Final Exam
Thursday, October 10, 2013

Directions

- You may use a calculator for this exam. You are not permitted to access user-generated files in this calculator.
- This is an 80 minute exam containing 2 questions and 90 points (60 for Question 1 and 30 for Question 2).
- Points are divided evenly among all parts of a question. There are 5 parts to Question 1 and 3 parts to Question 2.
- Justify your answers by showing work.
- If you feel that additional assumptions are required to answer the question, state these assumptions clearly.
- Write all your work and solutions on the exam sheets.
- If you run out of room on the front page, you may use the back or separate pages, but clearly indicate where your work is located.
- Extra paper is available if needed.

1. **Neoclassical growth model with population growth (60 points)**

   Assume that there is a set of homogeneous consumers in an economy with an infinite time horizon. The preferences are

$$\sum_{t=0}^{\infty} \beta^t u\left(c_t, l_t\right),$$

   where c_t is the level of consumption at time t and l_t is the level of leisure at time t. Each consumer has a unit of time in each period that can be split between labor and leisure.

   The set of homogeneous consumers has unit mass in the initial period: $H(0) = 1$. Thereafter, the population grows at the rate n, meaning that $1 + n = \frac{H(t+1)}{H(t)}$ for all time periods t, where $H(t)$ is the mass of consumers in time period t.

   There is a single firm in the economy with production function of the standard Cobb-Douglas form:

$$f\left(K_t, N_t\right) = \left(K_t\right)^{\theta} \left(N_t\right)^{1-\theta} \text{ for } \theta \in (0, 1),$$

   where K_t is the total capital on loan from the consumers to the firm in period t, and N_t is the total labor supply in period t.

   (a) Define an Arrow-Debreu equilibrium (ADE). Include in your answer careful definitions of the consumption set $X\left(k_0\right)$ as a function of the initial capital k_0, and the production possibilities set Y.

(b) Consider a market structure in which the consumer must satisfy a budget constraint in each and every period. Write down the consumer's problem (CP) in the sequence of markets equilibrium, easy version (SMEE).

(c) What assumptions are required on the parameters of the model so that the Bellman equation is well-defined?

(d) Define a recursive competitive equilibrium (RCE). Make sure that you clearly distinguish between aggregate state variables and individual state variables.

(e) Consider a change from a constant population growth rate of n to a larger constant growth rate of $\tilde{n} > n$. What are the effects of this change on the per-capita consumption? Include in your response a discussion of how the factor prices R and w change.

2. **Dynamic programming (30 points)**

Consider the dynamic problem of a pumpkin patch with land to grow one pumpkin. The weight of the pumpkin on that land in period t is given by the variable $k_t \geq 0$ (the weight is in lbs.).

The growth function for the pumpkin is given by $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, meaning that $k_{t+1} = \phi(k_t)$ is the weight of a pumpkin one year from now, as a function of the current weight. Assume that ϕ is differentiable, strictly increasing, and satisfies $\lim_{k \to \infty} D\phi(k_t) = 0$.

Every year at Halloween, a pumpkin can be sold at a price of \$$p$ per pound. The pumpkin patch owner must decide at this time of year whether to sell the pumpkin or to allow it to continue to grow. If the pumpkin is sold, the owner can replant at cost \$$c$ and the weight of the new pumpkin in the following year will be $\phi(0)$ (a newly planted pumpkin does not weigh anything).

The discount factor between one year and the next is $\beta \in (0, 1)$.

The Bellman equation can be expressed as:

$$V(k) = \max\{pk - c + \beta V(\phi(0)), \beta V(\phi(k))\}.$$

(a) Define a compact set A such that $k \in A$. Define the mapping T such that

$$TV(k) = \max\{pk - c + \beta V(\phi(0)), \beta V(\phi(k))\} \quad \forall k \in A.$$

Show that $T : CB(A) \to CB(A)$.

(b) Verify that T is a contraction.

(c) State the Contraction Mapping Theorem, and use it to conclude that the Bellman equation is well-defined.