Final Exam
Thursday, October 11, 2012

Directions

- You may use a calculator for this exam. You are not permitted to access user-generated files in this calculator.
- This is an 80 minute exam containing 2 questions and 120 points (80 for Question 1 and 40 for Question 2).
- Points are divided evenly among all parts of a question.
- Justify your answers by showing work.
- If you feel that additional assumptions are required to answer the question, state these assumptions clearly.
- Write all your work and solutions on the exam sheets.
- If you run out of room on the front page, you may use the back or separate pages, but clearly indicate where your work is located.
- Extra paper is available if needed.

1. **Growth model with constant technological growth (80 points)**

Assume that there is one representative consumer in an economy with an infinite time horizon. The preferences are

$$\sum_{t=0}^{\infty} \beta^t u(c_t, l_t),$$

where c_t is the level of consumption at time t and l_t is the level of leisure at time t. The utility function is assumed to be of the Cobb-Douglas form:

$$u(c_t, l_t) = (c_t)^{\eta} (l_t)^{1-\eta}.$$

The consumer has a unit of time in each period that can be split between labor and leisure.

Assume that technological change increases the supply of effective labor units each period by the factor $(1+\lambda) > 1$. The production function will be a standard Cobb-Douglas production function:

$$f\left(K_t, (1+\lambda)^t \cdot N_t\right) = (K_t)^{\theta} \left((1+\lambda)^t \cdot N_t\right)^{1-\theta} \text{ for } \theta \in (0,1).$$

Notice that with technological growth, the same labor input $N_t = N_{t+1}$ results in higher output in period $t+1$, because the labor is operating more effectively.

(a) Define an Arrow-Debreu equilibrium (ADE). Include in your answer careful definitions of the consumption set $X(k_0)$ as a function of the initial capital k_0, and the production possibilities set Y.

(b) State the Second Basic Welfare Theorem, including all the assumptions required for the theorem to be true. State in one sentence why this theorem is useful for this model.

(c) Consider a market structure in which the consumer owns the capital and rents it to the firms in each period. Write down the consumer's problem (CP) in the sequence of markets equilibrium, easy version (SMEE).

(d) Define a recursive competitive equilibrium (RCE) for the market structure defined above. Make sure that you clearly distinguish between aggregate state variables and individual state variables. To get the recursive form, it may be helpful to scale certain variables in period t by the factor $\frac{1}{(1+\lambda)^t}$.

(e) Write down a mapping that can be used to show that the value function introduced in Part (d) is well-defined (that is, the value function V on the left-hand side of the recursive consumer's problem is the same as the value function V on the right-hand side). What assumptions do we require to ensure that the defined mapping is a contraction?

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2. **Kuhn-Tucker conditions (40 points)**

   Consider the following optimization problem:

$$(P) \quad \begin{array}{lll} \max\limits_{x \in X} & f(x) & \\ subj.\ to & g_j(x) \geq 0 & j = 1, ..., m \end{array} .$$

   Assume that $X \subseteq \mathbb{R}^n$ is convex and both $f : X \to \mathbb{R}$ and $g = \begin{pmatrix} g_1 \\ : \\ g_m \end{pmatrix} : X \to \mathbb{R}^m$ are concave functions.

   (a) Write down the Kuhn-Tucker conditions associated with the problem (P).

   (b) Prove that the Kuhn-Tucker conditions are sufficient conditions for an optimal solution to the problem (P).

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