# General Equilibrium Models[1]

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[1]This manuscript is dedicated to Dave Cass, both for his immeasurable contributions to general equilibrium theory and for his inspirational work in graduate education and advising.

# Contents

# Preface

These notes have been prepared to serve as the theoretical foundation for graduate courses in theoretical economics. These notes have previously been used as the preliminary material for a 2nd year Ph.D. course on monetary theory (Purdue University, Spring 2012). They are perhaps best suited for a reference for any theorist wanting a quick refresher in general equilibrium theory.

The models of general equilibrium have been adopted by macroeconomics (Real Business Cycle school of macroeconomics), finance, and other fields. Given the importance of these models, it is unfortunate that a concise (and precise) source does not exist to summarize the predictions of properties of general equilibrium theory. That is, such a source does not exist until now. The present manuscript introduces the two fundamental general equilibrium models (the static Arrow-Debreu Model and the dynamic General Financial Model).

The material is a compilation of (i) the course material from the general equilibrium PhD course taught by Dave Cass, (ii) the major results in the field of general equilibrium in the past 30 years (yes, we've advanced beyond an Arrow-Debreu or Walrasian equilibrium setting), and (iii) the insights from the author about the use of these models in applications.

This manuscript is organized as follows. Chapter 1 contains the mathematical prerequisites for the course. Students are presumed to have a sufficient grasp of proof-based mathematics (the logic of a mathematical proof) and a working knowledge of real analysis and matrix algebra. To check if you are equipped to handle the material in this manuscript, verify that you can complete each of the following tasks:

- Define what it means for a set to be convex.
- Define what it means for a set to be open (and closed).
- Define what it means for a set to be bounded.

- In Euclidean space, does the definition of compactness imply closed and bounded? Vice versa?
- Define what it means for a function to be concave (and strictly concave).
- Define what it means for a function to be continuous.
- State and prove the Extreme Value Theorem.
- Define what it means for a function to be differentiable.
- State and prove the Mean Value Theorem.

For a refresher on any of these mathematical tasks, I encourage you to review the notes on real analysis on my teaching website: http://www.matthew-hoelle.com/teaching.html.

Chapter 2 introduces the Arrow-Debreu model, the static general equilibrium model of pure exchange where the markets are anonymous and perfectly competitive. This is the canonical model of an economic market. I demonstrate that an equilibrium of the model satisfies three properties: existence, optimality, and regularity. Chapter 3 introduces the general financial model, an extension of the Arrow-Debreu model to a dynamic and stochastic framework. Facing uncertainty in the future periods, households are permitted to trade financial assets. I analyze how the predictions of the model compare under the settings of complete financial markets and incomplete financial markets.

Chapter 4 adds money to the general financial model in Chapter 3, by specifying that the assets pay out in the unit of account (e.g., the currency of a country). The exogenous money supply pins down the price level, using the Quantity Theory of Money. I show that with complete financial markets, monetary policy is neutral, whereas with incomplete financial markets, monetary policy may induce a change in the real allocation.

Each chapter contains its own reference list if readers wish to review a particular topic in greater depth by reading the primary sources. Any proofs that are too long or complicated to justify being placed in the heart of a chapter are placed in the penultimate section of a chapter. The longer proofs are included to provide a complete account of the theory, but (as they are quite technical in nature) can typically be skipped by all but the most courageous readers.

In the ultimate section of each chapter, I have included a handful of exercises. These exercises not only review the main theoretical contributions, but also offer important applications of that theory. Solutions to the exercises can be found in the Appendix. Use this resource as a complement to learning, instead of a replacement.

**Notation** The following is a list of mathematical terminology and notation utilized throughout this manuscript:

- $\forall$ is the universal quantifier; it is the symbol "for all"
- $\exists$ is the existential quantifier; it is the symbol "there exists"
- $\exists!$ is the unique existence quantifier; it is the symbol "there exists a unique"
- 'iff' refers to 'if and only if'; it is used when two statements are necessary and sufficient for each other
- 'wlog' refers to 'without loss of generality'
- Unless otherwise indicated, $x \in \mathbb{R}^n$ is a column vector. Its tranpose $x^T$ is a row vector.
- The value $\|x\|$ denotes the Euclidean norm of the vector $x \in \mathbb{R}^n$.
- The space $\mathbb{R}^{m,n}$ refers to the space of $m \times n$ real matrices.
- For $x, y \in \mathbb{R}^n$, $x \geq y$ iff $x_i \geq y_i$ for $i = 1, ..., n$.
- For $x, y \in \mathbb{R}^n$, $x > y$ iff $x \geq y$ and $x \neq y$.
- For $x, y \in \mathbb{R}^n$, $x >> y$ iff $x_i > y_i$ for $i = 1, ..., n$.
- The set $A \backslash B = \{x \in A : x \notin B\}$.
- The set $N_\epsilon(x)$ is the open neighborhood of radius $\epsilon$ around $x$.
- The set $N_\epsilon^*(x)$ is the deleted open neighborhood of radius $\epsilon$ around $x$, that is, $N_\epsilon^*(x) = N_\epsilon(x) \backslash \{x\}$.
- $f$ is $C^0$ if it is continuous and is $C^n$ (for any $n \in \mathbb{N}$) if its $n^{th}$ derivative exists and is continuous.
- The set $intA$ denotes the interior of $A$:

$$intA = \{x \in A : \exists \epsilon > 0 \text{ such that } N_\epsilon(x) \subset A\}.$$

- The set $A'$ denotes the set of limit points of $A$ :

$$A' = \{x : \forall \epsilon > 0,\ N_{\epsilon}^{*}(x) \cap A \neq \emptyset\}.$$

- The set $clA$ denotes the closure of $A$ :

$$clA = A \cup A'.$$

- $\Delta^{n-1}$ is the $(n-1)$ −dimensional simplex defined as

$$\Delta^{n-1} = \{x \in \mathbb{R}_{+}^{n} : \textstyle\sum_{i=1}^{n} x_i = 1\}.$$

# Chapter 1

# Mathematical Prerequisites

The topics to be covered in this chapter are (1) Farkas Lemma (Section 1.1) with Duality as an important application, (2) Kuhn-Tucker theorem (Section 1.2), an important economic result that uses Farkas Lemma in its proof, (3) correspondences (Section 1.3) as used to prove equilibrium existence, and (4) differential topology (Section 1.4) as used to prove equilibrium regularity.

## 1.1 Farkas Lemma

Before stating the Farkas Lemma, I first state and prove the Separating Hyperplane Theorem. As a corollary of this, I state the Supporting Hyperplane Theorem. Interestingly enough, the Supporting Hyperplane Theorem is used in the proof of the Second Basic Welfare Theorem (Section 2.4). Finally, I state the Farkas Lemma and then apply it to the problem of Duality.

### 1.1.1 Hyperplane Theorems

The following theorem, the Separating Hyperplane Theorem, is used to prove (a) Supporting Hyperplane Theorem and (b) Farkas Lemma.

**Theorem 1.1** *Separating Hyperplane Theorem*

*Suppose $Z \subseteq \mathbb{R}^n$ is nonempty, closed, and convex, and that $y \in \mathbb{R}^n$, but $y \notin Z$. Then there exists $q \in \mathbb{R}^n \backslash \{0\}$ such that $q^T y > q^T z \quad \forall z \in Z$.*

The Separating Hyperplane Theorem can be depicted in Figure 1.1 in the separate 'Figures' document. In Figure 1.1, the boundary of the set is a solid line, indicating that the set

$Z$ is closed.

**Proof.** Pick any $z' \in Z$ and define $Z' = \{z \in Z : \|z - y\| \leq \|z' - y\|\}$. The set $Z'$ is compact (as it is closed and bounded). The function $\|z - y\|$ is a continuous function of the variable $z$.

Applying the Extreme Value Theorem, $\exists z^* = \arg\min_{z \in Z'} \|z - y\|$. For any $z \in Z$, as $y \notin Z$, then $\|z - y\| > 0$. Thus $\|z^* - y\| > 0$.

Take any $z \in Z$ and define the real-valued function $g : [0, 1] \to \mathbb{R}$ as follows:

$$g(\theta) = ((1-\theta) z^* + \theta z - y)^T ((1-\theta) z^* + \theta z - y).$$

The function is minimized at 0, namely $g(\theta) \geq g(0) \quad \forall \theta \in [0, 1]$. Using the definition of derivative, $Dg(0) \geq 0$. We can rewrite the function

$$g(\theta) = (\theta (z - z^*) + z^* - y)^T (\theta (z - z^*) + z^* - y).$$

Taking the derivative yields

$$Dg(0) = (z^* - y)^T (z - z^*) \geq 0.$$

Using algebra, $(y - z^*)^T (z - z^*) \leq 0$ or $(y - z^*)^T z \leq (y - z^*)^T z^* \quad \forall z \in Z$.

Define the separating hyperplane as $q = (y - z^*)$. Therefore, $q^T z^* \geq q^T z \quad \forall z \in Z$.

As $\|z^* - y\|^2 > 0$, then $(z^* - y)^T (z^* - y) > 0$ or $(y - z^*)^T (y - z^*) > 0$. This implies $q^T y > q^T z^*$. From above, $q^T y > q^T z^* \geq q^T z \quad \forall z \in Z$. This finishes the argument. ■

As a straightforward corollary of the Separating Hyperplane Theorem, we can now state the Supporting Hyperplane Theorem. As previously mentioned, this theorem is used in the proof of the Second Basic Welfare Theorem (Section 2.4).

**Corollary 1.1** *Supporting Hyperplane Theorem*

*Suppose $Z \subseteq \mathbb{R}^n$ is nonempty and convex, and that $y \in Z$, but $y \notin intZ$.*

*Then there exists a $q \in \mathbb{R}^n \backslash \{0\}$ such that $q^T y \geq q^T z \quad \forall z \in Z$.*

The Supporting Hyperplane Theorem can be depicted in Figure 1.2. The boundary of the set is a dashed line, indicating that the set $Z$ need not be closed.

### 1.1.2 Farkas Lemma

We are now prepared to state the Farkas Lemma (which as a result of its importance in economics has been promoted to a 'Theorem').

**Theorem 1.2** *Farkas Lemma*

*Let* $a_i \in \mathbb{R}^n \backslash \{0\}$ *for* $i = 1, ..., m,$ *matrix* $A = [a_1 ... a_m] \in \mathbb{R}^{n,m},$ *and*

$$Z = \left\{ z \in \mathbb{R}^n : z = A\alpha \text{ for some } \alpha \in \mathbb{R}^m_+ \right\}.$$

*For* $z^* \in \mathbb{R}^n$, *exactly one of the following conditions holds:*

*(i)* $z^* \in Z$.

*(ii)* $\exists q^* \in \mathbb{R}^n \backslash \{0\}$ *such that* $q^{*T} z^* > 0 \geq q^{*T} z \quad \forall z \in Z$.

The theorem is illustrated in the two-panel Figure 1.3. In the left panel labeled Farkas (i), the element $z^* \in Z$. In the right panel labeled Farkas (ii), $z^* \notin Z$ and $\exists q^* \in \mathbb{R}^n \backslash \{0\}$ such that $q^{*T} z^* > 0 \geq q^{*T} z \quad \forall z \in Z$.

### 1.1.3 Application: Duality

As an application of Farkas Lemma, let us state and prove the Duality Theorem. The Duality Theorem in this manuscript will be stated in terms of a linear objective function and linear constraints (I will refer to this as the "Linear Duality Theorem"). In general, a Duality Theorem shows the equivalence between a maximization problem and the related minimization problem (which is called the "dual" of the maximization problem). In order to obtain a Duality Theorem for nonlinear objective functions and nonlinear constraint functions, we must take derivatives of all nonlinear functions. Recall that a derivative is a linear mapping. Thus, a Duality Theorem for an objective function and constraint functions that are differentiable and concave (not necessarily linear) can be equivalently represented in the form of a Linear Duality Theorem (after taking derivatives).

We begin by writing down the maximization problem:

$$\begin{array}{lll} \text{maximize} & & x^T \gamma \\ x \geq 0 & \text{subject to} & x^T A \leq \beta^T \end{array}.$$

The dual minimization problem is then given by:

$$\begin{array}{lcl} \text{minimize} & & y^T\beta \\ y \geq 0 & \text{subject to} & Ay \geq \gamma \end{array}.$$

**Definition 1.1** *A linear program is feasible if there exists a vector satisfying the constraints.*

**Definition 1.2** *A feasible vector is an optimal vector if it maximizes or minimizes the linear form. The value of this max or min is called the value.*

**Theorem 1.3** *Duality*

*If both the maximization problem and its dual are feasible, then both have optimal vectors and the values of the two are the same.*

The following Lemma is used in the proof of Theorem 1.3.

**Lemma 1.1** *To prove Theorem 1.3, it suffices to find $(x, y) \geq 0$ that satisfies the following three equations:*

$$x^T A \leq \beta^T. \tag{1.1}$$

$$Ay \geq \gamma. \tag{1.2}$$

$$x^T\gamma - y^T\beta \geq 0. \tag{1.3}$$

**Proof.** From (1.1), $x^T Ay \leq \beta^T y$. From (1.2), $x^T Ay \geq x^T\gamma$. The previous two inequalities together imply $x^T\gamma \leq \beta^T y$. If (1.3) is satisfied, then $x^T\gamma = \beta^T y$.

Suppose, for contradiction, that $(x, y)$ are both feasible, but at least one is not optimal, wlog $x$. Then there exists feasible $z$ such that

$$\begin{array}{rcl} z^T\gamma & > & x^T\gamma. \\ z^T\gamma & \leq & \beta^T y. \end{array}$$

Using the equality $x^T\gamma = \beta^T y$, then $z^T\gamma \leq \beta^T y = x^T\gamma$, which is a contradiction. ■

We are now prepared to prove the Duality Theorem.

**Proof.** Suppose, for contradiction, there does not exist $(x, y) \geq 0$ satisfying (1.1), (1.2), and (1.3). The equations can be rewritten as:

$$\begin{array}{lll} \beta = A^T x + Iv & \text{for} & v \geq 0. \\ \gamma = Ay - Iw & \text{for} & w \geq 0. \\ 0 = \gamma^T x - \beta^T y - u & \text{for} & u \geq 0. \end{array}$$

Thus, there does not exist $\alpha = \begin{pmatrix} x \\ v \\ y \\ w \\ u \end{pmatrix} \geq 0$ such that

$$\begin{pmatrix} \beta \\ \gamma \\ 0 \end{pmatrix} = \begin{bmatrix} A^T & I & 0 & 0 & 0 \\ 0 & 0 & A & -I & 0 \\ \gamma^T & 0 & -\beta^T & 0 & -1 \end{bmatrix} \begin{pmatrix} x \\ v \\ y \\ w \\ u \end{pmatrix}.$$

Let's apply the Farkas Lemma with the following relations between the statement of the Farkas Lemma and this proof:

| Farkas Lemma | | This proof |
|---|---|---|
| $q^*$ | $\Longleftrightarrow$ | $q^*$ |
| $z^*$ | $\Longleftrightarrow$ | $\begin{pmatrix} \beta \\ \gamma \\ 0 \end{pmatrix}$ |
| $A$ | $\Longleftrightarrow$ | $\begin{bmatrix} A^T & I & 0 & 0 & 0 \\ 0 & 0 & A & -I & 0 \\ \gamma^T & 0 & -\beta^T & 0 & -1 \end{bmatrix}$. |
| $\alpha$ | $\Longleftrightarrow$ | $\begin{pmatrix} x \\ v \\ y \\ w \\ u \end{pmatrix}$ |

Define

$$Z = \left\{ z : z = \begin{bmatrix} A^T & I & 0 & 0 & 0 \\ 0 & 0 & A & -I & 0 \\ \gamma^T & 0 & -\beta^T & 0 & -1 \end{bmatrix} \begin{pmatrix} x \\ v \\ y \\ w \\ u \end{pmatrix} \text{ for some } \begin{pmatrix} x \\ v \\ y \\ w \\ u \end{pmatrix} \geq 0 \right\}.$$

The implications of our proof by contradiction is that $\begin{pmatrix} \beta \\ \gamma \\ 0 \end{pmatrix} \notin Z$. This says that part (i) of the Farkas Lemma does not hold, meaning that part (ii) must. Thus, there is $q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \neq 0$ such that $q^{*T} \begin{pmatrix} \beta \\ \gamma \\ 0 \end{pmatrix} > 0 \geq q^{*T} z \;\; \forall z \in Z$. The strict inequality on the left-hand side can be written as:

$$q_1^T \beta + q_2^T \gamma > 0. \tag{1.4}$$

As $q^{*T} z \leq 0 \;\; \forall z \in Z$, then $\left(q_1^T, q_2^T, q_3\right) \begin{bmatrix} A^T & I & 0 & 0 & 0 \\ 0 & 0 & A & -I & 0 \\ \gamma^T & 0 & -\beta^T & 0 & -1 \end{bmatrix} \leq 0$ (letting $z \in Z$ be determined by the unit vectors from $\alpha = \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix}$ to $\alpha = \begin{pmatrix} \vec{0} \\ 1 \end{pmatrix}$):

$$q_1^T A^T + q_3 \gamma^T \leq 0. \tag{1.5}$$

$$q_1^T \leq 0. \tag{1.6}$$

$$q_2^T A - q_3 \beta^T \leq 0. \tag{1.7}$$

$$-q_2^T \leq 0. \tag{1.8}$$

$$-q_3 \leq 0. \tag{1.9}$$

Two steps remain. The first is to show that $q_3 > 0$. If not, $q_3 = 0$, then (1.5) and (1.1) imply $q_1^T \beta \leq 0$ (recall that $q_1^T \leq 0$). Additionally, from (1.7) and (1.2), $q_2^T \gamma \leq 0$. The last two inequalities contradict (1.4).

Now with $q_3 > 0$, from (1.5):

$$\left(-\frac{q_1^T}{q_3}\right) A^T \left(\frac{q_2}{q_3}\right) \geq \gamma^T \left(\frac{q_2}{q_3}\right). \tag{1.10}$$

From (1.7):

$$\left(-\frac{q_2^T}{q_3}\right) A \left(-\frac{q_1}{q_3}\right) \leq \beta^T \left(-\frac{q_1}{q_3}\right). \tag{1.11}$$

After taking the transpose of (1.11) and adding it to (1.10), I obtain $\gamma^T \left(\frac{q_2}{q_3}\right) + \beta^T \left(\frac{q_1}{q_3}\right) \leq 0$, which is a contradiction of (1.4). This is a contradiction, allowing us to conclude that there must exist $(x, y) \geq 0$ satisfying (1.1), (1.2), and (1.3). ■

## 1.2 Kuhn-Tucker Theorem

### 1.2.1 Definitions

Prior to the statement of the theorem, I introduce two definitions. The first, of a concave function, should be familiar to the students. The second may be new. The convention throughout the entire manuscript is that for a function $f : \mathbb{R}^n \to \mathbb{R}$, the values of the derivative mapping $Df : \mathbb{R}^n \to \mathbb{R}$ are row vectors. Thus, $Df(x)$ is a $1 \times n$ row vector. Extending this idea, for $g : \mathbb{R}^n \to \mathbb{R}^m$, the values of the derivative mapping are $m \times n$ Jacobian matrices. Thus, $Dg(x)$ is a $m \times n$ matrix.

