dots & \lambda^{h_{i^*}}(s^*) \sum_{h \in H_{i^*}} (e_1^h(s^*) - x_1^h(s^*)) \end{pmatrix}$$

has full column rank.

Proof. See Section A.5.4. ■

Take as given a *financial equilibrium* $\left((x^h, \lambda^h)_{h \in H}, (\delta^h, \mu^h)_{h \notin \{h_1, \dots, h_{i^*}\}}, p \right)$. Given parameters $\theta = (e^h, u^h)_{h \in H}$, the variables $\hat{\xi} = \left((\hat{x}^h, \hat{\lambda}^h)_{h \in H}, \hat{p} \right)$ and feasible transfers $\tau = (\tau^h)_{h \in H}$ (with respect to the partition $(H_i)_{i \in I}$) are such that $\hat{x} = (\hat{x}^h)_{h \in H}$ is a constrained feasible allocation iff $\Gamma(\hat{\xi}, \tau, \theta) = 0$, where $\Gamma : \prod_{h \in H} (X^h \times \mathbb{R}^{s^*+1}) \times \mathbb{R}^{g^* - (s^*+1)} \times \mathbb{R}^{h^*(s^*+1)} \times \Theta \rightarrow \mathbb{R}^m$ is defined by the $m = h^*(g^* + s^* + 1) + g^* - (s^* + 1) + i^*(s^* + 1)$ equations:

$$(\hat{\xi}, \tau, \theta) \mapsto \begin{pmatrix} FOCx \\ BC \\ MCx \\ FC\tau \end{pmatrix} = \begin{pmatrix} \left((Du^h(\hat{x}^h) - \hat{\lambda}^h \hat{P})^T \right)_{h \in H} \\ \left(\hat{P} (e^h - \hat{x}^h) + \tau^h \right)_{h \in H} \\ \left(\sum_{h \in H} (e_{\setminus i^*}^h(s) - \hat{x}_{\setminus i^*}^h(s)) \right)_{s \in S} \\ \left(\sum_{h \in H_i} \tau^h \right)_{i \in I} \end{pmatrix}.$$

The analysis is conducted evaluating functions at $\tau^* = (\tau^{*h})_{h \in H}$, where $\forall i \in I$, $\tau^{*h_i} = - \sum_{h \in H_i \setminus \{h_i\}} \delta(h)$ and $\tau^{*h} = \delta(h) \forall h \in H_i \setminus \{h_i\}$. If $\Gamma(\hat{\xi}, \tau^*, \theta) = 0$ and $\Phi(\xi, \theta) = 0$, then

$$\hat{\xi} = \left((x^h, \lambda^h)_{h \in H}, p \right).$$

Define the $(h^* + m) \times (h^*(g^* + s^* + 1) + g^* - (s^* + 1) + h^*(s^* + 1))$ matrix Ψ_0 :

$$\Psi_0(\hat{\xi}, \tau, \theta) = \begin{pmatrix} D_{\hat{\xi}}U(\hat{x}) & 0 \\ D_{\hat{\xi}}\Gamma(\hat{\xi}, \tau, \theta) & D_{\tau}\Gamma(\hat{\xi}, \tau, \theta) \end{pmatrix}.$$

From Proposition 1 of Citanna et al. (1998), if Ψ_0 has full row rank, $\exists \hat{\xi} \neq \left((x^h, \lambda^h)_{h \in H}, p \right)$ s.t. $\hat{\xi}$ satisfies $\Gamma(\hat{\xi}, \tau, \theta) = 0$ (for some τ) and $U(\hat{x}) > U(x)$. To have full row rank, there must exist fewer rows than columns, so $h^* + i^*(s^* + 1) \leq h^*(s^* + 1)$ or

$$h^* \leq (h^* - i^*)(s^* + 1). \quad (22)$$

As the number of *branches* is bounded, $i^* \leq \frac{h^*}{2}$ (as discussed in Subsection 3.2), then (22) is trivially satisfied:

$$(h^* - i^*)(s^* + 1) \geq \frac{h^*}{2}(s^* + 1) \geq h^*.$$

If the matrix Ψ_0 has more columns than rows, I remove some columns (it does not matter which) in order to obtain a square matrix Ψ . This matrix Ψ does not have full rank iff $\exists \nu \in \mathbb{R}^{h^*+m}$ s.t. $\Phi'(\hat{\xi}, \tau^*, \nu, \theta) = 0$ where

$$\Phi'(\hat{\xi}, \tau^*, \nu, \theta) = \begin{pmatrix} \Psi^T \nu \\ \nu^T \nu / 2 - 1 \end{pmatrix}.$$