**Definition 1.3** *Given $X \subseteq \mathbb{R}^n$ is convex, a differentiable function $f : X \to \mathbb{R}$ is concave iff $Df(x^*)(x - x^*) + f(x^*) \geq f(x)$.*

**Definition 1.4** *Given $X \subseteq \mathbb{R}^n$ is convex, a differentiable function $f : X \to \mathbb{R}$ is quasi-concave iff $f(x) - f(x^*) \geq 0 \implies Df(x^*)(x - x^*) \geq 0$.*

A concave function is quasi-concave, but not vice-versa. Figure 1.4 provides an example of a function that is quasi-concave, but not concave.

The following equivalent definitions of concave and quasi-concave may prove useful in the sequel and highlight why concavity is a stronger assumption.

**Definition 1.5** *A function $f$ is concave iff $\forall x, y \in X$ and $\forall \theta \in [0, 1]$,*

$$f(\theta x + (1 - \theta) y) \geq \theta f(x) + (1 - \theta) f(y).$$

**Definition 1.6** *A function $f$ is quasi-concave iff* $\forall x, y \in X$ *and* $\forall \theta \in [0,1]$,

$$f(\theta x + (1-\theta) y) \geq \min\{f(x), f(y)\}.$$

### 1.2.2 Constraint Qualification

Define the programming problem $(P)$ as:

$$\begin{array}{llll} (P) & \text{maximize} & f(x) & \text{with multipliers} \\ & \text{subject to} & g_j(x) \geq 0 \quad j = 1, ..., m & \lambda_j \end{array}.$$

The Lagrange multipliers $\lambda = (\lambda_1, ..., \lambda_m)$ are taken to be a row vector. The vector-valued function $g$ is defined as $g = \begin{pmatrix} g_1 \\ \vdots \\ g_m \end{pmatrix}$. To state the Kuhn-Tucker Theorem, we must assume that the so-called "Constraint Qualification" holds. Examples of the Constraint Qualification are:

- The constraint functions $g_j$ are linear functions of $x$.
- If $x^*$ is an optimal solution to $(P)$, then $rankDg(x^*) = m^*$, where $m^*$ is the number of constraints that bind at the optimal solution (wlog, $g_j(x^*) = 0$ for $j = 1, ..., m^*$).

Basically, the Constraint Qualification is any sufficient condition for the following statement:

$$x^* \text{ is an optimal solution to } (P) \implies x^* \text{ is an optimal solution to } (LP),$$

where the linear programming problem $(LP)$ is:

$$\begin{array}{llll} (LP) & \text{maximize} & Df(x^*)x & \text{with multipliers} \\ & \text{subject to} & Dg_j(x^*)(x - x^*) \geq 0 \quad j = 1, ..., m^* & \lambda_j \end{array},$$

and as above, the first $m^*$ constraints bind at the optimal solution $x^*$.

**Proof.** Assume that the first suggested Constraint Qualification holds. The following argument shows that if $x$ is an optimal solution to $(P)$, then $x$ is an optimal solution to $(LP)$. Similar methods can be used to show that the second suggested Constraint Qualification yields the same implication.

The proof only requires the additional assumption that $X$ is a convex set.

Suppose that $x^*$ is not an optimal solution to $(LP)$. Then $\exists y \in X$ such that $Df(x^*)y > Df(x^*)x^*$ and $Dg_j(x^*)(y - x^*) \geq 0$ for $j = 1, ..., m^*$. Since the functions $g_j$ are linear, the dot products $Dg_j(x^*)y = g_j(y)$ and $Dg_j(x^*)x^* = g_j(x^*)$ (the derivative vector $Dg_j(x^*)$ is independent of $x^*$). This implies that $g_j(y) - g_j(x^*) \geq 0$ or simply that $g_j(y) \geq 0$ since $g_j(x^*) = 0$ for $j = 1, ..., m^*$, by definition.

Define $z^\theta = \theta y + (1 - \theta)x^*$ for any $\theta \in (0, 1)$. Since $X$ is convex, $z^\theta \in X$. Since $g_j(y) \geq 0$, $g_j(x^*) = 0$ for $j = 1, ..., m^*$, and $g_j$ are linear functions, then for any $\theta \in (0, 1)$, $g_j(z^\theta) \geq 0$ for $j = 1, ..., m^*$.

By definition, $g_j(x^*) > 0$ for $j = m^* + 1, ...., m$. There exists values of $\theta$ near 0 such that $g_j(z^\theta) > 0$ for $j = m^* + 1, ...., m$. All told, for values of $\theta$ near 0, $g_j(z^\theta) \geq 0$ for $j = 1, ..., m$.

Observe that if $Df(x^*)\left(z^\theta - x^*\right) > 0$ for any $z^\theta \in \mathbb{R}^n$, then by the definition of the derivative,

$$\lim_{\theta \to 0} \frac{\left| f\left(z^\theta\right) - f\left(x^*\right) - Df\left(x^*\right)\left(z^\theta - x^*\right) \right|}{\left| z^\theta - x^* \right|} = 0.$$

Since the supposition is that $Df(x^*)y > Df(x^*)x^*$, then for any $\theta \in (0, 1)$, $Df\left(x^*\right)\left(z^\theta - x^*\right) > 0$. This implies that for values of $\theta$ near 0, $f\left(z^\theta\right) - f\left(x^*\right) > 0$, or $f\left(z^\theta\right) > f\left(x^*\right)$. Combined with the facts that for values of $\theta$ near 0, $z^\theta \in X$ and $g_j(z^\theta) \geq 0$ for $j = 1, ..., m$, the vector $x^*$ is not an optimal solution to $(P)$. This finishes the proof by contraposition. ■

### 1.2.3 Kuhn-Tucker Theorem

The Kuhn-Tucker Theorem can now be stated.

**Theorem 1.4** *Given $X \subseteq \mathbb{R}^n$ is convex, assume that $f : X \to \mathbb{R}$ is differentiable and concave. Further assume that $g_j : X \to \mathbb{R}$ are differentiable and quasi-concave $\forall j = 1, ..., m$, and the Constraint Qualification is satisfied. Then $x^*$ is an optimal solution to the programming problem $(P)$ iff $x^*$ satisfies the Kuhn-Tucker conditions:*

*First Order Conditions*

$$Df(x^*) + \lambda Dg(x^*) = 0_{1\times n}.$$

*Complimentary Slackness Conditions*

$$\lambda g(x^*) = 0_{1\times 1}, \text{ with } \lambda \geq 0 \text{ and } g(x) \geq 0.$$

The proof proceeds in two steps: Sufficiency and Necessity. The easier of the two steps is to show that the Kuhn-Tucker conditions are sufficient for an optimal solution to $(P)$, so we begin there.

**Proof. Sufficiency of Kuhn-Tucker conditions**

We utilize a proof by contradiction. Suppose that $x^*$ satisfies the Kuhn-Tucker conditions, but is not an optimal solution to $(P)$. Then $\exists y$ such that $g_j(y) \geq 0 \;\; \forall j = 1, ..., m$ and $f(y) > f(x^*)$. Suppose that the first $m^*$ constraints bind at $x^*$ (wlog). Then $g_j(y) \geq g_j(x^*) = 0$ $\forall j = 1, ..., m^*$. Using the quasi-concavity of $g_j$, then $Dg_j(x^*)(y - x^*) \geq 0 \;\forall j = 1, ..., m^*$. From the concavity of $f$, $Df(x^*)(y - x^*) > 0$. Then for any values $\lambda_1, ..., \lambda_{m^*} \geq 0$, the equation

$$Df(x^*)(y - x^*) + \sum_{j=1}^{m^*} \lambda_j \cdot Dg_j(x^*)(y - x^*) > 0.$$

As the constraints $g_j(x^*) > 0$ for $j > m^*$, the Complimentary Slackness Conditions imply $\lambda_j = 0$ for $j > m^*$. Thus, we have (in vector notation):

$$(Df(x^*) + \lambda Dg(x^*))(y - x^*) > 0.$$

This contradicts the First Order Conditions: $Df(x^*) + \lambda Dg(x^*) = 0$. ■

To show that the Kuhn-Tucker conditions are necessary for an optimal solution to $(P)$, I make use of the Farkas Lemma. The proof is as follows.

**Proof. Necessity of Kuhn-Tucker conditions**

As the Constraint Qualification is satisfied by assumption, then $x^*$ is an optimal solution to $(LP)$. For simplicity, I define $c^T = Df(x^*) \in \mathbb{R}^n$, $b = Dg(x^*)x^* \in \mathbb{R}^m$, and $A = Dg(x^*) \in \mathbb{R}^{m,n}$. The linear programming problem $(LP)$ is then equivalent to the simpler linear problem:

$$\begin{array}{llllll} (LP2) & \text{maximize} & c^T x & & \text{with multipliers} & \\ & \text{subject to} & A_j x - b_j \geq 0 & j = 1, ..., m & \lambda_j & . \end{array}$$

Suppose, wlog, that the first $m^*$ constraints bind, $A_j x^* - b_j = 0$ for $j = 1, ..., m^*$. Define $\lambda_j = 0$ for $j > m^*$.

As $x^*$ is an optimal solution to $(LP2)$, there does not exist $x$ such that $c^T x - c^T x^* > 0$ and $A_j x - b_j \geq A_j x^* - b_j = 0 \;\; \forall j = 1, ..., m^*$. Thus, there does not exist $x \in \mathbb{R}^n$ such that:

$$(x - x^*)^T c > 0 \geq (x - x^*)^T (-A_j)^T \quad \forall j = 1, ..., m^*.$$

Let's apply the Farkas Lemma with the following relations between the statement of the Farkas Lemma and this proof:

| Farkas Lemma | | This proof |
|---|---|---|
| $q^* \in \mathbb{R}^n \backslash \{0\}$ | $\Longleftrightarrow$ | $(x - x^*) \in \mathbb{R}^n \backslash \{0\}$ |
| $z^* \in \mathbb{R}^n$ | $\Longleftrightarrow$ | $c \in \mathbb{R}^n$ |
| $A \in \mathbb{R}^{n,m^*}$ | $\Longleftrightarrow$ | $\left[-(A_1)^T \ldots\ldots - (A_{m^*})^T\right]$ |
| $\alpha \in \mathbb{R}_+^{m^*}$ | $\Longleftrightarrow$ | $\lambda^T \in \mathbb{R}_+^{m^*}$ |

Defining

$$Z = \left\{ z \in \mathbb{R}^n : z = \left[-(A_1)^T \ldots\ldots - (A_{m^*})^T\right] \cdot \lambda^T \text{ for some } \lambda^T \in \mathbb{R}_+^{m^*} \right\},$$

then $(-A_j)^T \in Z \;\; \forall j = 1, ..., m^*$. As there does not exist $(x - x^*)$ such that (ii) of Farkas Lemma holds (given the implication $(x - x^*)^T c > 0 \geq (x - x^*)^T z \;\; \forall z \in Z$), then (i) must hold $(c \in Z)$. Rewriting $Z$ as

$$Z = \left\{ z \in \mathbb{R}^n : z = -\left( \lambda \cdot \begin{bmatrix} A_1 \\ \vdots \\ A_{m^*} \end{bmatrix} \right)^T \text{ for some } \lambda^T \in \mathbb{R}_+^{m^*} \right\},$$

then $c \in Z$ implies that there exists $\lambda^T \in \mathbb{R}_+^{m^*}$ such that $c^T = -\lambda \begin{bmatrix} A_1 \\ \vdots \\ A_{m^*} \end{bmatrix}$. From above, recall that we defined $\lambda_j = 0$ for $j > m^*$. Thus, we have $c^T + \lambda A = 0$. These are the Kuhn-Tucker First Order Conditions:

$$Df(x^*) + \lambda Dg(x^*) = c^T + \lambda A = 0.$$

Additionally, we have that $\forall j = 1, ..., m : \lambda_j \geq 0$ and $\lambda_j = 0$ if $A_j x^* - b_j > 0$. These are the Complimentary Slackness Conditions. This finishes the argument. ■

## 1.3 Correspondences

A correspondence is a multi-valued mapping. If a correspondence is single-valued over its entire domain, then it is a function. Correspondences are crucial in economics, because demand and best responses may not always be single-valued. Specifically, there may exist multiple variables that comprise the demand or the best response.

The value of a correspondence is a set. When dealing with functions, the value of the function is also a set, albeit a singleton.

Over the entire domain, if the value of a correspondence is compact, then the correspondence is said to be *compact-valued*.

Over the entire domain, if the value of a correspondence is convex, then the correspondence is said to be *convex-valued*.

Over the entire domain, if the value of a correspondence is nonempty, then the correspondence is said to be *well-defined* (or *nonempty-valued*).

Some more advanced definitions are provided in the next subsection. With these, I can discuss the main results related to correspondences.

### 1.3.1 Definitions

Let $U \subseteq \mathbb{R}^m$ be the domain and $W \subseteq \mathbb{R}^n$ be the codomain of the correspondence $\phi$, denoted $\phi : U \rightrightarrows W$.

**Definition 1.7** *The correspondence $\phi : U \rightrightarrows W$ is upper hemi-continuous (uhc) iff for any $u^\nu \in U$, $w^\nu \in \phi(u^\nu)$ for $\nu \in \mathbb{N}$ such that $\lim_{\nu\to\infty} u^\nu = u \in U$, there is a subsequence, wlog the original sequence, such that $\lim_{\nu\to\infty} w^\nu = w \in \phi(u)$.*

If $W$ is a compact set (as will often be the case in economics), we know that there exists a subsequence, wlog the original sequence, such that $\lim_{\nu\to\infty} w^\nu = w$. All that remains to be shown for the definition of uhc is that $w \in \phi(u)$.

Consider any $u \in U$. We can certaintly specify a sequence $u^\nu \in U$ such that $u^\nu = u \ \ \forall \nu \in \mathbb{N}$. By the definition of uhc, all sequences $w^\nu \in \phi(u^\nu) = \phi(u)$ have a convergent subsequence (wlog the original sequence) $\lim_{\nu\to\infty} w^\nu = w \in \phi(u)$. As all sequences $w^\nu \in \phi(u^\nu) = \phi(u)$ have a convergent subsequence, then the set $\phi(u)$ is compact. Thus, $\phi$ is a compact-valued correspondence. So uhc implies compact-valued.[1]

[1] In some textbooks, compact-valued is assumed (redundantly) along with uhc.

Additionally, for a correspondence that takes all values in the codomain, i.e., $\phi(u) = W$ $\forall u \in U$, if $W$ is compact, then $\phi$ is uhc.

**Definition 1.8** *The correspondence $\phi : U \rightrightarrows W$ is lower hemi-continuous (lhc) iff for any $u^\nu \in U$ for $\nu \in \mathbb{N}$ such that $\lim\limits_{\nu\to\infty} u^\nu = u \in U$ and $w \in \phi(u)$, there is a sequence $w^\nu \in \phi(u^\nu)$ for $\nu \in \mathbb{N}$ such that $\lim\limits_{\nu\to\infty} w^\nu = w$.*

If the correspondence $\phi$ takes all values in the codomain, i.e., $\phi(u) = W \;\; \forall u \in U$, then $\phi$ is lhc.

**Definition 1.9** *The correspondence $\phi : U \rightrightarrows W$ is continuous iff it is upper hemi-continuous and lower hemi-continuous.*

Figure 1.5 illustrates the distinction between uhc and lhc. The left panel shows a continuous correspondence (both uhc and lhc). The middle panel shows a correspondence that is uhc, but not lhc. The right panel shows a correspondence that is lhc, but not uhc.

### 1.3.2 Berge's Theorem

There are two major results associated with correspondences: Berge's Maximum Theorem and Kakutani's Fixed Point Theorem. The proof of Kakutani is not included in this manuscript.

**Theorem 1.5** *Berge's Maximum Theorem*

*Suppose $X \subseteq \mathbb{R}^n$, $A \subseteq \mathbb{R}^m$, $f : X \times A \to \mathbb{R}$ is continuous, $X^f : A \rightrightarrows X$ is well-defined and continuous, and*

$$X^o : A \rightrightarrows X \;\; s.t. \;\; \alpha \longmapsto \underset{x \in X^f(\alpha)}{\arg\max} f(x, \alpha).$$

*Then $X^o$ is uhc.*

**Proof.** Suppose that $X^o$ is not uhc. Then for some sequence $\alpha^\nu \in A$, $x^\nu \in X^o(\alpha^\nu)$ for $\nu \in \mathbb{N}$ such that $\lim\limits_{\nu\to\infty} \alpha^\nu = \alpha \in A$, it is not true that $\lim\limits_{\nu\to\infty} x^\nu = x \in X^o(\alpha)$. As $X^f$ is uhc, there is a subsequence, wlog the original sequence, such that $\lim\limits_{\nu\to\infty} x^\nu = x \in X^f(\alpha)$. Then, it must be that $x \notin X^o(\alpha)$, even though $x^\nu \in X^o(\alpha^\nu)$ for $\nu \in \mathbb{N}$. As $X^f$ is lhc, then for every $x' \in X^f(\alpha)$, in particular $x' \in X^f(\alpha)$ such that $f(x, \alpha) < f(x', \alpha)$, there exists a sequence $x'^\nu \in X^f(\alpha^\nu)$ for $\nu \in \mathbb{N}$ such that $\lim\limits_{\nu\to\infty} x'^\nu = x'$. Since $f$ is continuous,

$$\lim_{\nu\to\infty} f(x^\nu, \alpha^\nu) = f(x, \alpha) < f(x', \alpha) = \lim_{\nu\to\infty} f(x'^\nu, \alpha^\nu).$$

As $x'^{\nu} \in X^f(\alpha^{\nu})$ for $\nu \in \mathbb{N}$, then this contradicts that $x^{\nu} \in X^o(\alpha^{\nu})$ for $\nu \in \mathbb{N}$. ■

**Corollary 1.2** *Suppose $X \subseteq \mathbb{R}^n$, $A \subseteq \mathbb{R}^m$, $f : X \times A \to \mathbb{R}$ is continuous, $X^f : A \rightrightarrows X$ is well-defined and continuous, and*

$$V : A \to X \ s.t. \ \alpha \longmapsto \max_{x \in X^f(\alpha)} f(x, \alpha).$$

*Then $V$ is continuous.*

**Proof.** Since $X^o$ is uhc, then for some sequence $\alpha^{\nu} \in A$, $x^{\nu} \in X^o(\alpha^{\nu})$ for $\nu \in \mathbb{N}$ such that $\lim_{\nu \to \infty} \alpha^{\nu} = \alpha \in A$, there is a subsequence, wlog the original sequence, such that $\lim_{\nu \to \infty} x^{\nu} = x \in X^o(\alpha)$. Since $(x^{\nu}, \alpha^{\nu}) \to (x, \alpha)$ and $f$ is continuous, then $f(x^{\nu}, \alpha^{\nu}) \to f(x, \alpha)$. Since $x^{\nu} \in X^o(\alpha^{\nu})$ and $x \in X^o(\alpha)$, then this implies that $V(\alpha^{\nu}) \to V(\alpha)$. This completes the argument. ■

### 1.3.3 Kakutani's Fixed Point Theorem

By the Extreme Value Theorem, $X^o$ is well-defined. Further, if $f$ is quasi-concave (Definition 1.4), then $X^o$ is convex-valued (details are left to the reader).

**Theorem 1.6** *Kakutani's Fixed Point Theorem*

*Let $\Gamma : X \rightrightarrows X$ be a well-defined, convex-valued, and uhc correspondence and $X$ be a compact, convex, and nonempty set. Then $\exists x \in X$ such that $x \in \Gamma(x)$.*

### 1.3.4 Application: Dynamic Programming

This application requires the use of some basic results related to metric spaces. If the only metric space you have worked with is the Euclidean space, then this material might be out of reach. This digression will be brief, so please bear with me as I take the steps to introduce dynamic programming. The mathematics behind dynamic programming includes Berge's Maximum Theorem.

The standard dynamic optimization problem is:

$$\text{Given } x_0 \in X, \ \underset{(x_n)_{n \in \mathbb{N}}}{max} \ f(x_0, x_1) + \sum_{i=1}^{\infty} \beta^i f(x_i, x_{i+1}) \text{ such that } x_n \in \Gamma(x_{n-1})$$

$\forall n \in \mathbb{N}$.

The following assumptions are made:

Assumption 1: $f$ is continuous and bounded.

Assumption 2: $\Gamma$ is continuous (uhc and lhc).

Assumption 3: $\beta \in (0,1)$.

Assumption 4: $X$ is compact.

Rather than finding an entire optimal sequence all at once, we can find a sequence of optimal solutions. To do this, we define the recursive programming problem as follows:

For all $n \in \mathbb{N}$, given $x_{n-1} \in X$, $\underset{x_n}{max}\ \ f(x_{n-1}, x_n) + \beta V(x_n)$ such that $x_n \in \Gamma(x_{n-1})$.

We need to verify that the function $V : X \to \mathbb{R}$ satisfies

$$V(x_{n-1}) = \max_{x_n} f(x_{n-1}, x_n) + \beta V(x_n)$$

for all $n \in \mathbb{N}$ such that $x_n \in \Gamma(x_{n-1})$.

Define the mapping $T$ such that

$$(TV)(x) = \max_{y \in \Gamma(x)} \{f(x, y) + \beta V(y)\} \text{ for all } x \in X.$$

Assume that $V \in CB(X)$, where $CB(X)$ is the set of bounded and continuous functions with compact domain $X$. This means that $V$ is a continuous function. Using Assumptions 1 and 2, the Corollary to Berge's Theorem dictates that $TV$ is a continuous function. Since $X$ is compact, then $TV$ is also a bounded function. Consequently, $TV \in CB(X)$.

So the mapping $T$ is a self-map $CB(X) \to CB(X)$.

The space $CB(X)$ is a complete metric space. A complete metric space is one in which all Cauchy sequences in the set converge in the set.

We want to claim that the mapping $T$ is a contraction.

**Definition 1.10** *A self map $\Phi : K \to K$ is a contraction if there exists $\delta \in (0,1)$ such that $d(\Phi(x), \Phi(y)) \leq \delta d(x, y)$ for all $x, y \in K$, where $d(\cdot)$ is the metric for the metric space $K$.*

Rather than verify directly that $T$ is a contraction, we will use the Blackwell sufficient conditions.

**Lemma 1.2** *If* $\Phi : K \to K$ *is increasing and there exists* $\delta \in (0,1)$ *such that*

$$\Phi(f+\alpha) \leq \Phi(f) + \delta\alpha$$

*for all* $(f,\alpha) \in K \times \mathbb{R}_+$, *then* $\Phi$ *is a contraction.*

Using these sufficient conditions, is the mapping $T$ a contraction? Let's consider each condition in turn:

1. Increasing

   If $V \geq W$, where $V, W \in CB(X)$, this implies $V(x) \geq W(x) \;\; \forall x \in X$. So, if $V(x) \geq W(x) \;\; \forall x \in X$, then $T(V)(x) \geq T(W)(x) \;\; \forall x \in X$. This is by the definition of $T(V)(x) = \max_{y \in \Gamma(x)} \{f(x,y) + \beta V(y)\}$ as any $y^* = \arg\max_{y \in \Gamma(x)} \{f(x,y) + \beta W(y)\}$ can also be selected for $\arg\max_{y \in \Gamma(x)} \{f(x,y) + \beta V(y)\}$ (if not selected, then a higher max is achieved for $T(V)(x)$). Thus, $T$ is increasing.

2. $\exists \delta \in (0,1)$ such that $\Phi(f+\alpha) \leq \Phi(f) + \delta\alpha$

   By definition, $T(V+\alpha)(x) = \max_{y \in \Gamma(x)} \{f(x,y) + \beta V(y) + \beta\alpha\}$ for all $x \in X$. By definition, $\max_{y \in \Gamma(x)} \{f(x,y) + \beta V(y) + \beta\alpha\} = T(V)(x) + \beta\alpha$. As $\beta < 1$ by Assumption 3, then the condition is satisfied.

So, $T$ is a contraction. How does this help us? We can apply a different fixed point theorem, called the Contraction Mapping Theorem.

**Theorem 1.7** *Contraction Mapping Theorem*

*If* $\Phi : K \to K$ *is a contraction and* $K$ *is a complete metric space, then there exists a unique* $k^* \in K$ *such that* $\Phi(k^*) = k^*$.

As $T$ is a contraction and $CB(X)$ is a complete metric space, then the Contraction Mapping Theorem guarantees that there exists a unique fixed point $V \in CB(X)$ such that $T(V) = V$.

This results in the so-called Bellman equation:

$$V(x) = \max_{y \in \Gamma(x)} \{f(x,y) + \beta V(y)\} \text{ for all } x \in X.$$

Not only does a value function $V : X \to \mathbb{R}$ exist that is consistent with the Bellman equation, but this value function is unique from the statement of the Contraction Mapping Theorem.

This implies that any solution to standard dynamic optimization problem, stated as

$$\text{Given } x_0 \in X, \underset{(x_n)_{n\in\mathbb{N}}}{max} \quad f(x_0, x_1) + \sum_{i=1}^{\infty} \beta^i f(x_i, x_{i+1}) \text{ such that } x_n \in \Gamma(x_{n-1})$$

$\forall n \in \mathbb{N}$,

is also a solution to the Bellman equation, and vice versa.

## 1.4 Differential Topology

I first introduce the concept of measure and properness before moving into the basic results for the analysis of differentiable equations.

### 1.4.1 Measure

The set $X^0 \subseteq \mathbb{R}^n$ has *zero measure* if for every $\epsilon > 0$, $X^0$ can be covered by $n-$dimensional cubes whose total volume is less than $\epsilon$. For any open set $X \subseteq \mathbb{R}^n$, the subset $X^* \subseteq X$ is a *generic subset* of $X$ iff:

1. $X^*$ is open.
2. The complement relative to $X$, $(X^*)_X^C = X \backslash X^*$ has zero measure.

If a property holds over a generic subset, the property is said to hold *generically.*

### 1.4.2 Properness

For $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$, the function $f : X \to Y$ is *proper* iff:

1. $f$ is continuous.
2. For any $Y' \subseteq Y$ compact, the set $f^{-1}(Y')$ is compact.

Using the definition of properness, we obtain the following property (of great value in the coming sections).

**Lemma 1.3** *Properness Property*

*Suppose that for $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$, the function $f : X \to Y$ is proper. If $X' \subseteq X$ is closed (relative to $X$), then $f(X') \subseteq Y$ is closed (relative to $Y$).*

**Proof.** Consider a sequence $y^\nu \in f(X')$ for $\nu \in \mathbb{N}$ such that $\lim_{\nu\to\infty} y^\nu = y'$. As $y^\nu \in f(X')$ for $\nu \in \mathbb{N}$, then there exists $(x^\nu)_{\nu\in\mathbb{N}}$ such that $x^\nu \in X'$ and $y^\nu = f(x^\nu)\ \forall \nu \in \mathbb{N}$. Define the set $Y' = \left\{(y^\nu)_{\nu\in\mathbb{N}}, y'\right\}$. The set $Y'$ is compact. As $f$ is proper, then $f^{-1}(Y')$ is compact. As $x^\nu \in f^{-1}(Y')\ \ \forall \nu \in \mathbb{N}$, then there exists a subsequence, wlog the original sequence, such that $\lim_{\nu\to\infty} x^\nu = x' \in f^{-1}(Y')$. As $X'$ is closed, then $x' \in X'$. As $f$ is continuous, $y' = f(x')$, so $y' \in f(X')$. This verifies that $f(X')$ is closed. ■

### 1.4.3 Regular vs. Critical values

The following framework is utilized when analyzing differentiable equations:

- $\Xi \subset \mathbb{R}^J$ is the set of variables; it is open with typical element $\xi$.
- $\Theta \subset \mathbb{R}^K$ is the set of parameters; it is open with typical element $\theta$.
- $H = \Xi \times \Theta$ is the set of variables and parameters, with typical element $\eta = (\xi, \theta)$.
- $\Phi : H \to \mathbb{R}^L$ is the system of equations characterizing equilibria; it is $C^1$.

The set of solutions are $\eta \in H$ such that $\Phi(\eta) = 0$. The results that we will obtain will depend upon whether (i) $J = L$ (equal number of variables and equations) or (ii) $J < L$ (more equations than variables). For the case of $J = L$ (as discussed in the remainder of the section), the results that we obtain (provided the assumptions are met) can be viewed as 'positive.' For the case of $J < L$ (as considered in Exercise 3 at the end of the chapter), the results that we obtain (provided the exact same assumptions are met) can be viewed as 'negative.'

The following are important mathematical definitions:

- $M = \{\eta \in H : \Phi(\eta) = 0\}$, the set of solutions.
- $\pi$ is the projection of $M$ onto $\Theta$.
- $\eta \in M$ is a critical point if $rank D_\xi \Phi(\eta) < L$, a regular point if $rank D_\xi \Phi(\eta) = L$.

- $\theta \in \Theta$ is a critical value if $\pi^{-1}(\theta)$ contains at least one critical point. Define $\Theta^c$ as the set of critical values.

- Any $\theta \in \Theta$ that is not a critical value must be a regular value. Define $\Theta^r = \Theta \backslash \Theta^c$ as the set of regular values.

The terms critical/regular point and critical/regular value are with respect to the projection $\pi$. Notice that if no solution exists for some parameter $\theta^*$, that is, $\pi^{-1}(\theta^*) = \emptyset$, then $\theta^*$ is a regular value (any value that is not a critical value is a regular value, by definition). Figure 1.6 illustrates the concepts of critical point, regular point, critical value, and regular value.

### 1.4.4 Results

This section contains the required mathematical results. The results are stated for the case of $J = L$. The following section combines the results and provides a simple routine in order to prove the important regularity result known as FLU (finite and locally unique).

**Theorem 1.8** *Closedness Theorem*

*If $\pi$ is proper, then $\Theta^c$ is closed, so that $\Theta^r$ is open.*

**Proof.** The set $\{0\}$ is closed and the function $\Phi$ is continuous, so $\Phi^{-1}(0) = M$ is closed (relative to $H$). With $J = L$, we can define the determinant $\det : M \to \mathbb{R}$ on the matrices $D_\xi \Phi(\eta)$. The determinant is continuous and the set $\{0\}$ is closed, so the set of critical points, those $\eta \in M$ such that $\eta \in \det^{-1}(0)$, is closed. As $\pi$ is proper, then by the Properness Property, the set of critical values $\Theta^c$ is closed. ■

Concerning notation, the derivative matrix $D_\xi \Phi(\eta)$ considers the derivatives with respect to the variables $\xi$. As $J = L$ (number of variables equals number of equations), then $D_\xi \Phi(\eta)$ is a square matrix. The derivative matrix $D\Phi(\eta)$ considers the derivatives with respect to both the variables $\xi$ and the parameters $\theta$. This matrix could be more accurately written as follows: $D\Phi(\eta) = D_{\xi,\theta}\Phi(\eta)$. This matrix now has more columns than rows (as the number of variables + the number of parameters is greater than the number of equations).

**Theorem 1.9** *Transversality Theorem*

*If, for every $\eta \in M$, $rankD\Phi(\eta) = L$, then $\Theta^c$ has zero measure, so that $\Theta^r$ has full measure.*

The proof of the theorem is too involved to include in these notes, but it is basically an application of Sard's Theorem. Consider what the theorem says. The set of parameters $\theta$ that are both (a) critical (that is, $rankD_{\xi}\Phi(\eta) < L$) and (b) satisfy the rank condition $rankD\Phi(\eta) = L$ is a set of measure zero. This brings us to the aptly named Stack of Records Theorem.

**Theorem 1.10** *Stack of Records Theorem*

*If $\pi$ is proper, $\theta' \in \Theta^r$, and $\pi^{-1}(\theta') \neq \emptyset$, then*

*1. $\pi^{-1}(\theta') = \{\eta_i' : i = 1, ..., I \text{ with } I < \infty\}$.*

*2. There is an open neighborhood $\Theta'$ around $\theta'$ and open neighborhoods $H_i'$ around $\eta_i'$ and $C^1$ mappings $\phi_i : \Theta' \to H_i' \cap M \quad \forall i = 1, ..., I$ such that for any $\theta \in \Theta'$ and any $\eta \in \pi^{-1}(\theta)$, there exists a unique $i$ such that $\eta = \phi_i(\theta)$.*

Figure 1.7 illustrates the result.

**Proof.** Consider any $\eta' \in \pi^{-1}(\theta')$. Since $rankD_{\xi}\Phi(n) = L$ and $J = L$, then we can apply the Implicit Function Theorem. Thus, there exists a neighborhood $\Theta'$ of $\theta$ and a neighbohood $H'$ of $\eta'$ and a $C^1$ mapping $\phi : \Theta' \to H'$ such that for any $\theta \in \Theta'$, $\eta \in H' \cap M$ iff $\eta = \phi(\theta)$. Thus, $\pi^{-1}(\theta') \subset \underset{\eta' \in \pi^{-1}(\theta')}{\cup} H'$, an open covering. Since $\pi$ is proper and $\{\theta'\}$ is compact, then $\pi^{-1}(\theta')$ is compact, so the cover $\underset{\eta' \in \pi^{-1}(\theta')}{\cup} H'$ must have a finite subcover $\underset{i=1,...,I}{\cup} H_i'$. That is, $\pi^{-1}(\theta') = \{\eta_i' : i = 1, ..., I \text{ with } I < \infty\}$.

By taking the neighborhood $\Theta'$ to be small enough, then $H_i' \cap H_j' = \emptyset \quad \forall i \neq j$. Thus, for any $\theta \in \Theta'$, each $\eta \in \pi^{-1}(\theta)$ has a unique $\phi_i$. ■

### 1.4.5 Finite Local Uniqueness

For any system of equations $\Phi(\eta) = 0$, we seek to verify the following two properties:

- $\pi$ is proper.
- For every $\eta \in M$, $rankD\Phi(\eta) = L$ (called the rank condition).

Then over a generic subset of parameters (the set of regular values $\Theta^r$), there are a finite number of solutions (that is, $\pi^{-1}(\theta)$ is a finite set $\forall \theta \in \Theta^r$) and the solutions are locally unique. By locally unique, we mean that for any $\theta \in \Theta^r$ with a solution $\eta \in \pi^{-1}(\theta)$, there is only one solution $\eta' \in \pi^{-1}(\theta')$ in a neighborhood around $\eta$, where $\theta'$ lies in a neighborhood around $\theta$.

This is the result Finite Local Uniqueness (FLU). It implies that small changes in the parameters do not change the number of solutions nor do they allow for any "jumps".

## 1.5 Exercises

1. (Applying Farkas Lemma)

   Let $R$ be a $m \times n$ matrix and $\theta \in \mathbb{R}^n$. Use Farkas Lemma to prove the following implication.

$$\begin{aligned} \text{There does not exist } \theta \text{ such that } R\theta &> 0 \\ &\Downarrow \\ \exists\lambda &\in \mathbb{R}^m_{++} \text{ s.t. } \lambda^T R = 0. \end{aligned}$$

   (Note: The other implication is a simple one-line proof. Make sure that you are proving the correct implication [i.e., the multi-page behemoth of a proof]).

2. (Applying Kakutani's Fixed Point Theorem)

   We will use the theory of correspondences to prove the existence of a Nash equilibrium. Consider a game with $I$ players. Each player $i$ has a finite number of actions $\left\{a_1^i, ...., a_{J_i}^i\right\}$. The set of strategies for each player $i$ is then the simplex $\Delta^{J_i - 1}$ of dimension $J_i - 1$. The strategies are simply the probabilities that each player assigns to each of the finite number of actions. For simplicity, denote the strategy set for each player as $S^i$ with element $s^i$. These sets are nonempty, compact, and convex.

   If player $i's$ payoff value for the action profile $a = (a_1, ..., a_i, ..., a_I)$ is $p^i(a)$, then the objective function for each player is (using the convention $s = (s_1, ..., s_i, ..., s_I)$):

$$u^i(s) = \sum\nolimits_a p^i(a) \cdot \prod\nolimits_i s^i\left(a^i\right),$$

   where $s^i\left(a^i\right)$ is the probability that player $i$ selects the action $a^i$. This objective function is quasi-concave in $s$ (as it's linear).

   Denote $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_I)$ as the vector of strategies (not including the strategy for player $i$) and $\underset{j \neq i}{\times} S^j = S^1 \times ... \times S^{i-1} \times S^{i+1} \times ... \times S^I$ as the Cartesian product of the strategy sets (not including the strategy set for player $i$). Define $F^i : \underset{j \neq i}{\times} S^j \rightrightarrows S^i$ as the set of "feasible" strategies. Notice that $F^i(s_{-i}) = S^i \ \ \forall s_{-i} \in \underset{j \neq i}{\times} S^j$.

Define $BR^i : \underset{j \neq i}{\times} S^j \rightrightarrows S^i$ as the best response correspondence for player $i$. Use the Berge's Maximum Theorem and Kakutani's Fixed Point Theorem to prove that the self-map $\left(BR^1, ..., BR^I\right) : \underset{i}{\times} S^i \rightrightarrows \underset{i}{\times} S^i$ has a fixed point. By definition, a fixed point is a Nash equilibrium.

3. (A Different Application of Differential Topology)

   Let $\Xi \subset \mathbb{R}^J$ be the set of variables (with typical element $\xi$), $\Theta \subset \mathbb{R}^K$ be the set of parameters (with typical element $\theta$), $H = \Xi \times \Theta$ (so $\eta = (\xi, \theta)$), and $\Phi : H \to \mathbb{R}^L$ be the $C^1$ system of equations. Assume that $J < L$. Suppose that the projection $\pi$ is proper and the rank condition holds (that is, $rank D\Phi(\eta) = L$). Using the mathematical results from Section 1.4, what conclusions can be drawn? The conclusions will be of the form, "over a generic subset of parameters (the set of regular values $\Theta^r$), then...".

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# Chapter 2

# Arrow-Debreu Model

This chapter will introduce the canonical general equilibrium model of pure-exchange in the static setting. The equilibrium is commonly called a Walrasian equilibrium. However, the existence of the equilibrium was not shown until the early 1950s in joint work by the Nobel laureates Kenneth Arrow and Gerard Debreu. For this reason, I use the term "Arrow-Debreu equilibrium."

After introducing the model and defining an equilibrium, I can prove the fundamental properties of an Arrow-Debreu equilibrium. First, I prove existence (using the results for correspondences from Section 1.3). Second, I prove the First Basic Welfare Theorem. Next, I prove the Second Basic Welfare Theorem (using the Supporting Hyperplane Theorem from Section 1.1). Finally, I will verify the regularity of the Arrow-Debreu equilibria, namely that the equilibria are finite and locally unique (using the FLU result from Section 1.4).