For simplicity, $\nu^T = (\Delta u^T, \Delta x^T, \Delta \lambda^T, \Delta p^T, \Delta \tau^T) \in \mathbb{R}^{h^*+m}$ is defined so that each subvector corresponds to an equation (row) in Ψ :

$$\begin{aligned} \Delta u^T &\iff D_{\hat{\xi}}U(\hat{x}) \\ \Delta x^T &\iff FOCx \\ \Delta \lambda^T &\iff BC \\ \Delta p^T &\iff MCx \\ \Delta \tau^T &\iff FC\tau. \end{aligned}$$

Theorem 6 is complete upon showing that for a generic choice of $\theta \in \Theta$, there does not exist (ξ, ν) s.t.

$$\begin{aligned} \Phi(\xi, \theta) &= 0 \\ \Phi' \left(\left((x^h, \lambda^h)_{h \in H}, p \right), \tau^*, \nu, \theta \right) &= 0 \end{aligned} \quad (23)$$

From Proposition 3 of Citanna et al. (1998), it suffices to show that for $\tau = \tau^*$ and $\hat{\xi} = \left((x^h, \lambda^h)_{h \in H}, p \right)$, the matrix

$$M = \left(\begin{array}{c} \Psi^T \\ \nu^T \end{array} \quad D_\theta \Phi' \right) \text{ has full row rank.} \quad (24)$$

In Case I, I suppose $(\Delta x^h)^T \neq 0 \forall h \in H$ and show that (24) holds over a generic subset of Θ . In Case II, I suppose $(\Delta x^h)^T = 0$ for some $h \in H$ and show that (23) does not have a solution for all $(e^h)_{h \in H}$ in a generic subset of \mathcal{E} .

A.5.1 Case I: $(\Delta x^h)^T \neq 0 \forall h \in H$

Lemma 6 For $\tau = \tau^*$, $D_u \Phi' = \left(\begin{array}{c} d(\dot{A}^h) \\ \vec{0} \end{array} \right)$, where $d(\dot{A}^h)$ has full row rank and corresponds to the rows for derivatives with respect to $(x^h)_{h \in H}$.

Proof. See Lemma 2 of Citanna et al. (1998) or Lemma 4 of Hoelle (2012). ■

In the matrix M , rows correspond to the variables $((x^h, \lambda^h)_{h \in H}, p)$, feasible transfers $\tau = (\tau^h)_{h \in H}$ (with respect to the partition $(H_i)_{i \in I}$), and vector v^T , in that order. I employ the same (c, r, d) convention as in the proof of Theorem 5.

$$M = \left(\begin{array}{cccccc} d(Du^h(x^h)^T) & d(D^2u^h(x^h)) & d(-P^T) & c(-\Lambda^T) & 0 & d(\dot{A}^h) \\ 0 & d(-P) & 0 & 0 & 0 & 0 \\ 0 & r(-(\Lambda_2^h)^T) & r((Z^h)^T) & 0 & 0 & 0 \\ 0 & 0 & d(I_{s^*+1}) & 0 & \Upsilon & 0 \\ r(\Delta u^h) & r((\Delta x^h)^T) & r((\Delta \lambda^h)^T) & \Delta p^T & \Delta \tau^T & 0 \end{array} \right),$$

where $(\Lambda, \Lambda_2^h, Z^h)$ were defined in the proof of Theorem 5. The submatrix Υ is the block diagonal matrix of size $h^*(s^*+1) \times i^*(s^*+1)$ in which each block $\Upsilon(i)$ is the $\#H_i(s^*+1) \times (s^*+1)$ submatrix $\Upsilon(i) = \left[I_{s^*+1} \quad \dots \quad I_{s^*+1} \right]^T$.

A subset of the equations $\nu^T \Psi = 0$, specifically corresponding to the columns of Ψ for derivatives with respect to $(x^h, \lambda^h)_{h \in H}$, are given by:

$$\begin{aligned} \left(\Delta u^h D u^h(x^h) + (\Delta x^h)^T D^2 u^h(x^h) - (\Delta \lambda^h)^T P - \Delta p^T \Lambda \right)_{h \in H} &= 0. \quad (25.a) \\ \left(-(\Delta x^h)^T P^T \right)_{h \in H} &= 0. \quad (25.b) \end{aligned} \quad (25)$$

Lemma 7 $(\Delta u^h, \Delta p^T) \neq 0 \quad \forall h \in H$.

Proof. Suppose not, that is $(\Delta u^h, \Delta p^T) = 0$ for some h . From (25.a), $(\Delta x^h)^T D^2 u^h(x^h) - (\Delta \lambda^h)^T P = 0$. Together with (25.b), $(\Delta x^h)^T D^2 u^h(x^h) \Delta x^h = 0$, which implies $(\Delta x^h)^T = 0$ (using Assumption 3). This is a contradiction of Case I. ■

From Lemmas 6 and 7, the first and last row blocks of M are linearly independent from the others. By the definition of Λ_2^h , the $[h^*(s^* + 1) + g^* - (s^* + 1)] \times h^*g^*$ submatrix

$$\begin{pmatrix} d(-P) \\ r(-(\Lambda_2^h)^T) \end{pmatrix} = \begin{pmatrix} -P & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & -P \\ -(\Lambda_2^h)^T & \dots & -(\Lambda_2^{h^*})^T \end{pmatrix} \text{ is a full rank matrix. The submatrix } d(I_{s^*+1})$$

has full row rank. Thus, M has full rank, so (24) has been met. This finishes the argument in Case I.

A.5.2 Case II: $(\Delta x^h)^T = 0$ for some $h \in H$

I will show that over a generic subset of \mathcal{E} , (23) has no solution. Suppose $\exists h' \in H$ such that $(\Delta x^{h'})^T = 0$. From (25.a) and Φ , I obtain

$$\begin{aligned} \Delta u^{h'} D u^{h'}(x^{h'}) - (\Delta \lambda^{h'})^T P - \Delta p^T \Lambda &= 0 \\ D u^{h'}(x^{h'}) - \lambda^{h'} P &= 0, \end{aligned}$$

which together imply that $\Delta p^T = 0$ and $(\Delta \lambda^{h'})^T = \Delta u^{h'} \lambda^{h'}$. For all other $h \neq h'$, postmultiply $\Delta u^h D u^h(x^h)$ by Δx^h and use Φ and (25.b) to get $\Delta u^h D u^h(x^h) \Delta x^h = 0$. Next, postmultiply (25.a) by Δx^h and use the previous fact and (25.b) to arrive at $(\Delta x^h)^T D^2 u^h(x^h) \Delta x^h = 0$. By Assumption 3, $(\Delta x^h)^T = 0 \quad \forall h \neq h'$. Thus $\forall h \in H$, $(\Delta \lambda^h)^T = \Delta u^h \lambda^h$.

The equations $\nu^T \Psi = 0$ corresponding to the columns of Ψ for derivatives with respect to $(\tau^h)_{h \in H}$ imply that $\forall i \in I$ and $\forall h \in H_i$:

$$(\Delta \lambda^h)^T + \Delta \tau^i = 0. \quad (26)$$

From (26), $(\Delta\lambda^h)^T = (\Delta\lambda^{h_i})^T \quad \forall h \in H_i$ and $\forall i \in I$. Therefore,

$$\Delta u^h \lambda^h = \Delta u^{h_i} \lambda^{h_i} \quad \forall h \in H_i \text{ and } \forall i \in I. \quad (27)$$