Some proofs are relegated to the end of the chapter. I prefer to spend the main sections of the chapter motivating the results and discussing why certain assumptions are required and what is required to carry out the proofs.

## 2.1 The Model

I will be fairly pedantic about the notation used in the model. The reason is that the fundamentals of the model will be used throughout the manuscript, so proper notation can help to minimize confusion.

The first component of the environment is households. Let $\mathbf{H} = \{1, ..., H\}$ be the set of households, where the number of households is finite ($H < \infty$). A typical element of the set $\mathbf{H}$ will be $h$. Households trade and consume $G$ goods. Let $\mathbf{G} = \{1, ..., G\}$ be the set of goods, with typical element $g$. Denote $x_g^h$ as the consumption of good $g$ by household $h$. Then $x^h = \left(x_g^h\right)_{g\in\mathbf{G}}$ is the vector of consumption by household $h$. Finally, $x = \left(x^h\right)_{h\in\mathbf{H}}$ is the vector of consumption by all households. This is also referred to as the allocation of the economy.

This model is one of pure exchange. Production is not considered in this manuscript. Each household begins with a vector of endowments. As with consumption, let $e_g^h$ denote the endowment of good $g$ for household $h$, where $e^h = \left(e_g^h\right)_{g\in\mathbf{G}}$ is the vector of endowments of household $h$ and $e = \left(e^h\right)_{h\in\mathbf{H}}$ is the vector of all endowments.

Each household must select consumption within its consumption set. The consumption set is denoted $X^h$ and must be a subset of the nonnegative orthant, $X^h \subseteq \mathbb{R}_+^G$. We assume that the endowments $e^h$ belong to the consumption set, $e^h \in X^h$. This allows for the possibility of a no-trade equilibrium (also called autarchy).

The objective function of a household is called the utility function $u^h : X^h \to \mathbb{R}$. These household primitives (parameters of the model) are listed below.

- $X^h \subseteq \mathbb{R}_+^G$ is the consumption set. We typically assume that $X^h$ is closed (relative to the set $\mathbb{R}_+^G$) and convex.

- $u^h : X^h \to \mathbb{R}$ is the utility function. We typically assume that $u^h$ is continuous, locally non-satiated, and quasi-concave.

- $e^h \in X^h$ is the endowment vector. We typically assume that $e^h >> 0$.

**Definition 2.1** *The function $u^h : X^h \to \mathbb{R}$ is locally non-satiated if $\forall x \in X^h$ and $\forall \epsilon > 0$, there exists $y \in X^h \cap N_\epsilon(x)$ such that $u^h(y) > u^h(x)$.*

The second component of the environment is the market institutions. In the Arrow-Debreu model, markets are perfectly competitive, meaning that all households take prices as given and do not account for how their actions affect the market prices. Each good has a market price $p_g$. The vector of all prices is $p = (p_g)_{g\in\mathbf{G}} \in \mathbb{R}^G \backslash \{0\}$. The convention is that the price vector $p$ is a row vector. As can be shown (see Exercise 1), the assumptions on utility guarantee that $p > 0$. An equilibrium, as will be evident from the definition, exhibits nominal price indeterminacy. To remove this, we normalize the prices so that $p \in \Delta^{G-1}$.

We are now prepared to define an Arrow-Debreu equilibrium.

**Definition 2.2** *An Arrow-Debreu equilibrium is $\left(\left(x^h\right)_{h\in\mathbf{H}}, p\right)$ such that*

1. *$\forall h \in \mathbf{H}$, given $p$, $x^h$ is an optimal solution to the household problem $(HP)$*

$$(HP) \qquad \begin{array}{ll} \text{maximize} & u^h\left(x^h\right) \\ \text{subject to} & x^h \in X^h \\ & p\left(e^h - x^h\right) \geq 0 \end{array} .$$

2. *Markets clear*

$$\sum_{h \in \mathbf{H}} x_g^h = \sum_{h \in \mathbf{H}} e_g^h \quad \forall g \in \mathbf{G}.$$

## 2.2 Existence

Which assumptions are required to guarantee that an Arrow-Debreu equilibrium exists for all parameters $\left(e^h, u^h\right)_{h \in \mathbf{H}}$? These assumptions are listed below. After the statement of the theorem, I will discuss why the third assumption is required.

Assumption E1: $X^h = \mathbb{R}_+^G \ \ \forall h \in \mathbf{H}$.

Assumption E2: $u^h : X^h \to \mathbb{R}$ is continuous, locally non-satiated, and quasi-concave $\forall h \in \mathbf{H}$.

Assumption E3: $e^h >> 0 \ \ \forall h \in \mathbf{H}$.

Assumption E4: For some household $h'$, $u^{h'}$ is non-decreasing.

**Theorem 2.1** *Under Assumptions E1-E4, an Arrow-Debreu equilibrium $\left(\left(x^h\right)_{h \in \mathbf{H}}, p\right)$ exists.*

**Proof.** For the complete proof, see Section 2.6. ■

The proof method proceeds as follows. We begin by defining the budget correspondence $b^h : \Delta^{G-1} \rightrightarrows X^h \ \ \forall h \in \mathbf{H}$ such that $b^h(p) = \left\{x^h \in X^h : p\left(e^h - x^h\right) \geq 0\right\}$. Using this definition, the demand correspondence can be defined as $d^h : \Delta^{G-1} \rightrightarrows X^h$ such that $d^h(p) = \underset{x^h \in b^h(p)}{\arg\max} \ u^h\left(x^h\right)$. If the conditions of Berge's Maximum Theorem are satisfied, namely that $b^h$ is well-defined and continuous, then the correspondence $d^h$ is upper hemi-continuous. Using Assumption E2 (continuous and quasi-concave), the correspondence $d^h$ is also well-defined and convex-valued. Thus, we can apply the Kakutani's Fixed Point Theorem to show that a fixed point exists. Further details are left to Section 2.6.

From the outline of the proof, the key step is to verify that the conditions of Berge's Maximum Theorem are satisfied. It is straightforward to show that the budget correspondence $b^h$ is well-defined and upper hemi-continuous. But the conditions of Berge's Maximum Theorem require that the correspondence is also lower hemi-continuous. In order for this to be achieved, we must satisfy the equilibrium condition $pe^h > 0$. Since $p > 0$, an assumption of $e^h > 0$ does not suffice as leaves open the possibility that $pe^h = 0$ (namely, any good with a strictly positive price $[p_g > 0]$ may have a zero endowment $\left[e_g^h = 0\right]$). So we require that $e^h >> 0$, which is Assumption E3.

What is so special about requiring $pe^h > 0$? Suppose otherwise. That is, suppose $(p_1, p_2) = (0, 1)$. This is depicted in Figure 2.1 in the companion 'Figures' document. In both panels, the initial endowment is labeled as the point $e$ (we drop the household superscript for convenience). Given the prices $(p_1, p_2) = (0, 1)$, the choice $x$ is budget feasible, $x \in b^h(p)$.

Consider any sequence $p^\nu \to p$, specifically $p^\nu = \left(\frac{1}{\nu}, 1 - \frac{1}{\nu}\right)$ for $\nu \in \mathbb{N}$. Equilibrium prices (see Exercise 1) can never contain a negative element: $p^\nu > 0$. The definition of lower hemi-continuity requires that a sequence $x^\nu$ exists such that (i) $x^\nu \to x$ and (ii) $x^\nu \in b^h(p^\nu)$ for $\nu \in \mathbb{N}$. However, as can be seen in the right panel of Figure 2.1, no such sequence can be found (budget feasible choices are those lying to the lower-left of the dotted budget lines). Thus, the correspondence $b^h$ is not lower-hemicontinuous unless $pe^h > 0$.[1]

## 2.3 First Basic Welfare Theorem

The First Basic Welfare Theorem states that all Arrow-Debreu equilibria are Pareto optimal. Let's define a Pareto optimal allocation.

**Definition 2.3** *A feasible allocation* $\left(x^h\right)_{h \in \mathbf{H}}$ *is such that*

1. $x^h \in X^h \ \forall h \in \mathbf{H}$ *and*

[1] A weaker assumption can be made in order to guarantee $pe^h > 0$. This weaker assumption was derived by Lionel McKenzie and termed irreducibility. Formally, his assumption contains two parts:

1. $e^h > 0 \ \ \forall h \in \mathbf{H}$ and $\sum_{h \in \mathbf{H}} e^h >> 0$
2. $\forall \mathbf{H}_1, \mathbf{H}_2 \neq \emptyset$ such that $\mathbf{H}_1 \cap \mathbf{H}_2 = \emptyset$ and $\mathbf{H}_1 \cup \mathbf{H}_2 = \mathbf{H}$ and any allocation $\left(x^h\right)_{h \in \mathbf{H}}$, there exists $\left(y^h\right)_{h \in \mathbf{H}_1} \geq 0$ and $\left(\tilde{x}^h\right)_{h \in \mathbf{H}_2} \geq 0$ such that (a) $\sum_{h \in \mathbf{H}_1} y_g^h = 0$ whenever $\sum_{h \in \mathbf{H}_1} e_g^h = 0$, (b) $\sum_{h \in \mathbf{H}_2} \tilde{x}^h \leq \sum_{h \in \mathbf{H}_1} \left(y^h + e^h\right) + \sum_{h \in \mathbf{H}_2} e^h$, and (c) $\left(u^h\left(\tilde{x}^h\right)\right)_{h \in \mathbf{H}_2} > \left(u^h\left(x^h\right)\right)_{h \in \mathbf{H}_2}$.

Basically, the second condition says that each household has a strictly positive endowment of at least one good that is "desired" by some other household (forcing the price of that good to be strictly positive).

2. $\sum_{h\in\mathbf{H}} x_g^h = \sum_{h\in\mathbf{H}} e_g^h \quad \forall g \in \mathbf{G}$.

For simplicity, define the set of feasible allocations as $FA$.

**Definition 2.4** *A Pareto optimal allocation* $\left(x^h\right)_{h\in\mathbf{H}}$ *is such that there does not exist a feasible allocation* $\left(y^h\right)_{h\in\mathbf{H}}$ *where* $u^h\left(y^h\right) \geq u^h\left(x^h\right)$ $\forall h \in \mathbf{H}$ *and* $u^{h'}\left(y^{h'}\right) > u^{h'}\left(x^{h'}\right)$ *for some* $h'$.

To find the Pareto optimal allocations, we have two options. First, we can assume that $u^{h'}$ is strictly increasing for some household $h'$ and $\forall h \neq h'$, $u^h$ is continuous. Then $\left(x^{h*}\right)_{h\in\mathbf{H}}$ is a Pareto optimal allocation iff $\left(x^{h*}\right)_{h\in\mathbf{H}}$ is an optimal solution to the following nonlinear programming problem:

$$\begin{array}{lll} (PO_{h'}) & \text{maximize} & u^{h'}\left(x^{h'}\right) \\ & \text{subject to} & u^h\left(x^h\right) \geq u^h\left(x^{h*}\right) \ \forall h \neq h' \\ & & \left(x^h\right)_{h\in\mathbf{H}} \in FA \end{array} .$$

The proof is in Section 2.6.

The second option employs concavity. If $\left(x^{h*}\right)_{h\in\mathbf{H}}$ is a Pareto optimal allocation and $u^h$ is concave $\forall h \in \mathbf{H}$, then there exists $\left(\mu^h\right)_{h\in\mathbf{H}} \in \Delta^{H-1}$ such that $\left(x^{h*}\right)_{h\in\mathbf{H}}$ is an optimal solution to the following nonlinear programming problem:

$$\begin{array}{lll} (PO) & \text{maximize} & \sum_{h\in\mathbf{H}} \mu^h \cdot u^h\left(x^h\right) \\ & \text{subject to} & \left(x^h\right)_{h\in\mathbf{H}} \in FA \end{array} .$$

Conversely, if there exists $\left(\mu^h\right)_{h\in\mathbf{H}} \in int\left(\Delta^{H-1}\right)$ such that $\left(x^{h*}\right)_{h\in\mathbf{H}}$ is an optimal solution to $(PO)$, then it is a Pareto optimal allocation. The proof is in Section 2.6.

The assumptions required to prove the First Basic Welfare Theorem are:

Assumption F1: $u^h : X^h \to \mathbb{R}$ is locally non-satiated $\forall h \in \mathbf{H}$.

**Theorem 2.2** *Under Assumption F1, all Arrow-Debreu equilibrium allocations* $\left(x^h\right)_{h\in\mathbf{H}}$ *are Pareto optimal.*

**Proof.** See Section 2.6. ■

The assumption of locally non-satiated utility implies that all budget constraints hold with equality: $p\left(e^h - x^h\right) = 0 \ \forall h \in \mathbf{H}$. In this case, why do we define an Arrow-Debreu equilibrium with inequalities in the budget constraints? The reason is that the First Basic Welfare Theorem no longer holds when we define an Arrow-Debreu equilibrium with equalities in the budget constraint.

**Definition 2.5** *An Arrow-Debreu equilibrium with equalities in the budget constraint is* $\left(\left(x^h\right)_{h\in\mathbf{H}}, p\right)$ *such that*

1. $\forall h \in \mathbf{H}$, *given* $p$, $x^h$ *is an optimal solution to the household problem* $(HP=)$

$$(HP=) \qquad \begin{array}{ll} \textit{maximize} & u^h\left(x^h\right) \\ \textit{subject to} & x^h \in X^h \\ & p\left(e^h - x^h\right) = 0 \end{array} .$$

2. *Markets clear*

$$\sum\nolimits_{h\in\mathbf{H}} x_g^h = \sum\nolimits_{h\in\mathbf{H}} e_g^h \quad \forall g \in \mathbf{G}.$$

Why does the First Basic Welfare Theorem not hold for an Arrow-Debreu equilibrium with equalities in the budget constraint? Consider the following counterexample. Assume the economy has 2 households and 2 goods, so we can illustrate the economy in an Edgeworth box. The consumption sets for both households are given by:

$$\begin{aligned} X^1 &= \left\{x^1 \in \mathbb{R}_+^2 : x_2^1 \geq \frac{1}{x_1^1}\right\}. \\ X^2 &= \mathbb{R}_+^2. \end{aligned}$$

These consumption sets are closed and convex (satisfying Assumption S1 below, which are typical assumptions for $\left(X^h\right)_{h\in\mathbf{H}}$). The utility functions for both households are given by:

$$\begin{aligned} u^1\left(x^1\right) &= -x_1^1. \\ u^2\left(x^2\right) &= x_1^2. \end{aligned}$$

Both utility functions are locally non-satiated (satisfying Assumption F1). The endowments are labeled $e$ in Figure 2.2. The shaded region above the curve $x_2^1 = \frac{1}{x_1^1}$ denotes $X^1$. There are two price vectors that comprise Arrow Debreu equilibria with equalities in the budget constraints: (i) $p^* >> 0$ and (ii) $p^{**} = (1,0)$. Given the equilibrium price vector (i) $p^* >> 0$, the equilibrium allocation is $x^*$ near the top-left corner of the Edgeworth box. Given the equilibrium price vector (ii) $p^{**} = (1,0)$, the equilibrium allocation lies anywhere along the darkened vertical line through the endowment $e$. In both cases, the noted equilibrium allocations are the optimal solutions to the household maximization problem $(HP =)$ (equalities in the budget constraints).

The First Basic Welfare Theorem requires that each of these equilibrium allocations is Pareto optimal. Yet, the only Pareto optimal allocation is $x^*$. All of the allocations along the darkened vertical line can be Pareto improved (both households made better off) by moving to the left in the Edgeworth box.

## 2.4 Second Basic Welfare Theorem

The Second Basic Welfare Theorem serves as a partial converse to the First Basic Welfare Theorem. The Second Basic Welfare Theorem states that for any Pareto optimal allocation, equilibrium prices can be found so that that allocation is an Arrow-Debreu equilibrium allocation.

The assumptions required to prove the Second Basic Welfare Theorem are:

Assumption S1: $X^h \subseteq \mathbb{R}_+^G$ is closed and convex $\forall h \in \mathbf{H}$.

Assumption S2: $u^h : X^h \to \mathbb{R}$ is continuous and quasi-concave $\forall h \in \mathbf{H}$.

Assumption S3: For some household $h'$, wlog $h' = 1$, $X^1$ is unbounded above and $u^1$ is increasing.

**Theorem 2.3** *Under Assumptions S1-S3, if* $\left(e^h\right)_{h\in\mathbf{H}} = \left(x^{h*}\right)_{h\in\mathbf{H}}$ *is a Pareto optimal allocation and* $x^{h*} \in intX^h$ $\forall h \in \mathbf{H}$, *then there is* $p^* > 0$ *such that* $\left(\left(x^{h*}\right)_{h\in\mathbf{H}}, p^*\right)$ *is an Arrow-Debreu equilibrium.*

**Proof.** See Section 2.6. ■

If we want to consider a Pareto optimal allocation $\left(x^{h*}\right)_{h\in\mathbf{H}}$ that differs from the endowment $\left(e^h\right)_{h\in\mathbf{H}}$, then the statement of the theorem includes "an imposed tax/subsidy scheme $\left(\tau^h\right)_{h\in\mathbf{H}}$ with $\sum_{h\in\mathbf{H}} \tau^h = 0$ such that $\tau^h = p^*x^{h*} - p^*e^h$ $\forall h \in \mathbf{H}$." If $\tau^h > 0$, the transfer is a subsidy and that amount is added to the household's income. If $\tau^h < 0$, the transfer is a tax and the amount is subtracted from the household's income.

Notice that in the statement of Theorem 2.3, we require $x^{h*} \in intX^h$ $\forall h \in \mathbf{H}$. Why is this? Why can't the Second Basic Welfare Theorem be valid for an allocation on the boundary of some household's consumption set? Consider the following counter-example.

The economy contains 2 households and 2 goods, so the allocations can be depicted in the Edgeworth box. The consumption sets for both households are given by:

$$X^1 = \left\{x^1 \in \mathbb{R}^2_+ : x^1_2 \geq 1 - x^1_1\right\}.$$
$$X^2 = \mathbb{R}^2_+.$$

These consumption sets are closed and convex (satisfying Assumption S1). The utility functions for both households are given by:

$$u^1\left(x^1\right) = \min\left\{x^1_1, x^1_2\right\}.$$
$$u^2\left(x^2\right) \text{ is "smooth" } (C^2, \text{ strictly concave, strictly increasing}).$$

Both utility functions are continuous and quasi-concave (satisfying Assumption S2). Consider Figure 2.3, where $X^1$ is the shaded region above the curve $x^1_2 = 1 - x^1_1$. Figure 2.3 contains three indifference curves for $h = 1$ and two for $h = 2$.

The allocation $E$ is Pareto optimal. Why? We cannot find a feasible allocation that makes both households better off than they currently are at $E$.

But does there exist a price $p^* > 0$ such that the allocation $E$ is an Arrow-Debreu equilibrium allocation? Well, the only possible equilibrium price is $p^*$ as drawn in Figure 2.3 (tangent to the indifference curve for household $h = 2$). But given this equilibrium price $p^*$, the optimal consumption choice by household $h = 1$ would be $E'$, not $E$.