The following is the equation from $\nu^T \Psi = 0$ corresponding to derivatives with respect to p :

$$\sum_{h \in H} \Delta\lambda^h(s) (e_{l^*}^h(s) - x_{l^*}^h(s))^T = 0 \quad \forall s \in S. \quad (28)$$

For the analysis to hold at this point, I must use Assumption 6: $l^* > 1$. From (28), only consider the first physical commodity, $l = 1$. Using the equality $(\Delta\lambda^h)^T = \Delta u^h \lambda^h$ and (27), (28) simplifies to $\sum_{i \in I} \left[\Delta u^{h_i} \lambda^{h_i}(s) \sum_{h \in H_i} (e_1^h(s) - x_1^h(s)) \right] = 0 \quad \forall s \in S$, which can be written in matrix form as:

$$\begin{pmatrix} \lambda^1(0) \sum_{h \in H_1} (e_1^h(0) - x_1^h(0)) & \dots & \lambda^{h_{i^*}}(0) \sum_{h \in H_{i^*}} (e_1^h(0) - x_1^h(0)) \\ \vdots & & \vdots \\ \lambda^1(s^*) \sum_{h \in H_1} (e_1^h(s^*) - x_1^h(s^*)) & \dots & \lambda^{h_{i^*}}(s^*) \sum_{h \in H_{i^*}} (e_1^h(s^*) - x_1^h(s^*)) \end{pmatrix} \begin{pmatrix} \Delta u^1 \\ \vdots \\ \Delta u^{h_{i^*}} \end{pmatrix} = 0. \quad (29)$$

By assumption in Theorem 6, $i^* \leq s^* + 1$. From Lemma 5 and both (27) and (29), $\Delta u^h = 0 \quad \forall h \in H$. This implies $(\Delta\lambda^h)^T = 0 \quad \forall h \in H$ and $\Delta\tau^T = 0$, from (26). The entire vector $\nu^T = 0$, which violates the $\nu^T \nu / 2 = 1$ equation in Φ' . I conclude that for a generic choice of $\theta \in \Theta$, (23) has no solution. This completes the proof of Theorem 6.

A.5.3 Proof of Lemma 4

This proof is independent of the proof of Corollary 2, so notation will be recycled. For any

$s \in S$, I define the system of equations $\Phi^* : \Xi \times \Theta \rightarrow \mathbb{R}^{n+1}$ as $\Phi^* = \begin{pmatrix} \Phi \\ \sum_{h \in H'} (x_1^h(s) - e_1^h(s)) \end{pmatrix}$.

Define the equilibrium projection $\pi^* : \Xi \times \Theta \rightarrow \Theta$ so that $\theta \in \pi^*(\xi, \theta)$ iff $\Phi^*(\xi, \theta) = 0$.

Lemma 8 *The set of regular values of π^* is a generic subset of Θ .*

Given Lemma 8 and Φ^* with a greater number of equations than unknowns, for all $(e^h)_{h \in H}$ in a generic subset of \mathcal{E} , the equations $\Phi(\xi, \theta) = 0$ and $\sum_{h \in H'} (x_l^h(s) - e_l^h(s)) = 0$ cannot both hold. This implies that for all $(e^h)_{h \in H}$ in a generic subset of \mathcal{E} , $\sum_{h \in H'} (x_l^h(s) - e_l^h(s)) \neq 0$, finishing the argument.

Proof of Lemma 8 It suffices to show that (a) π^* is proper and (b) $D_{\xi,e}\Phi^*(\xi, \theta)$ has full row rank $n + 1$ whenever $\Phi^*(\xi, \theta) = 0$. The proof of part (a) is standard and the details can be found in Villanacci et al. (2002). For part (b), using the definition of ν from the proof of Theorem 5, set $(\nu^T, \Delta r) D_{\xi,e}\Phi^*(\xi, \theta) = 0$, for $\Delta r \in \mathbb{R}$. The proof is complete upon showing that $(\nu^T, \Delta r) = 0$.

From the columns corresponding to derivatives with respect to $(x^h)_{h \in H}$ and $(e^h)_{h \in H}$,

$$(\Delta x^h)^T D^2 u^h(x^h) - (\Delta \lambda^h)^T P - \Delta p^T \Lambda = 0 \quad h \notin H' \quad (30.a)$$

$$(\Delta x^h)^T D^2 u^h(x^h) - (\Delta \lambda^h)^T P - \Delta p^T \Lambda - \Delta r G = 0 \quad h \in H' \quad (30.b)$$

$$(\Delta \lambda^h)^T P + \Delta p^T \Lambda = 0 \quad h \in \{h_1, \dots, h_{i^*}\}, h \notin H' \quad (30.c)$$

$$(\Delta \lambda^h)^T P + \Delta p^T \Lambda + \Delta r G = 0 \quad h \in \{h_1, \dots, h_{i^*}\}, h \in H' \quad (30.d)$$

$$(\Delta \lambda^h)^T P + \Delta p^T \Lambda + \Delta \mu^h \lambda^h P = 0 \quad h \notin \{h_1, \dots, h_{i^*}\}, h \notin H' \quad (30.e)$$

$$(\Delta \lambda^h)^T P + \Delta p^T \Lambda + \Delta \mu^h \lambda^h P + \Delta r G = 0 \quad h \notin \{h_1, \dots, h_{i^*}\}, h \in H' \quad (30.f)$$

(30)

The g^* -dimensional row vector G is defined such that $G_1(s) = 1$ and $G_l(s') = 0 \forall (l, s') \neq (1, s)$.

Using the **first** step from the proof of Theorem 5, $\Delta p^T = 0$, $(\Delta \lambda^h)^T = 0 \forall h \in \{h_1, \dots, h_{i^*}\}$, and $(\Delta x^h)^T = 0 \forall h \in \{h_1, \dots, h_{i^*}\}$. Using the **second** step from the proof of Theorem 5, $(\Delta x^h)^T = 0 \forall h \notin \{h_1, \dots, h_{i^*}\}$. Using (30.a) and (30.b), $\Delta r = 0$ and $(\Delta \lambda^h)^T = 0 \forall h \notin \{h_1, \dots, h_{i^*}\}$. From the **third** step of the proof of Theorem 5, $(\Delta \delta^h)^T = 0$ and $\Delta \mu^h = 0 \forall h \notin \{h_1, \dots, h_{i^*}\}$. This completes the argument.

A.5.4 Proof of Lemma 5

This proof is independent of the proof of Lemma 4, so notation will be recycled. The matrix Z has full column rank iff $\begin{pmatrix} Z\omega \\ \omega^T \omega \end{pmatrix} = 0$ for any $\omega \in \mathbb{R}^{i^*}$. For this reason, I define the system

of equations $\Phi^* : \Xi \times \mathbb{R}^{i^*} \times \Theta \rightarrow \mathbb{R}^{n+i^*+1}$ as $\Phi^* = \begin{pmatrix} \Phi \\ Z\omega \\ \omega^T \omega - 1 \end{pmatrix}$. Define the equilibrium

projection $\pi^* : \Xi \times \mathbb{R}^{i^*} \times \Theta \rightarrow \Theta$ so that $\theta \in \pi^*(\xi, \omega, \theta)$ iff $\Phi^*(\xi, \omega, \theta) = 0$.

Lemma 9 *The set of regular values of π^* is a generic subset of Θ .*

Given Lemma 9 and Φ^* with a greater number of equations ($n + i^* + 1$) than unknowns (ξ, ω) , for all $(e^h)_{h \in H}$ in a generic subset of \mathcal{E} , the equations $\Phi(\xi, \theta) = 0$, $Z\omega = 0$, and $\omega^T \omega = 1$ cannot all hold. This implies that for all $(e^h)_{h \in H}$ in a generic subset of \mathcal{E} , the matrix Z has full column rank.

Proof of Lemma 9 It suffices to show that (a) π^* is proper and (b) $D_{\xi,\omega,e}\Phi^*(\xi,\omega,\theta)$ has full row rank $n + i^* + 1$ whenever $\Phi^*(\xi,\omega,\theta) = 0$. The proof of part (a) is standard and the details can be found in Villanacci et al. (2002). For part (b), using the definition of ν from the proof of Theorem 5, set $(\nu^T, \Delta z^T, \Delta\omega) D_{\xi,\omega,e}\Phi^*(\xi,\omega,\theta) = 0$, where Δz^T corresponds to the rows for equation $Z\omega = 0$ and $\Delta\omega$ corresponds to the row for equation $\omega^T\omega = 1$. The proof is complete upon showing that $(\nu^T, \Delta z^T, \Delta\omega) = 0$.