Thus, the Second Basic Welfare Theorem relies importantly on the assumption that $x^{h*} \in intX^h \ \ \forall h \in \mathbf{H}$.

## 2.5 Regularity

When we state that the Arrow-Debreu equilibria are regular, what we mean is that the number of equilibria is finite and that each is locally unique. Recall the Finite Local Uniqueness result from Section 1.4.

The assumptions required to prove Regularity are:

Assumption R1: $X^h = \mathbb{R}^G_{++} \ \ \forall h \in \mathbf{H}$.

Assumption R2: $u^h : X^h \to \mathbb{R}$ is $C^2$, differentiably strictly increasing (meaning that $Du^h\left(x^h\right) >> 0$), differentiably strictly concave (meaning that $D^2u^h\left(x^h\right)$

is a negative definite matrix), and satisfies the boundary condition (meaning that $\forall x \in \mathbb{R}^G_{++}$, $cl\left\{y : u^h(y) \geq u^h(x)\right\} \subseteq \mathbb{R}^G_{++}$) $\forall h \in \mathbf{H}$.

Assumption R3: $e^h >> 0 \ \ \forall h \in \mathbf{H}$.

**Theorem 2.4** *Under Assumptions R1-R3, all Arrow-Debreu equilibria satisfy Finite Local Uniqueness.*

The remainder of the section walks through the proof of Theorem 2.4, with useful properties of the model sprinkled in.

Recall that the household problem $(HP)$ is given by

$$(HP) \qquad \begin{array}{ll} \text{maximize} & u^h\left(x^h\right) \\ \text{subject to} & x^h \in X^h \\ & p\left(e^h - x^h\right) \geq 0 \end{array} .$$

Under Assumption R2 (namely, the boundary condition), the condition $x^h \in X^h$ never binds. Applying the Kuhn-Tucker Theorem, the following conditions are necessary and sufficient conditions for an optimal solution to $(HP)$ :

$$\begin{aligned} Du^h\left(x^h\right) - \lambda^h p &= 0 \\ p\left(e^h - x^h\right) &= 0, \end{aligned}$$

where $\lambda^h \in \mathbb{R}$ is the Lagrange multiplier associated with the budget constraint $p\left(e^h - x^h\right) \geq 0$ in the problem $(HP)$. The convention is that $Du^h\left(x^h\right)$ is a row vector (as is the case with $p$). Under Assumption R2 (namely, $Du^h\left(x^h\right) >> 0$), the budget constraint binds and both $\lambda^h > 0$ and $p >> 0$.

If $\left(\left(x^h\right)_{h\in\mathbf{H}}, p\right)$ satisfies the definition of an Arrow-Debreu equilibrium, then so does $\left(\left(x^h\right)_{h\in\mathbf{H}}, \kappa p\right)$ for any scalar $\kappa > 0$. This is called nominal indeterminacy, as a continuum of prices exist that satisfy the equilibrium definition. In the household problem $(HP)$, only relative prices matter, not the absolute price level. To remove this nominal indeterminacy, we impose the price normalization $p_G = 1$.[2] Thus, the only remaining price variables are $p_{\backslash G} = (p_1, ..., p_{G-1}) \in \mathbb{R}^{G-1}_{++}$. What we show with regularity is that there is no real determinacy; namely that the Arrow-Debreu equilibria are determinate (not a continuum).

[2] This is a different price normalization than previously used in this chapter.

Walras' Law (obtained by summing over all households' budget constraints) states:

$$\sum_{h\in\mathbf{H}} p\left(e^h - x^h\right) = 0.$$

As $p >> 0$, if $\sum_{h\in\mathbf{H}}\left(e^h_g - x^h_g\right) = 0\ \forall g < G$, then it must also be the case that $\sum_{h\in\mathbf{H}}\left(e^h_G - x^h_G\right) = 0$. Thus, the market clearing condition $\sum_{h\in\mathbf{H}}\left(e^h_G - x^h_G\right) = 0$ is redundant given that the market clearing conditions hold $\forall g < G$. I use the notation

$$\sum_{h\in\mathbf{H}}\left(e^h_{\backslash G} - x^h_{\backslash G}\right) = \left(\sum_{h\in\mathbf{H}}\left(e^h_1 - x^h_1\right), ..., \sum_{h\in\mathbf{H}}\left(e^h_{G-1} - x^h_{G-1}\right)\right)^T$$

for the $(G-1)$ −dimensional column vector.

Define the system of equations $\Phi : \underset{h\in\mathcal{H}}{\times}\left(\mathbb{R}^{G+1}_{++}\right) \times \mathbb{R}^{G-1}_{++} \to \mathbb{R}^{H(G+1)+G-1}$ as:

$$\Phi\left(\left(x^h, \lambda^h\right)_{h\in\mathbf{H}}, p\right) = \begin{pmatrix} \begin{pmatrix} \left[Du^h\left(x^h\right) - \lambda^h p\right]^T \\ p\left(e^h - x^h\right) \end{pmatrix}_{h\in\mathbf{H}} \\ \sum_{h\in\mathbf{H}}\left(e^h_{\backslash G} - x^h_{\backslash G}\right) \end{pmatrix},$$

where $\left(x^h, \lambda^h\right)_{h\in\mathbf{H}} \in \mathbb{R}^{H(G+1)}_{++}$ and $p \in \mathbb{R}^{G-1}_{++}$. By definition, $\Phi\left(\left(x^h, \lambda^h\right)_{h\in\mathbf{H}}, p\right) = 0$ iff $\left(\left(x^h\right)_{h\in\mathbf{H}}, p\right)$ is an Arrow-Debreu equilibrium.

Recalling the proof method discussed in Section 1.4.5, Finite Local Uniqueness is proven if we can show that (i) $\pi$ is proper and (ii) $rank D\Phi\left(\left(x^h, \lambda^h\right)_{h\in\mathbf{H}}, p\right) = H(G+1) + G - 1$. The first condition is left to Exercise 6. We will now walk through the second condition.

The derivative $D\Phi\left(\left(x^h, \lambda^h\right)_{h\in\mathbf{H}}, p\right)$ means that we are taking derivatives with respect to $\left(x^h\right)_{h\in\mathbf{H}}$, $\left(\lambda^h\right)_{h\in\mathbf{H}}$, $p$, and $\left(e^h\right)_{h\in\mathbf{H}}$ (all variables and all parameters). If we show that the rank condition holds by only taking derivatives with respect to $\left(x^h\right)_{h\in\mathbf{H}}$, $\left(\lambda^h\right)_{h\in\mathbf{H}}$, $p$, and $e^1$, then we have finished the argument.

The derivative matrix $M = D_{\left(x^h,\lambda^h\right)_{h\in\mathbf{H}},p,e^1}\Phi\left(\left(x^h, \lambda^h\right)_{h\in\mathbf{H}}, p\right)$ has $[H(G+1) + G - 1]$ rows and $[H(G+1) + G - 1 + G]$ columns. The rows of $M$ correspond to the equations in $\Phi\left(\left(x^h, \lambda^h\right)_{h\in\mathbf{H}}, p\right)$, while the columns correspond to the variables (or parameters) that we are taking derivatives with respect to. The derivative matrix is given by:

$$M = \begin{bmatrix} D^2u^1\left(x^1\right) & -p^T & 0 & 0 & 0 & 0 & \begin{pmatrix} -\lambda^1 I_{G-1} \\ 0 \end{pmatrix} & 0 \\ -p & 0 & 0 & 0 & 0 & 0 & \left(e^1_{\setminus G} - x^1_{\setminus G}\right)^T & p \\ 0 & 0 & \dots & \dots & 0 & 0 & \vdots & 0 \\ 0 & 0 & \dots & \dots & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & D^2u^H\left(x^H\right) & -p^T & \begin{pmatrix} -\lambda^H I_{G-1} \\ 0 \end{pmatrix} & 0 \\ 0 & 0 & 0 & 0 & -p & 0 & \left(e^H_{\setminus G} - x^H_{\setminus G}\right)^T & 0 \\ \begin{pmatrix} -I_{G-1} & 0 \end{pmatrix} & 0 & \dots & \dots & \begin{pmatrix} -I_{G-1} & 0 \end{pmatrix} & 0 & 0 & \begin{pmatrix} I_{G-1} & 0 \end{pmatrix} \end{bmatrix}.$$

To show that the matrix $M$ has full rank (there are more columns than rows, so we have to show that the matrix $M$ has full row rank), we set $\nu^T M = 0$ and must verify $\nu^T = 0$, where $\nu \in \mathbb{R}^{H(G+1)+G-1}$ corresponds to equations in $\Phi$ :

$$\nu = \begin{pmatrix} \Delta x^1 \\ \Delta\lambda^1 \\ \vdots \\ \Delta x^h \\ \Delta\lambda^h \\ \vdots \\ \Delta p \end{pmatrix} \begin{pmatrix} FOC1 \\ BC1 \\ \vdots \\ FOCh \\ BCh \\ \vdots \\ MC \end{pmatrix}.$$

The equations $\nu^T M = 0$ are given by:

$$\begin{array}{ll} \left(\Delta x^1\right)^T D^2u^1\left(x^1\right) - \Delta\lambda^1 p - \Delta p^T \left[\begin{array}{cc} I_{G-1} & 0 \end{array}\right] = 0 & (A.1.a) \\ -\left(\Delta x^1\right)^T p^T = 0 & (A.1.b) \\ \vdots & \\ \left(\Delta x^h\right)^T D^2u^h\left(x^h\right) - \Delta\lambda^h p - \Delta p^T \left[\begin{array}{cc} I_{G-1} & 0 \end{array}\right] = 0 & (A.1.c) \\ -\left(\Delta x^h\right)^T p^T = 0 & (A.1.d) \\ \vdots & \\ -\sum_{h\in\mathbf{H}} \lambda^h \left(\Delta x^h_{\setminus G}\right)^T + \sum_{h\in\mathbf{H}} \Delta\lambda^h \left(e^h_{\setminus G} - x^h_{\setminus G}\right)^T = 0 & (A.1.e) \\ \Delta\lambda^1 p + \Delta p^T \left[\begin{array}{cc} I_{G-1} & 0 \end{array}\right] = 0 & (A.1.f) \end{array}. \qquad (2.1)$$

Let's show that $\nu^T = 0$ in three steps:

1. $(A.1.f)$ can be written as

$$\Delta\lambda^1 (p_1, ..., p_{G-1}, 1) + (\Delta p_1, ..., \Delta p_{G-1}, 0) = 0.$$

   As $p >> 0$, then $\Delta\lambda^1 = 0$. Consequently, $\Delta p^T = 0$.

2. Use $(A.1.f)$ to simplify $(A.1.a)$ and then postmultiply the equation by $\Delta x^1$ :

$$\left(\Delta x^1\right)^T D^2 u^1 \left(x^1\right) \Delta x^1 = 0.$$

   As $u^1$ is differentiably strictly concave, the Hessian matrix $D^2 u^1 \left(x^1\right)$ is negative definite. Thus, $\left(\Delta x^1\right)^T = 0$.

3. For any household $h > 1$, postmultiply $(A.1.c)$ by $\Delta x^h$ :

$$\left(\Delta x^h\right)^T D^2 u^h \left(x^h\right) \Delta x^h - \Delta\lambda^h p \Delta x^h = 0.$$

   From $(A.1.d)$, $p\Delta x^h = \left(\left(\Delta x^h\right)^T p^T\right)^T = 0$. As $u^h$ is differentiably strictly concave, the Hessian matrix $D^2 u^h \left(x^h\right)$ is negative definite. Thus, $\left(\Delta x^h\right)^T = 0$. From $(A.1.c)$, $\Delta\lambda^h = 0$.

Thus, $\nu^T = \left(\left(\Delta x^1\right)^T, \Delta\lambda^1, ..., \left(\Delta x^h\right)^T, \Delta\lambda^h, ..., \Delta p^T\right) = 0$, finishing the argument.

## 2.6 Proofs

### 2.6.1 Proof of Theorem 2.1

The existence proof is divided into 6 parts.

#### Part 1: Price and consumption space

The price space $\Delta^{G-1}$ is compact, convex, and nonempty. Define the bounded consumption set $\bar{X}^h = \left\{x^h \in X^h : x_g^h \leq 2 \cdot \sum_{h \in \mathbf{H}} e_g^h \; \forall g \in \mathcal{G}\right\}$. The set $\bar{X}^h$ is compact, convex, and nonempty. We show in Part 6 that equilibrium consumption $x^h$ is an optimal solution to $(HP)$

iff $x^h$ is an optimal solution to:

$$(\overline{HP}) \qquad \begin{array}{ll} \text{maximize} & u^h\left(x^h\right) \\ \text{subject to} & x^h \in \bar{X}^h \\ & p\left(e^h - x^h\right) \geq 0 \end{array} .$$

**Part 2: Demand correspondence**

Define the budget correspondence $b^h : \Delta^{G-1} \rightrightarrows \bar{X}^h \;\; \forall h \in \mathbf{H}$ such that

$$b^h(p) = \left\{x^h \in X^h : p\left(e^h - x^h\right) \geq 0\right\}.$$

The correspondence is well-defined (consider an element $e^h \in b^h(p)$).

**Claim 2.1** *$b^h$ is upper hemi-continuous.*

**Proof.** Consider sequences $p^\nu$ and $x^\nu$ such that $x^\nu \in b^h(p^\nu) \;\; \forall \nu \in \mathbb{N}$. Let $p^\nu \to p$ and $x^\nu \to x$. To prove upper hemi-continuity, we must prove that $x \in b^h(p)$. Suppose otherwise, that is, $x \notin b^h(p)$. Then, either $x_g < 0$ for some good $g$ or $p\left(e^h - x\right) < 0$. Then, by continuity, for some $\nu$, either $x_g^\nu < 0$ for some good $g$ or $p^\nu\left(e^h - x^\nu\right) < 0$. This contradicts that $x^\nu \in b^h(p^\nu)$ $\forall \nu \in \mathbb{N}$. ■

**Claim 2.2** *$b^h$ is lower hemi-continuous.*

**Proof.** Consider a sequence $p^\nu$ such that $p^\nu \to p$ and $x \in b^h(p)$. To prove lower hemi-continuity, we must find a sequence $x^\nu$ such that $x^\nu \in b^h(p^\nu)$ for $\nu \in \mathbb{N}$ and $x^\nu \to x$. There are two cases to consider.

*Case I*: $p\left(e^h - x\right) > 0$. Then $\exists \nu^*$ such that $\forall \nu \geq \nu^*$, $p^\nu\left(e^h - x\right) > 0$. Define $x^\nu = x$ $\forall \nu \geq \nu^*$. Then, $x^\nu \to x$ and $x^\nu \in b^h(p^\nu) \;\; \forall \nu \geq \nu^*$.

*Case II*: $p\left(e^h - x\right) = 0$. I will define $x^\nu = \theta^\nu \cdot x$ for $\nu \in \mathbb{N}$, where

$$\theta^\nu = \left\{\begin{array}{cc} \frac{p^v e^h}{p^v x} & \text{if } \frac{p^v e^h}{p^v x} < 1 \text{ and } p^v x \neq 0 \\ 1 & \text{otherwise} \end{array}\right\}.$$

As $p^\nu \to p$ and $p\left(e^h - x\right) = 0$, then $\exists v^*$ such that $\forall \nu \geq \nu^*$, $p^\nu x > 0$. As $p^\nu \to p$, then $\frac{p^v e^h}{p^v x} \to 1$. Thus, $\forall \nu \geq \nu^*$, $x^\nu = \theta^\nu \cdot x \to x$. By definition, $p^\nu x^\nu = \theta^\nu \cdot p^\nu x = 1 \cdot p^v x \leq p^v e^h$ if $\frac{p^v e^h}{p^v x} \geq 1$ and $p^\nu x^\nu = \theta^\nu \cdot p^\nu x = \left(\frac{p^v e^h}{p^v x}\right) p^v x = p^v e^h$ if $\frac{p^v e^h}{p^v x} < 1$. Thus, $x^\nu \in b^h(p^\nu) \;\; \forall \nu \geq \nu^*$. ■

Thus, $b^h$ is upper hemi-continuous and lower hemi-continuous. Define the demand correspondence $d^h : \Delta^{G-1} \rightrightarrows X^h$ such that

$$d^h(p) = \underset{x^h \in b^h(p)}{\arg\max} \; u^h\left(x^h\right).$$

From Berge's Maximum Theorem, $d^h$ is upper hemi-continuous. As $u^h$ is continuous and $\bar{X}^h$ is compact, the Extreme Value Theorem implies that $d^h$ is well-defined. As $u^h$ is quasi-concave and $\bar{X}^h$ is convex, $d^h$ is convex-valued. To see this, let $x, y \in d^h(p)$. Then $x, y \in X^h$ and both $p\left(e^h - x\right) \geq 0$ and $p\left(e^h - y\right) \geq 0$. Therefore, $\forall \lambda \in [0,1]$, the choice $\lambda x + (1-\lambda) y \in X^h$ (convexity) and $p\left(e^h - (\lambda x + (1-\lambda) y)\right) \geq 0$ (budget constraints are linear). Further, $u^h(\lambda x + (1-\lambda) y) \geq \min\left\{u^h(x), u^h(y)\right\} = u^h(x) = u^h(y)$ (definition of quasi-concavity). Thus, $\lambda x + (1-\lambda) y \in d^h(p)$.

**Part 3: Price correspondence**

Define the price correspondence $\rho : \underset{h \in \mathbf{H}}{\times} \bar{X}^h \rightrightarrows \Delta^{G-1}$ such that $z = \left(z^1, ..., z^H\right) \mapsto \underset{p \in \Delta^{G-1}}{\arg\max}\; p \cdot \sum_{h \in \mathbf{H}} \left(z^h - e^h\right)$. The correspondence is well-defined, using the Extreme Value Theorem. The correspondence is convex-valued as the objective function is quasi-concave (in $p$) and $\Delta^{G-1}$ is convex.

**Claim 2.3** *$\rho$ is upper hemi-continuous.*

**Proof.** Consider sequences $z^\nu$ and $p^\nu$ such that $p^\nu \in \rho(z^\nu)$ for $\nu \in \mathbb{N}$. Let $z^\nu \to z$ and $p^\nu \to p$. To prove upper hemi-continuity, we must show that $p \in \rho(z)$. Suppose otherwise, that is, $p \notin \rho(z)$. Then, there exists $\hat{p} \in \Delta^{G-1}$ such that $\hat{p} \cdot \sum_{h \in \mathbf{H}} \left(z^h - e^h\right) > p \cdot \sum_{h \in \mathbf{H}} \left(z^h - e^h\right)$. Then for some $\nu$, $\hat{p}^\nu \cdot \sum_{h \in \mathbf{H}} \left(z^h - e^h\right) > p^\nu \cdot \sum_{h \in \mathbf{H}} \left(z^h - e^h\right)$. This contradicts that $p^\nu \in \rho(z^\nu)$ for $\nu \in \mathbb{N}$. ■

**Part 4: Fixed point**

Define $\Gamma : \underset{h \in \mathbf{H}}{\times} \bar{X}^h \times \Delta^{G-1} \rightrightarrows \underset{h \in \mathbf{H}}{\times} \bar{X}^h \times \Delta^{G-1}$ as the Cartesian product of $\left(d^h\right)_{h \in \mathbf{H}}$ and $\rho$. The set $\underset{h \in \mathbf{H}}{\times} X^h \times \Delta^{G-1}$ is compact, convex, and nonempty. As the correspondences $d^h$ (for any $h$) and $\rho$ are upper hemi-continuous, convex-valued, and well-defined, then so is the Cartesian product.

From Kakutani's Fixed Point Theorem, there exists a fixed point $\left(\left(x^h\right)_{h \in \mathbf{H}}, p^*\right) \in \Gamma\left(\left(x^h\right)_{h \in \mathbf{H}}, p^*\right)$.

**Part 5: Market clearing**

As $\left(u^h\right)_{h\in\mathbf{H}}$ are locally non-satiated, then $p^*\left(x^h - e^h\right) = 0\ \forall h \in \mathbf{H}$. Walras' Law is then given by $p^* \sum_{h\in\mathbf{H}} \left(x^h - e^h\right) = 0$. This together with the definition of the price correspondence implies $\sum_{h\in\mathbf{H}} \left(x_g^h - e_g^h\right) \leq 0\ \forall g \in \mathbf{G}$. Otherwise, if $\sum_{h\in\mathbf{H}} \left(x_{g'}^h - e_{g'}^h\right) > 0$ for some $g'$, then the maximum of the price correspondence is $p^* \sum_{h\in\mathbf{H}} \left(z^h - e^h\right) > 0$ by setting $p_{g'}^* = 1$.

If $\sum_{h\in\mathbf{H}} \left(x_g^h - e_g^h\right) < 0$, then $p_g^* = 0$ (see Walras' Law). For all households $h \neq h'$, define $x^{h*} = x^h$. For $h'$, define $x^{h'*} = x^{h'} - \sum_{h\in\mathbf{H}} \left(x^h - e^h\right)$. As $u^{h'}$ is non-decreasing (Assumption 4), then $\left(x^{h*}\right)_{h\in\mathbf{H}}$ are optimal solutions to the household problems $(HP)$. Further, market clearing is satisfied as:

$$\sum_{h\in\mathbf{H}} x^{h'*} = \sum_{h\neq h'} x^h + \left[x^{h'} - \sum_{h\in\mathbf{H}} \left(x^h - e^h\right)\right] = \sum_{h\in\mathbf{H}} e^h.$$

**Part 6: Innocuous to bound consumption**

I want to show that equilibrium consumption $x^h$ is an optimal solution to $(HP)$ iff $x^h$ is an optimal solution to:

$$(\overline{HP}) \qquad \begin{array}{ll} \text{maximize} & u^h\left(x^h\right) \\ \text{subject to} & x^h \in \bar{X}^h \\ & p\left(e^h - x^h\right) \geq 0 \end{array} .$$

To do this, I use the assumptions that $u^h$ is quasi-concave and locally non-satiated.

If $x^h$ is an optimal solution to $(HP)$, then the market clearing requirement implies that $x^h$ is an optimal solution to $(\overline{HP})$.

For the other direction, suppose that $x^h$ is an equilibrium consumption choice under $(\overline{HP})$, but $\exists y' \in X^h \backslash \bar{X}^h$ such that $p\left(e^h - y'\right) \geq 0$ and $u^h\left(y'\right) > u^h\left(x^h\right)$. The market clearing requirement implies $x^h \in int\bar{X}^h$ and $p\left(e^h - x^h\right) = 0$. By continuity, $\exists y \in X^h \backslash \bar{X}^h$ such that $p\left(e^h - y\right) > 0$ and $u^h(y) > u^h\left(x^h\right)$. As $u^h$ is locally non-satiated, then there exists $x^* \in int\bar{X}^h$ such that $u^h(x^*) > u^h\left(x^h\right)$. As $u^h$ is quasi-concave, then $\forall \lambda \in [0,1]$, $u^h\left(\lambda y + (1-\lambda) x^*\right) > u^h\left(x^h\right)$. Further, provided that $\lambda$ is small enough, $\lambda y + (1-\lambda) x^* \in \bar{X}^h$. As $y$ satisfies $p\left(e^h - y\right) > 0$, then $p\left(e^h - (\lambda y + (1-\lambda) x^*)\right) > 0$. This contradicts that $x^h$ is an equilibrium consumption choice under $(\overline{HP})$, as the allocation $\lambda y + (1-\lambda) x^*$ satisfies the constraints and provides strictly higher utility.

### 2.6.2 Proof of Theorem 2.2

Before proving the First Basic Welfare Theorem, I prove the two statements about how to find Pareto optimal allocations as optimal solutions to programming problems.

**Optimal solution to** $(PO_{h'})$

Suppose that $\left(x^{h*}\right)_{h\in\mathbf{H}}$ is not an optimal solution to the programming problem $(PO_{h'})$. Then $\exists\left(y^{h}\right)_{h\in\mathbf{H}}$ such that $u^{h'}\left(y^{h'}\right) > u^{h'}\left(x^{h'*}\right)$, $u^{h}\left(y^{h}\right) \geq u^{h}\left(x^{h*}\right)$ $\forall h \neq h'$, and $\left(y^{h}\right)_{h\in\mathbf{H}} \in FA$. As $\left(y^{h}\right)_{h\in\mathbf{H}}$ is feasible, then $\left(x^{h*}\right)_{h\in\mathbf{H}}$ is not Pareto optimal.

Suppose that $\left(x^{h*}\right)_{h\in\mathbf{H}}$ is not Pareto optimal. Then $\exists\left(y^{h}\right)_{h\in\mathbf{H}} \in FA$ such that $u^{h}\left(y^{h}\right) \geq u^{h}\left(x^{h*}\right)$ $\forall h \in \mathbf{H}$ and $u^{k}\left(y^{k}\right) > u^{k}\left(x^{k*}\right)$ for some $k$. There are two cases to consider: (i) $k = h'$ and (ii) $k \neq h'$. In case (i) $k = h'$, the allocation $\left(x^{h*}\right)_{h\in\mathbf{H}}$ would not be an optimal solution to $(PO_{h'})$ by definition. In case (ii) $k \neq h'$, as $u^{h'}$ is strictly increasing and $u^{k}$ is continuous, then for some $\epsilon > 0$, define a new allocation $\hat{y}^{h} = y^{h}$ $\forall h \notin \{h', k\}$, $\hat{y}^{h'} = y^{h'} + (\epsilon, ..., \epsilon)$, and $\hat{y}^{k} = y^{k} - (\epsilon, ..., \epsilon)$. Then $\left(\hat{y}^{h}\right)_{h\in\mathbf{H}}$ is such that $u^{h'}\left(\hat{y}^{h'}\right) > u^{h'}\left(y^{h'}\right) = u^{h'}\left(x^{h'*}\right)$, $u^{k}\left(\hat{y}^{k}\right) > u^{k}\left(x^{k*}\right)$, and $u^{h}\left(\hat{y}^{h}\right) \geq u^{h}\left(x^{h*}\right)$ $\forall h \notin \{h', k\}$. Thus, $\left(x^{h*}\right)_{h\in\mathbf{H}}$ is not an optimal solution to the programming problem $(PO_{h'})$.

**Optimal solution to** $(PO)$

Suppose that $\left(x^{h*}\right)_{h\in\mathbf{H}}$ is not Pareto optimal. Then $\exists\left(y^{h}\right)_{h\in\mathbf{H}} \in FA$ such that $u^{h}\left(y^{h}\right) \geq u^{h}\left(x^{h*}\right)$ $\forall h \in \mathbf{H}$ and $u^{h'}\left(y^{h'}\right) > u^{h'}\left(x^{h'*}\right)$ for some $k$. Therefore, for any $\left(\mu^{h}\right)_{h\in\mathbf{H}} >> 0$, $\sum_{h\in\mathbf{H}} \mu^{h} \cdot u^{h}\left(y^{h}\right) > \sum_{h\in\mathbf{H}} \mu^{h} \cdot u^{h}\left(x^{h*}\right)$. Thus, $\left(x^{h*}\right)_{h\in\mathbf{H}}$ is not an optimal solution to the programming problem $(PO)$.

Suppose that $\left(x^{h*}\right)_{h\in\mathbf{H}}$ is Pareto optimal. Define the set $U = \{u \in \mathbb{R}^{H} : u^{h} \leq u^{h}\left(x^{h}\right)$ $\forall h \in \mathbf{H}$, $\left(x^{h}\right)_{h\in\mathbf{H}} \in FA\}$. The point $u^{*} = \left(u^{1}\left(x^{1*}\right), ...., u^{H}\left(x^{H*}\right)\right) \in U$, but $u^{*} \notin intU$. The set $U$ is nonempty. Select any points $v, w \in U$, where $v^{h} \leq u^{h}\left(x^{h}\right)$ $\forall h \in \mathbf{H}$ and $w^{h} \leq u^{h}\left(y^{h}\right)$ $\forall h \in \mathbf{H}$ for $\left(x^{h}\right)_{h\in\mathbf{H}}, \left(y^{h}\right)_{h\in\mathbf{H}} \in FA$. Then as $u^{h}$ is concave $\forall h \in \mathbf{H}$, any convex combination $\theta v + (1-\theta) w \in U$ as $\theta\left(x^{h}\right)_{h\in\mathbf{H}} + (1-\theta)\left(y^{h}\right)_{h\in\mathbf{H}} \in FA$ and

$$u^{h}\left(\theta x^{h} + (1-\theta) y^{h}\right) \geq \theta u^{h}\left(x^{h}\right) + (1-\theta) u^{h}\left(y^{h}\right) \geq \theta v^{h} + (1-\theta) w^{h}.$$

Thus, the set $U$ is convex. From the Supporting Hyperplane Theorem (Corollary 1.1. in Section 1.1), there exists a $q \in \mathbb{R}^{H} \backslash \{0\}$ such that $q^{T} u^{*} \geq q^{T} u$ $\forall u \in U$. We need to show that there exists a $q > 0$ such that $q^{T} u^{*} \geq q^{T} u$ $\forall u \in U$. Suppose not, that is, $q^{h} < 0$ for some

$h$. Then, we can specify a new vector $u^{h'} = u^{h'}\left(x^{h'*}\right) \forall h' \neq h$ and $u^h < u^h\left(x^{h*}\right)$. This new vector is an element of $U$, yet $q^T u^* < q^T u$. Thus, $q > 0$ and we define $\left(\mu^h\right)_{h\in\mathbf{H}} = \frac{q}{\|q\|} \in \Delta^{H-1}$. So, there exists $\left(\mu^h\right)_{h\in\mathbf{H}} \in \Delta^{H-1}$ such that $\left(x^{h*}\right)_{h\in\mathbf{H}}$ is an optimal solution to $(PO)$.

**Proof of First Basic Welfare Theorem**

Suppose that $\left(\left(x^h\right)_{h\in\mathbf{H}}, p\right)$ is an Arrow-Debreu equilibrium, but $\left(x^h\right)_{h\in\mathbf{H}}$ is not a Pareto optimal allocation. Then there exists a feasible allocation $\left(y^h\right)_{h\in\mathbf{H}}$ such that $u^h\left(y^h\right) \geq u^h\left(x^h\right) \forall h \in \mathbf{H}$, with $u^{h'}\left(y^{h'}\right) > u^{h'}\left(x^{h'}\right)$ for some $h'$.

Then $p\left(e^h - y^h\right) \leq p\left(e^h - x^h\right) = 0 \quad \forall h \in \mathbf{H}$. Why? If $p\left(e^h - y^h\right) > p\left(e^h - x^h\right) = 0$, then as $u^h$ is locally non-satiated, there exists $\hat{y}^h$ such that $p\left(e^h - \hat{y}^h\right) > p\left(e^h - x^h\right) = 0$ and $u^h\left(\hat{y}^h\right) > u^h\left(x^h\right)$. This contradicts that $x^h$ is an optimal solution to $(HP)$.

Likewise, if $u^h\left(y^h\right) > u^h\left(x^h\right)$ for some $h$, then $p\left(e^h - y^h\right) < p\left(e^h - x^h\right) = 0$. Otherwise, $p\left(e^h - y^h\right) \geq p\left(e^h - x^h\right) = 0$ and $x^h$ is not an optimal solution to $(HP)$.

Summing the budget constraints over all households:

$$p\sum_{h\in\mathbf{H}}\left(e^h - y^h\right) < p\sum_{h\in\mathbf{H}}\left(e^h - x^h\right) = 0.$$

As $p > 0$, this implies that $\sum_{h\in\mathbf{H}}\left(e^h_g - y^h_g\right) < 0$ for some good $g$. This contradicts that $\left(y^h\right)_{h\in\mathbf{H}}$ is a feasible allocation.

### 2.6.3 Proof of Theorem 2.3

The proof proceeds in two steps: first an application of the Supporting Hyperplane Theorem (Corollary 1.1 in Section 1.1) and second an application of the Duality Theorem (Theorem 1.3 in Section 1.1).

**Part 1: Supporting Hyperplane Theorem**

Define the set

$$Z = \left\{ z \in \mathbb{R}^G : z \leq \sum_{h\in\mathbf{H}}\left(e^h - x^h\right) \quad \text{where } \left(x^h\right)_{h\in\mathbf{H}} \text{ is such that } \left\{ \begin{array}{l} x^h \in X^h \text{ and} \\ u^h\left(x^h\right) \geq u^h\left(x^{h*}\right) \end{array} \right\} \forall h \in \mathbf{H} \right\}.$$

The set $Z$ is convex. Why? Consider any $a, b \in Z$. Then $a \leq \sum_{h\in\mathbf{H}}\left(e^h - x^h\right)$ and $u^h\left(x^h\right) \geq u^h\left(x^{h*}\right) \ \forall h \in \mathbf{H}$ where $x^h \in X^h \ \forall h \in \mathbf{H}$, and $b \leq \sum_{h\in\mathbf{H}}\left(e^h - y^h\right)$ and $u^h\left(y^h\right) \geq u^h\left(x^{h*}\right) \ \forall h \in \mathbf{H}$ where $y^h \in X^h \ \forall h \in \mathbf{H}$. As $u^h$ is quasi-concave, then $\forall \theta \in [0,1]$,

$u^h\left(\theta x^h+(1-\theta)y^h\right) \geq u^h\left(x^{h*}\right)$ and $\theta a+(1-\theta)\,b \leq \sum_{h\in\mathbf{H}}\left(e^h-\left[\theta x^h+(1-\theta)y^h\right]\right)$. Further, $\theta x^h+(1-\theta)y^h \in X^h\ \forall h\in\mathbf{H}$. Thus, $\theta a+(1-\theta)\,b\in Z$.

Considering the allocation $\left(x^h\right)_{h\in\mathbf{H}}=\left(x^{h*}\right)_{h\in\mathbf{H}}$, we know that $0\in Z$, but $0\notin intZ$. To prove the latter, suppose that $0\in intZ$. Then there exists $z>>0$ such that $z\in Z$. The allocation $\left(x^h\right)_{h\in\mathbf{H}}$ such that $x^1=x^{1*}+z$ and $x^h=x^{h*}\ \forall h\geq 2$ is the allocation corresponding to $z$. This implies $u^h\left(x^h\right)\geq u^h\left(x^{h*}\right)\ \forall h\geq 2$ and $u^1\left(x^1\right)>u^1\left(x^{1*}\right)$ (Assumption S3). This contradicts that $\left(x^{h*}\right)_{h\in\mathbf{H}}$ is a Pareto optimal allocation.

Applying the Supporting Hyperplane Theorem, there exists $p^*\in\mathbb{R}^G\backslash\{0\}$ such that $0\geq p^*z\ \ \forall z\in Z$. Further, $p^*>0$. Why? Suppose not, that is $p_g<0$ for some $g$. Consider the element $z$ such that $z_g=-1$ and $z_{g'}=0\ \forall g'\neq g$. Obviously, $z\in Z$, yet $p^*z>0$. This contradiction verifies that $p^*>0$.

**Part 2: Duality Theorem**

The conclusion from Part 1 is that for any allocation such that $x^h\in X^h$ and $u^h\left(x^h\right)\geq u^h\left(x^{h*}\right)\ \forall h\in\mathbf{H}$, $0\geq p^*\sum_{h\in\mathbf{H}}\left(x^h-e^h\right)$. In particular, for any household $\bar{h}$, we can define an allocation $\left(y^h\right)_{h\in\mathbf{H}}=\left(x^{\bar{h}},\left(x^{h*}\right)_{h\neq h'}\right)$. If $x^{\bar{h}}\in X^{\bar{h}}$ and $u^{\bar{h}}\left(x^{\bar{h}}\right)\geq u^{\bar{h}}\left(x^{\bar{h}*}\right)$, it must be $0\geq p^*\left(x^{\bar{h}}-e^{\bar{h}}\right)$. As this holds for all households $\bar{h}\in\mathbf{H}$, then $x^{h*}$ is an optimal solution to the following Expenditure Minimization problem $\forall h\in\mathbf{H}$ :

$$(ExpMin)\qquad\qquad \begin{array}{ll} \text{minimize} & p^*x^h \\ \text{subject to} & x^h\in X^h \\ & u^h\left(x^h\right)-u^h\left(x^{h*}\right)\geq 0 \end{array}\ .$$

I wish to show that this implies $x^{h*}$ is an optimal solution to the Utility Maximization problem $(HP)\ \forall h\in\mathbf{H}$. Suppose not. Then for some household $h$, there exists $y^h$ such that $p^*\left(e^h-y^h\right)\geq 0$, $y^h\in X^h$, and $u^h\left(y^h\right)>u^h\left(x^{h*}\right)$. Consider the consumption $\beta x^{h*}$ for some $\beta<1$. As $x^{h*}\in intX^h$, then for $\beta$ close to 1, $\beta x^{h*}\in X^h$. Further, $p^*\left(e^h-\beta x^{h*}\right)>0$. Then, defining the convex combination $\theta y^h+(1-\theta)\left(\beta x^{h*}\right)$, $p^*\left(e^h-\left[\theta y^h+(1-\theta)\left(\beta x^{h*}\right)\right]\right)>0$ $\forall\theta\in(0,1)$. By definition, $p^*e^h=p^*x^{h*}$, implying

$$p^*\left(\theta y^h+(1-\theta)\left(\beta x^{h*}\right)\right)<p^*x^{h*}.$$

By the continuity of $u^h$, for some $\theta$ close to 1, $u^h\left(\theta y^h+(1-\theta)\left(\beta x^{h*}\right)\right)>u^h\left(x^{h*}\right)$. Thus, $x^{h*}$ is not an optimal solution to the problem $(ExpMin)$. This completes the contrapositive argument.

## 2.7 Exercises

1. Show that if $u^h$ is locally non-satiated $\forall h \in \mathbf{H}$, then the Arrow-Debreu equilibrium prices satisfy $p > 0$.

2. Assume that $u^h$ is differentiable, differentiably strictly increasing, and concave $\forall h \in \mathbf{H}$. Using the results from the programming problems in this chapter, show that if $\left(\left(x^h\right)_{h\in\mathbf{H}}, p\right)$ is an Arrow-Debreu equilibrium, then the allocation $\left(x^h\right)_{h\in\mathbf{H}}$ is an optimal solution to the problem $(PO_{h'})$ for any $h'$. From Section 2.3, the allocation $\left(x^h\right)_{h\in\mathbf{H}}$ is then Pareto optimal. As this was the first proof of the First Basic Welfare Theorem, it is often called the "classical proof."