From the columns corresponding to derivatives with respect to $(x^h)_{h \in H}$ and $(e^h)_{h \in H}$,

$$\begin{aligned} (\Delta x^h)^T D^2 u^h(x^h) - (\Delta \lambda^h)^T P - \Delta p^T \Lambda - \Delta z^T \Gamma^i &= 0 & h = h_i, \forall i \in I & \quad (31.a) \\ (\Delta x^h)^T D^2 u^h(x^h) - (\Delta \lambda^h)^T P - \Delta p^T \Lambda &= 0 & h \notin \{h_1, \dots, h_{i^*}\} & \quad (31.b) \\ (\Delta \lambda^h)^T P + \Delta p^T \Lambda + \Delta z^T \Gamma^i &= 0 & h = h_i, \forall i \in I & \quad (31.c) \\ (\Delta \lambda^h)^T P + \Delta p^T \Lambda + \Delta \mu^h \lambda^h P &= 0 & h \notin \{h_1, \dots, h_{i^*}\} & \quad (31.d) \end{aligned} \quad (31)$$

All matrices except Γ^i were previously defined in the proof of Theorem 5. The $(s^* + 1) \times g^*$ matrix Γ^i is given by:

$$\Gamma^i = \begin{pmatrix} \left(\begin{array}{cc} \lambda^{h_i}(0)\omega^i & \vec{0} \end{array} \right) & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & \left(\begin{array}{cc} \lambda^{h_i}(s^*)\omega^i & \vec{0} \end{array} \right) \end{pmatrix}.$$

As in the **first** step of the proof of Theorem 5, (31.a) and (31.c) imply $(\Delta \lambda^h)^T = 0 \forall h \in \{h_1, \dots, h_{i^*}\}$ and $(\Delta x^h)^T = 0 \forall h \in \{h_1, \dots, h_{i^*}\}$. Additionally, (31.c) implies $\Delta p^{(l,s)} = 0 \forall (l,s) \notin \{(1,0), \dots, (1,s^*)\}$ and

$$\Delta p^{1,s} + \Delta z^s \lambda^{h_i}(s)\omega^i = 0 \quad \forall s \in S \text{ and } \forall i \in I. \quad (32)$$

Exactly as in the **second** step of the proof of Theorem 5, (31.b), (31.d), (20), and (21) imply $(\Delta x^h)^T = 0 \forall h \notin \{h_1, \dots, h_{i^*}\}$. Exactly as in the **third** step of the proof of Theorem 5, (31.b), (31.d), and (20) imply $(\Delta \lambda^h)^T = 0 \forall h \notin \{h_1, \dots, h_{i^*}\}$, $(\Delta \delta^h)^T = 0 \forall h \notin \{h_1, \dots, h_{i^*}\}$, and $\Delta \mu^h = 0 \forall h \notin \{h_1, \dots, h_{i^*}\}$. From the columns corresponding to derivatives with respect to $(\lambda^{h_i})_{i \in I}$:

$$\Delta z^s \omega^i \sum_{h \in H_i} (e_1^h(s) - x_1^h(s)) = 0 \quad \forall (i,s) \in I \times S.$$

From Lemma 4, over a generic subset of \mathcal{E} , $\sum_{h \in H_i} (e_1^h(s) - x_1^h(s)) \neq 0 \quad \forall (i,s) \in I \times S$. Since $\omega \neq 0$, then $\Delta z^s = 0 \quad \forall s \in S$. From (32), $\Delta p^{1,s} = 0 \forall s \in S$. From the columns corresponding to derivatives with respect to ω , the scalar $\Delta\omega = 0$. Thus $(\nu^T, \Delta z^T, \Delta\omega) = 0$, completing

the argument.

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