3. Consider an economy with two households and two goods. Both utility functions are differentiable, strictly increasing, and strictly concave, but there is a missing market for the second good. That is, each household can consume no more than its initial endowment of the second good. Using an Edgeworth box, show that the equilibrium allocations are typically Pareto suboptimal.

4. Consider an economy with two households and two goods. The two households have utility functions

$$
\begin{aligned}
u^1\left(x^1\right) &= \frac{1}{2}\log\left(x_1^1 + x_1^2\right) + \frac{1}{2}\log\left(x_2^1\right) \\
u^2\left(x^2\right) &= \frac{1}{2}\log\left(x_1^2\right) + \frac{1}{2}\log\left(x_2^2\right)
\end{aligned}
$$

   and endowments $e^1 = (1,2)$ and $e^2 = (2,1)$. Notice that household 1 cares about household 2's consumption of the first good $(x_1^2)$. Using the programming problems from this chapter, characterize the Pareto optimal allocations and then characterize the equilibrium allocations. From this, can you claim that the First Basic Welfare Theorem holds? Use an Edgeworth box to illustrate your argument.

5. Consider an economy with two households and two goods. Prove that if the utility functions are Cobb-Douglas (of the form $u^h\left(x^h\right) = \alpha_1 \log\left(x_1^h\right) + \alpha_2 \log\left(x_2^h\right)$), then there is a unique Arrow-Debreu equilibrium. Does this property hold for an economy with more than two households and more than two goods?

6. Show that the projection $\pi$ is proper. $\pi$ is the projection $\pi : \mathbb{R}_{++}^{H(G+1)+G-1} \times \mathbb{R}_{++}^{HG} \to \mathbb{R}_{++}^{HG}$ which maps $\left( \left(x^h\right)_{h\in\mathbf{H}}, p, \left(e^h\right)_{h\in\mathbf{H}} \right) \mapsto \left(e^h\right)_{h\in\mathbf{H}}$ such that $\Phi\left( \left(x^h\right)_{h\in\mathbf{H}}, p; \left(e^h\right)_{h\in\mathbf{H}} \right) = 0$.

7. Prove that for any endowments $\left(e^h\right)_{h\in\mathbf{H}}$ within an open set of allocations around a Pareto optimal allocation, the resulting Arrow-Debreu equilibrium is unique. To do this, you must verify that the matrix $D_{\left(x^h,\lambda^h\right)_{h\in\mathbf{H}},p}\Phi\left( \left(x^h, \lambda^h\right)_{h\in\mathbf{H}}, p \right)$ has full rank (square matrix) given that the initial endowment is a Pareto optimal allocation $\left(e^h\right)_{h\in\mathbf{H}} = \left(x^{h*}\right)_{h\in\mathbf{H}}$. Notice that the derivative matrix only contains derivatives with respect to variables, not the endowment $e^1$.

8. Using "A Different Application of Differential Topology" from Exercise 3 in Chapter 1, prove that over a generic subset of endowments, if $G > 1$, then $p_1 \neq p_2$.

9. As a general result of the ideas in Exercise 3 above, show that over a generic subset of endowments, if there is a missing market for one of the goods, then the allocation $\left(x^h\right)_{h\in\mathbf{H}}$ is not Pareto optimal. To attack this problem, use "A Different Application of Differential Topology" from Exercise 3 in Chapter 1 and consider that a necessary condition for Pareto optimality is $\frac{D_g u^h\left(x^h\right)}{D_{g'} u^h\left(x^h\right)} = \frac{D_g u^{h'}\left(x^{h'}\right)}{D_{g'} u^{h'}\left(x^{h'}\right)}$ $\forall\left(h, h', g, g'\right) \in \mathbf{H} \times \mathbf{H} \times \mathbf{G} \times \mathbf{G}$.

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# Chapter 3

# General Financial Model

This chapter explores a pure-exchange general equilibrium model in a dynamic setting. For simplicity, there are only two time periods, with a finite number of states of uncertainty in the final period. Models with a longer, but still finite, time horizon can be reduced to the two-period model. Available to the households are financial markets that allow for the transfer of wealth between the states of uncertainty. This model in which households trade not only commodities, but also financial assets, is labeled the general financial model. We will introduce the concept of incomplete markets in this chapter. Given the importance of this topic, theorists have taken to calling this model the GEI model, where GEI stands for "general equilibrium with incomplete markets". The equilibrium concept, typically called a GEI equilibrium or Radner equilibrium (the latter in recognition of the work of Roy Radner), will be referred to as a general financial equilibrium in this manuscript.

In this chapter, I first introduce the financial model with numeraire assets. Then I discuss the equilibrium properties under two conditions: complete markets and incomplete markets. I next consider how the results differ when considering a more general class of assets called real assets. One large proof is relegated to Section 3.5.

## 3.1 The Model

The model is dynamic with two time periods, $t = 0$ and $t = 1$. In the final period, one of $S$ possible states of uncertainty is realized, where $S < \infty$ is finite. Thus, the total number of states, including the initial period, is $S + 1$. Denote the set of states as $\mathbf{S} = \{0, 1, ..., S\}$ with typical element $s$. The uncertainty tree is depicted in Figure 3.1.

There are a finite number of households. Denote the set of households as $\mathbf{H}= \{1, ..., H\}$

with typical element $h$. In each state, households trade and consumer $L$ commodities. Define the commodity space as $\mathbf{L} = \{1, ..., L\}$. The number of commodities is finite. By definition, one commodity in a state is a different good than the same commodity in a different state (an avocado may be ripe today, but spoiled tomorrow). Thus, the total number of goods is $G = L(S+1)$. The convention is that $x_l^h(s)$ is the consumption by household $h$ of good $(l, s)$, or the $l^{th}$ commodity in state $s$, $x^h(s) = \left(x_l^h(s)\right)_{l\in\mathbf{L}}$ is the vector of consumption by household $h$ of all commodities in state $s$, and $x^h = \left(x^h(s)\right)_{s\in\mathbf{S}}$ is the vector of consumption by household $h$ of all goods.

As with the static model, the household primitives are given below.

- $X^h = \mathbb{R}_+^G$ is the consumption set.
- $u^h : X^h \to \mathbb{R}$ is the utility function. We typically assume that the function is continuous, locally non-satiated, and quasi-concave.
- $e^h \in X^h$ is the endowment. We typically assume that $e^h >> 0$.

Each good $(l, s)$ has a market price $p_l(s)$. The vector of all prices is $p = (p_l(s))_{(l,s)\in\mathbf{L}\times\mathbf{S}} \in \mathbb{R}^G\backslash\{0\}$. The convention is that the price vector $p(s)$ is a row vector $\forall s \in \mathbf{S}$. I define the $(S+1) \times G$ price matrix $P = \begin{bmatrix} p(0) & 0 & 0 & 0 \\ 0 & p(1) & 0 & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & 0 & p(S) \end{bmatrix}$.

Now, we consider the financial elements of the model. There are $J$ financial assets that a household can choose to buy or sell in the initial period.[1] The number of assets is finite ($J < \infty$). Denote the set of assets as $\mathbf{J} = \{1, ..., J\}$, with typical element $j$. The decision to buy or sell an asset in the initial period is influenced by the price of that asset. Denote the price of an asset $j$ in the initial period as $q_j$. Denote all asset prices as $q = (q_j)_{j\in\mathbf{J}}$. These prices are variables in the equilibrium and will be determined to satisfy market clearing conditions.

In the state of uncertainty $s$ in the final period, the asset $j$ has a fixed payout $r_j(s)$. The payouts of all assets are in terms of the same physical commodity, defined as the numeraire commodity, in all states. Without loss of generality, we label the numeraire commodity as commodity $l = L$. We typically assume that for some household $h'$, the utility $u^{h'}$ is strictly

[1] Both buying and selling the same asset is redundant in this model without transaction costs and without default.

increasing in the commodity $l = L$ in all states $s \in \mathbf{S}$. This way, the commodity price of the numeraire commodity is $p_L(s) > 0 \;\; \forall s \in \mathbf{S}$. To remove the nominal indeterminacy, we make the price normalization $p_L(s) = 1 \;\; \forall s \in \mathbf{S}$.

Further, the assets are assumed to have nonnegative payouts $r_j(s) \geq 0 \;\; \forall (j,s) \in \mathbf{J} \times \mathbf{S} \backslash \{0\}$, where $r_j = (r_j(s))_{s>0} > 0 \;\; \forall j \in \mathbf{J}$ . Let's collect the payouts of all assets in all states $s > 0$ into one payout matrix $R$. This payout matrix has $S$ rows and $J$ columns:

$$R = \begin{bmatrix} r_1 & ... & r_J \end{bmatrix} = \begin{bmatrix} r_1(1) & ... & r_J(1) \\ : & ... & : \\ r_1(S) & ... & r_J(S) \end{bmatrix}.$$

The lone financial primitive is the matrix of asset payouts.

- $R \in \mathbb{R}_+^{S,J}$ is the payout matrix. We typically assume that $R$ is nonnegative and has full column rank (though the final assumption is innocuous with respect to the real variables as we will see in Section 3.1.4).

Given $R$ and $q$, households determine their positions on all assets (also called their portfolio). As with the static general equilibrium model, the trading of assets is perfectly competitive and anonymous. That is, all households can select any asset positions that they wish to. There are no investment constraints and households do not take into consideration market clearing conditions when making their optimal choice.

Household $h$ chooses portfolio $z^h \in \mathbb{R}^J$, where the position for a particular asset $j$ is denoted $z_j^h \in \mathbb{R}$. If $z_j^h < 0$, then the household has sold the asset (often called short-selling). In this case, the asset has negative payout in the states $s > 0 : r_j(s) z_j^h \leq 0$ (the household has borrowed). If $z_j^h > 0$, then the household has purchased the asset. In this case, the asset has positive payout in the states $s > 0 : r_j(s) z_j^h \geq 0$ (the household has saved). We typically assume that the economy is closed, so that the assets are in zero net supply. That simply means that the sum of all assets sold equals the sum of all assets purchased.

### 3.1.1 Existence

We are now prepared to define a general financial equilibrium, which is the equilibrium concept in this financial model.

**Definition 3.1** *A general financial equilibrium is* $\left( (x^h, z^h)_{h \in \mathbf{H}}, p, q \right)$ *such that*

1. $\forall h \in \mathbf{H}$, *given* $(p, q)$, $\left(x^h, z^h\right)$ *is an optimal solution to the household problem* $(HP)$

$$(HP) \qquad \begin{array}{ll} \text{maximize} & u^h\left(x^h\right) \\ \text{subject to} & x^h \in X^h \\ & z^h \in \mathbb{R}^J \\ & P\left(e^h - x^h\right) + \begin{pmatrix} -q \\ R \end{pmatrix} z^h \geq 0 \end{array} .$$

2. *Markets clear*

$$\begin{aligned} \sum\nolimits_{h \in \mathbf{H}} x_l^h(s) &= \sum\nolimits_{h \in \mathbf{H}} e_l^h(s) \quad \forall (l, s) \in \mathbf{L} \times \mathbf{S}. \\ \sum\nolimits_{h \in \mathbf{H}} z_j^h &= 0 \quad \forall j \in \mathbf{J}. \end{aligned}$$

It is important to verify, under roughly the same conditions as the static model, that a financial equilibrium exists. The assumptions sufficient for existence are given by:

Assumption E1: $X^h = \mathbb{R}_+^G \;\; \forall h \in \mathbf{H}$.

Assumption E2: $u^h : X^h \to \mathbb{R}$ is continuous, non-decreasing, strictly increasing in the commodity $l = L$ in all states $s \in \mathbf{S}$, and quasi-concave $\forall h \in \mathbf{H}$.

Assumption E3: $e^h >> 0 \; \forall h \in \mathbf{H}$.

Assumption E4 (No Redundancy): The payout matrix $R$ has full column rank (that is, the assets are linearly independent).

Notice that Assumption E4 implies that $J \leq S$.

**Theorem 3.1** *Under Assumptions E1-E4, a financial equilibrium* $\left(\left(x^h, z^h\right)_{h \in \mathbf{H}}, p, q\right)$ *exists.*

**Proof.** The arguments from the static model apply across the board. All that remains is to show that households' portfolios are bounded, so that households' budget sets are compact. Given that consumption is bounded (using the argument in Subsection 2.6.1, Part 1), the budget constraints imply that the portfolio payouts $Rz^h$ are bounded. Given Assumption E4, the portfolio $z^h$ must also be bounded. We discuss the validity of Assumption E4 in Section 3.1.4. In particular, we can show that it is innocuous to make this assumption. By innocuous, I mean that making this assumption does not affect the real equilibrium variables.
■

### 3.1.2 Regularity

The assumptions required to prove Regularity are:

Assumption R1: $X^h = \mathbb{R}^G_{++} \;\; \forall h \in \mathbf{H}$.

Assumption R2: $u^h : X^h \to \mathbb{R}$ is $C^2$, differentiably strictly increasing (meaning that $Du^h\left(x^h\right) >> 0$), differentiably strictly concave (meaning that $D^2u^h\left(x^h\right)$ is a negative definite matrix), and satisfies the boundary condition (meaning that $\forall x \in \mathbb{R}^G_{++}$, $cl\left\{y : u^h(y) \geq u^h(x)\right\} \subseteq \mathbb{R}^G_{++}$) $\forall h \in \mathbf{H}$.

Assumption R3: $e^h >> 0 \;\; \forall h \in \mathbf{H}$.

Assumption R4 (No Redundancy): The payout matrix $R$ has full column rank (that is, the assets are linearly independent).

**Theorem 3.2** *Under Assumptions R1-R4, over a generic subset of endowments, all financial equilibria satisfy Finite Local Uniqueness.*

The Lagrange multipliers for the budget constraints are the elements of the $(S+1)$ −dimensional row vector $\lambda^h$. Applying the Kuhn-Tucker Theorem, the following conditions are necessary and sufficient conditions for an optimal solution to $(HP)$ :

$$\begin{aligned} Du^h\left(x^h\right) - \lambda^h P &= 0, \\ P\left(e^h - x^h\right) + \begin{pmatrix} -q \\ R \end{pmatrix} z^h &= 0, \\ \lambda^h \begin{pmatrix} -q \\ R \end{pmatrix} &= 0, \end{aligned}$$

where the third set of equations contains the first order conditions with respect to the portfolio $z^h$. From this third set (knowing that $\lambda^h >> 0$ from the first order conditions with respect to consumption and $r_j > 0 \;\; \forall j \in \mathbf{J}$ by Assumption R4), $q >> 0$.

Define the system of equations $\Phi : \underset{h\in\mathbf{H}}{\times}\left(\mathbb{R}^{G+S+1}_{++}\times\mathbb{R}^{J}\right)\times\mathbb{R}^{G-(S+1)}_{++}\times\mathbb{R}^{J}_{++}\rightarrow\mathbb{R}^{n}$ as:

$$\Phi\left(\left(x^h,\lambda^h,z^h\right)_{h\in\mathbf{H}},p,q\right)=\left(\begin{array}{c}\left(\begin{array}{c}\left[Du^h\left(x^h\right)-\lambda^h P\right]^T\\ P\left(e^h-x^h\right)+\left(\begin{array}{c}-q\\ R\end{array}\right)z^h\\ \left[\lambda^h\left(\begin{array}{c}-q\\ R\end{array}\right)\right]^T\end{array}\right)_{h\in\mathbf{H}}\\ \sum_{h\in\mathbf{H}}\left(e^h_{\backslash G}-x^h_{\backslash G}\right)\\ \sum_{h\in\mathbf{H}}z^h\end{array}\right),$$

where $n=H\left(G+S+1+J\right)+G-(S+1)+J$. Allow me to verify that the equilibrium variables belong to open sets: $x^h\in X^h=\mathbb{R}^G_{++}$, $\lambda^h\in\mathbb{R}^{S+1}_{++}$, and $z^h\in\mathbb{R}^J\ \ \forall h\in\mathbf{H}$. Further, $p\in\mathbb{R}^{G-(S+1)}_{++}$ following the price normalization $p_L\left(s\right)=1\ \forall s\in\mathbf{S}$ and $q\in\mathbb{R}^J_{++}$. By definition, $\Phi\left(\left(x^h,\lambda^h,z^h\right)_{h\in\mathbf{H}},p,q\right)=0$ iff $\left(\left(x^h,\lambda^h,z^h\right)_{h\in\mathbf{H}},p,q\right)$ is a general financial equilibrium.

Recalling the proof method discussed in Section 1.4.5 (in Chapter 1), Finite Local Uniqueness is proven if we can show that (i) $\pi$ is proper and

$$\text{(ii) } rankD\Phi\left(\left(x^h,\lambda^h,z^h\right)_{h\in\mathbf{H}},p,q\right)=n. \tag{3.1}$$

The first condition is shown in the same fashion as Exercise 6 in Chapter 2 (the details are left to the intrepid reader). We will now walk through the second condition.

The derivative $D\Phi\left(\left(x^h,\lambda^h,z^h\right)_{h\in\mathbf{H}},p,q\right)$ means that we are taking derivatives with respect to $\left(x^h\right)_{h\in\mathbf{H}}$, $\left(\lambda^h\right)_{h\in\mathbf{H}}$, $\left(z^h\right)_{h\in\mathbf{H}}$, $p$, $q$, and $\left(e^h\right)_{h\in\mathbf{H}}$ (both variables and parameters). If we show that the rank condition holds by only taking derivatives with respect to $\left(x^h\right)_{h\in\mathbf{H}}$, $\left(\lambda^h\right)_{h\in\mathbf{H}}$, $\left(z^h\right)_{h\in\mathbf{H}}$, and $e^1$, then we have finished the argument.

The derivative matrix

$$M=D_{\left(x^h,\lambda^h,z^h\right)_{h\in\mathbf{H}},e^1}\Phi\left(\left(x^h,\lambda^h,z^h\right)_{h\in\mathbf{H}},p,q\right)$$

has $H\left(G+S+1+J\right)+G-(S+1)+J$ rows and $H\left(G+S+1+J\right)+G$ columns. The rows of $M$ will correspond to the equations in $\Phi$, while the columns will correspond to the variables (or parameters) that we are taking derivatives with respect to. For simplicity, define $\Psi = \begin{pmatrix} -q \\ R \end{pmatrix}$ and $\Lambda = \begin{bmatrix} \begin{pmatrix} I_{L-1} & 0 \end{pmatrix} & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & \begin{pmatrix} I_{L-1} & 0 \end{pmatrix} \end{bmatrix} \in \mathbb{R}^{G-(S+1),G}$. The derivative matrix is given by:

$$M = \begin{bmatrix} D^2u^1\left(x^1\right) & -P^T & 0 & & & & & & & 0 \\ -P & 0 & \Psi & 0 & & & & 0 & & P \\ 0 & \Psi^T & 0 & & & & & & & 0 \\ & & & .. & .. & .. & & & & 0 \\ & 0 & & .. & .. & .. & & 0 & & 0 \\ & & & .. & .. & .. & & & & 0 \\ & & & & & & D^2u^H\left(x^H\right) & -P^T & 0 & 0 \\ & 0 & & & 0 & & -P & 0 & \Psi & 0 \\ & & & & & & 0 & \Psi^T & 0 & 0 \\ -\Lambda & 0 & 0 & .. & .. & .. & -\Lambda & 0 & 0 & \Lambda \\ 0 & 0 & I_J & .. & .. & .. & 0 & 0 & I_J & 0 \end{bmatrix}.$$

To show that the matrix $M$ has full rank (there are more columns than rows, so we have to show that the matrix $M$ has full row rank), we set $\nu^T M = 0$ and must verify $\nu^T = 0$, where $\nu \in \mathbb{R}^n$ corresponds to equations in $\Phi$ :

$$\nu = \begin{pmatrix} \Delta x^1 \\ \Delta\lambda^1 \\ \Delta z^1 \\ : \\ \Delta x^h \\ \Delta\lambda^h \\ \Delta z^h \\ : \\ \Delta p \\ \Delta q \end{pmatrix} \begin{pmatrix} FOCx1 \\ BC1 \\ FOCz1 \\ : \\ FOCxh \\ BCh \\ FOCzh \\ : \\ MCx \\ MCz \end{pmatrix}.$$

The equations $\nu^T M = 0$ are given by:

$$\begin{array}{ll}
(\Delta x^1)^T D^2u^1(x^1) - (\Delta\lambda^1)^T P - \Delta p^T \Lambda = 0 & (3.2.a) \\
-(\Delta x^1)^T P^T + (\Delta z^1)^T \Psi^T = 0 & (3.2.b) \\
(\Delta\lambda^1)^T \Psi + \Delta q^T = 0 & (3.2.c) \\
\vdots & \\
(\Delta x^h)^T D^2u^h(x^h) - (\Delta\lambda^h)^T P - \Delta p^T \Lambda = 0 & (3.2.d) \\
-(\Delta x^h)^T P^T + (\Delta z^h)^T \Psi^T = 0 & (3.2.e) \\
(\Delta\lambda^h)^T \Psi + \Delta q^T = 0 & (3.2.f) \\
\vdots & \\
\Delta\lambda^1 P - \Delta p^T \Lambda = 0 & (3.2.g)
\end{array} \quad (3.2)$$

Let's show that $\nu^T = 0$ in four steps:

1. From $(3.2.g)$, by the definition of $\Lambda$, $\Delta\lambda^1 = 0$. Consequently, $\Delta p^T = 0$.

2. Postmultiply $(3.2.a)$ by $\Delta x^1$ :

$$(\Delta x^1)^T D^2u^1(x^1)\,\Delta x^1 = 0.$$

   As $u^1$ is differentiably strictly concave, the Hessian matrix $D^2u^1(x^1)$ is negative definite. Thus, $(\Delta x^1)^T = 0$.

3. From $(3.2.b)$, as $\Psi = \begin{pmatrix} -q \\ R \end{pmatrix}$ and $R$ has full column rank (Assumption R4), then $\Psi^T$ has full row ranking leading to the implication: $(\Delta z^1)^T \Psi^T = 0 \Longrightarrow (\Delta z^1)^T = 0$. From $(3.2.c)$, $\Delta q^T = 0$.

4. For any household $h > 1$, postmultiply $(3.2.d)$ by $\Delta x^h$ :

$$(\Delta x^h)^T D^2u^h(x^h)\,\Delta x^h - (\Delta\lambda^h)^T P\Delta x^h = 0.$$

   From $(3.2.e)$, after postmultiplying by $\Delta\lambda^h$ and then taking the transpose:

$$(\Delta\lambda^h)^T P\Delta x^h = (\Delta\lambda^h)^T \Psi\Delta z^h.$$

   From $(3.2.f)$, $(\Delta\lambda^h)^T \Psi = 0$. Thus, $(\Delta x^h)^T D^2u^h(x^h)\,\Delta x^h = 0$. As $u^h$ is differentiably strictly concave, the Hessian matrix $D^2u^h(x^h)$ is negative definite. Thus, $(\Delta x^h)^T = 0$.

From (3.2.$d$), $\Delta\lambda^h = 0$. From (3.2.$e$), as $R$ has full column rank (Assumption R4), then $\left(\Delta z^h\right)^T \Psi^T = 0$ implies $\left(\Delta z^h\right)^T = 0$.

Thus, $\nu^T = \left(\left((\Delta x^1)^T, \left(\Delta\lambda^1\right)^T, (\Delta z^1)^T\right), ..., \left(\left(\Delta x^h\right)^T, \left(\Delta\lambda^h\right)^T, \left(\Delta z^h\right)^T\right), ..., \Delta p^T, \Delta q^T\right) =$ 0, finishing the argument.

### 3.1.3 No Arbitrage

Given $R$ and $q$, an arbitrage opportunity exists if $\exists z \in \mathbb{R}^J$ such that $\begin{pmatrix} -q \\ R \end{pmatrix} z > 0$. This is an arbitrage opportunity, because the household could hold the portfolio $\kappa z$ as $\kappa \to \infty$, and provide itself with unbounded wealth in some states (without having unbounded debt in other states).

Thus, the No Arbitrage condition is given by:

No Arbitrage: There does not exist $z \in \mathbb{R}^J$ such that $\begin{pmatrix} -q \\ R \end{pmatrix} z > 0.$

This condition is quite intuitive, but difficult to work with mathematically. What we would like to do is prove that the No Arbitrage condition is equivalent to the following condition involving state prices $\hat{\alpha} \in \mathbb{R}^{S+1}_{++}$ :

No Arbitrage II: There exists $\hat{\alpha} \in \mathbb{R}^{S+1}_{++}$ so that $\hat{\alpha} \begin{pmatrix} -q \\ R \end{pmatrix} = 0.$

State prices are 'price-like' variables, so the convention is that they are written as row vectors (similar to prices $p$ and $q$). As mentioned above, we believe this No Arbitrage II condition is equivalent to No Arbitrage I. Let's verify this.

1. No Arbitrage II implies No Arbitrage I

   Suppose that No Arbitrage I does not hold. Then $\exists z \in \mathbb{R}^J$ so that $\begin{pmatrix} -q \\ R \end{pmatrix} z > 0$.

   This implies that for any $\hat{\alpha} \in \mathbb{R}^{S+1}_{++}$, $\hat{\alpha} \begin{pmatrix} -q \\ R \end{pmatrix} z > 0$. This is a contradiction, as No Arbitrage II requires that for some $\hat{\alpha} \in \mathbb{R}^{S+1}_{++}$, $\hat{\alpha} \begin{pmatrix} -q \\ R \end{pmatrix} z = 0$.

2. No Arbitrage I implies No Arbitrage II

   Let us recall something that we verified in Chapter 1 (specifically, Exercise 1):

   Let $\begin{pmatrix} -q \\ R \end{pmatrix}$ be a $(S+1) \times J$ matrix and $z \in \mathbb{R}^J$. The Farkas Lemma can be used to prove the following implication.

$$\begin{array}{rcl} \text{There does not exist } z \text{ such that } \begin{pmatrix} -q \\ R \end{pmatrix} z & > & 0 \\ & \Downarrow & \\ \exists \hat{\alpha} & \in & \mathbb{R}^{S+1}_{++} \text{ s.t. } \hat{\alpha} \begin{pmatrix} -q \\ R \end{pmatrix} = 0. \end{array}$$

   This tells us exactly what we need to finish the argument.

You are asked to verify in Exercise 1 that No Arbitrage is a necessary condition of equilibrium.

Let's use the No Arbitrage II condition to see whether asset prices can be "No Arbitrage prices" or not. The method that I am set to introduce was developed by Thorsten Hens, and is henceforth known as the Hens method. The method is valid for economies with either $J = 2$ or $J = 3$ assets.

**Hens method: Two assets**

Consider an economy with $S \geq 2$ states of uncertainty and $J = 2$ assets. The payout matrix $R$ is given by:

$$R = \begin{bmatrix} r_1(1) & r_2(1) \\ : & : \\ r_1(S) & r_2(S) \end{bmatrix}.$$

From No Arbitrage II, the asset prices are given by $q = \alpha R$ for some $\alpha \in \mathbb{R}^S_{++}$. To see this, take the No Arbitrage II definition, which says that $\hat{\alpha} \begin{pmatrix} -q \\ R \end{pmatrix} = 0$ for some $\hat{\alpha} >> 0$. Then defining $\alpha = \left( \frac{\hat{\alpha}(1)}{\hat{\alpha}(0)}, \ldots, \frac{\hat{\alpha}(S)}{\hat{\alpha}(0)} \right)$, we obtain $-q + \alpha R = 0$. The Hens method proceeds in two steps.

1. Plot the pairs $(r_1(s), r_2(s))$ in $\mathbb{R}^2_+$ for all $s > 0$. As you can see in Figure 3.2, I

have plotted the pairs corresponding to the payout matrix $R = \begin{bmatrix} 0 & 1 \\ 4 & 2 \\ 3 & 3 \end{bmatrix}$ (the x-axis corresponds to $j = 1$ and the y-axis to $j = 2$).

2. The convex cone of this set $\{(r_1(1), r_2(1)), ..., (r_1(S), r_2(S))\}$ can then be found. In Figure 3.2, the convex cone is the shaded area.

3. If the asset prices $q = (q_1, q_2)$ belong to the interior of this convex cone, then they are No Arbitrage prices. Otherwise, they are not No Arbitrage prices.

It is important to notice that Step 3 requires the asset prices to lie in the **interior** of the cone.

The No Arbitrage condition is a necessary condition for a general financial equilibrium (Exercise 1). Consequently, any equilibrium asset prices must be No Arbitrage prices (the converse is not true). Exercise 2 provides practice with the Hens method with two assets.

### Hens method: Three assets

Consider an economy with $S \geq 3$ states of uncertainty and $J = 3$ assets. To apply the Hens method, we must take an initial step involving linear operations on the payout matrix $R$.

1. Given the payout matrix $R$, we need to perform appropriate linear operations on the column space of $\begin{pmatrix} -q \\ R \end{pmatrix}$ so that the equivalent matrix $R^*$ has one asset (wlog asset $j = 1$) with constant payouts of 1 in every state $s > 0$.

The equivalent asset price vector and payout matrix following the linear operations are $\begin{pmatrix} -q^* \\ R^* \end{pmatrix}$, where

$$R^* = \begin{bmatrix} 1 & r_2^*(1) & r_3^*(1) \\ 1 & : & : \\ 1 & r_2^*(S) & r_3^*(S) \end{bmatrix}.$$

As an example, suppose that $R = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 4 & 2 \\ 4 & 3 & 3 \end{bmatrix}$. Then the linear operation that I will perform will be $C1 \mapsto C1 - C3$, meaning that I replace the 1st column with the difference (1st column) - (3rd column). This results in the equivalent asset price vector and payout matrix:

$$\begin{pmatrix} -q^* \\ R^* \end{pmatrix} = \begin{bmatrix} -(q_1 - q_3) & -q_2 & -q_3 \\ 1 & 0 & 1 \\ 1 & 4 & 2 \\ 1 & 3 & 3 \end{bmatrix}$$

From No Arbitrage, the asset prices are given by $q^* = \alpha R^*$ for some $\alpha \in \mathbb{R}^S_{++}$. This means that $q_1^* = \alpha \cdot \overrightarrow{1}$ or $q_1^* = \sum_{s>0} \alpha(s)$. The Hens method continues with the following steps.

2. Plot the pairs $(r_2^*(s), r_3^*(s))$ in $\mathbb{R}^2_+$ for all $s > 0$. As you can see in Figure 3.3, I have plotted the pairs corresponding to the payout matrix $R^* = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 4 & 2 \\ 1 & 3 & 3 \end{bmatrix}$.

3. The convex hull of this set $\{(r_2^*(1), r_3^*(1)), ..., (r_2^*(S), r_3^*(S))\}$ can then be found. In Figure 3.3, the convex hull is the shaded area inside the triangle.

4. If the asset prices $\left(\frac{q_2^*}{q_1^*}, \frac{q_3^*}{q_1^*}\right)$ belong to the interior of this convex hull, then they are No Arbitrage prices. Otherwise, they are not No Arbitrage prices. Continuing with my example, as $q_1^* = q_1 - q_3$, the asset price pairs that I will actually be plotting are $\left(\frac{q_2}{q_1 - q_3}, \frac{q_3}{q_1 - q_3}\right)$. Obviously, any prices such that $q_1 - q_3 \leq 0$ automatically fail the No Arbitrage requirement.

Notice two elements of Step 4. First, we require that the asset prices are all divided by $q_1^*$. Second, we again require that the asset prices lie in the **interior** of the convex hull.

Exercise 3 provides practice with the Hens method and three assets.

### 3.1.4 No Redundancy

The No Redundancy condition assumes that the payout matrix $R$ has full column rank (that is, the assets are linearly independent). You are asked to verify in Exercise 4 that No Redundancy is an innocuous assumption (in terms of the real variables) to make.

## 3.2 Complete Markets

Throughout this section, we assume that the No Arbitrage condition and the No Redundancy condition both hold. The assumption of "Complete Markets" states that $J = S$ (equal number of assets and states of uncertainty). The payout matrix $R$ is now a full rank square matrix (hence, invertible). Let's see what this gets us.

Examples of payout matrices with complete markets are given by:

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

I define the column span of any payout matrix as

$$\langle R \rangle = \left\{ y \in \mathbb{R}^S : y = Rz \text{ for some } z \in \mathbb{R}^J \right\}. \tag{3.3}$$

For instance, under the assumption of complete markets, $\langle R \rangle = \mathbb{R}^S$.

Given complete markets and the equilibrium asset prices $q \in \mathbb{R}^J_{++}$, there exists a unique vector of state prices. Recall the No Arbitrage pricing conditions:

$$q = \alpha R,$$

where $\alpha = \left( \frac{\hat{\alpha}(1)}{\hat{\alpha}(0)}, ..., \frac{\hat{\alpha}(S)}{\hat{\alpha}(0)} \right)$. Given that $R$ has full rank, then the state prices are uniquely defined as $\alpha = qR^{-1}$.

Suppose that the market clearing conditions hold for all commodities:

$$\sum\nolimits_{h \in \mathbf{H}} \left( e^h - x^h \right) = 0.$$

Walras' Law states that $P \sum\nolimits_{h \in \mathbf{H}} \left( e^h - x^h \right) + \begin{pmatrix} -q \\ R \end{pmatrix} \sum\nolimits_{h \in \mathbf{H}} z^h = 0$ (sum of budget constraints equals zero). Given commodity market clearing and full column rank $R$, then $\sum\nolimits_{h \in \mathbf{H}} z^h = 0$. Thus, the market clearing conditions for assets are guaranteed to hold when the market clearing conditions for commodities hold.

The following theorem (called Arrow's equivalency theorem) proves that a general financial equilibrium with complete markets is allocation-equivalent to an Arrow-Debreu equilibrium.

**Theorem 3.3** *Arrow's Equivalency Theorem*

*Under Assumptions E1-E4, if* $J = S$ *and* $\left(\left(x^h, z^h\right)_{h\in\mathbf{H}}, p, q\right)$ *is a general financial equilibrium, then* $\left(\left(x^h\right)_{h\in\mathbf{H}}, \rho\right)$ *is an Arrow-Debreu equilibrium where* $\rho = (1, qR^{-1}) \cdot P$.

*Under Assumptions E1-E3, if* $\left(\left(x^h\right)_{h\in\mathbf{H}}, \rho\right)$ *is an Arrow-Debreu equilibrium, then for any full rank payout matrix* $R$, $\exists \left(z^h\right)_{h\in\mathbf{H}}$ *such that for* $\alpha(s) = \frac{\rho_L(s)}{\rho_L(0)}$ $\forall s > 0$ *and* $q = \alpha R$, *the vector* $\left(\left(x^h, z^h\right)_{h\in\mathbf{H}}, p, q\right)$ *is a general financial equilibrium.*

**Proof. General financial $\Longrightarrow$ Arrow-Debreu**

As the markets are complete, $\exists!$ state prices $\alpha = qR^{-1}$ as discussed above. Let $1 \in \mathbb{R}^L$ be the $L-$dimensional row vector with 1 for each element. This will be the state price for $s = 0$. Then the state prices for all $s \in \mathcal{S}$ are $(1, \alpha)$. Recall that the price matrix $P = \begin{bmatrix} p(0) & 0 & 0 & 0 \\ 0 & p(1) & 0 & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & 0 & p(S) \end{bmatrix}$ is a $(S+1) \times G$ matrix. Premultiply $P$ by the state prices to obtain the Arrow-Debreu prices $\rho \in \mathbb{R}^G_+$ :

$$\rho = (1, \alpha) \cdot P.$$

In particular, $\rho_l(0) = p_l(0)$ $\forall l \in \mathbf{L}$ and $\rho_l(s) = \alpha(s) \cdot p_l(s)$ $\forall (l, s) \in \mathbf{L} \times \mathbf{S} \backslash \{0\}$.

The proof requires two steps. First, let $\left(x^h, z^h\right)$ be budget feasible under the general financial equilibrium budget constraints:

$$\left(x^h, z^h\right) \in B^h(p, q) = \left\{(x, z) : P\left(e^h - x\right) + \begin{pmatrix} -q \\ R \end{pmatrix} z \geq 0\right\}.$$

Premultiply the budget constraints by the state prices $(1, \alpha)$. Then the budget constraints reduce to

$$(1, \alpha) P\left(e^h - x\right) + (1, \alpha) \begin{pmatrix} -q \\ R \end{pmatrix} z \geq 0.$$

From No Arbitrage II, $(1, \alpha) \begin{pmatrix} -q \\ R \end{pmatrix} = 0$. Thus, we have $\rho\left(e^h - x\right) \geq 0$. So $x^h \in b^h(\rho) = \left\{x : \rho\left(e^h - x\right) \geq 0\right\}$.

For the second step, suppose that $y^h$ is budget feasible under the Arrow-Debreu equilibrium budget set $b^h(\rho)$. Define $\alpha(s) = \frac{\rho_L(s)}{\rho_L(0)}$ $\forall s > 0$ and $p_l(s) = \frac{\rho_l(s)}{\rho_L(s)}$ $\forall (l, s) \in \mathbf{L} \times \mathbf{S}$. For any full rank $R$, define $q = \alpha R$. There exists

$$z^h = R^{-1} \begin{pmatrix} p(1)\left(e^h(1) - x^h(1)\right) \\ \vdots \\ p(S)\left(e^h(S) - x^h(S)\right) \end{pmatrix}.$$

By construction, $\left(x^h, z^h\right) \in B^h(p,q)$ for the selected full rank payout matrix $R$.

To conclude, we pose the question: if $\left(x^h, z^h\right) \in \underset{(x,z)\in B^h(p,q)}{\arg\max}\ u^h(x)$, then since $x^h \in b^h(\rho)$, is it also true that $x^h \in \underset{x\in b^h(\rho)}{\arg\max}\ u^h(x)$? Suppose not, that is, $\exists y \in b^h(\rho)$ such that $u^h(y) > u^h\left(x^h\right)$. If $y \in b^h(\rho)$, then $\exists \hat{z} \in \mathbb{R}^J$ such that $(y, \hat{z}) \in B^h(p,q)$. As $u^h(y) > u^h\left(x^h\right)$ and $(y,\hat{z}) \in B^h(p,q)$, then this contradicts that $\left(x^h, z^h\right) \in \underset{(x,z)\in B^h(p,q)}{\arg\max}\ u^h(x)$.

**Arrow-Debreu $\Longrightarrow$ General financial**

Choose any full rank payout matrix $R$. The proof requires three steps.

If $x^h$ is budget feasible under the Arrow-Debreu equilibrium budget set $b^h(\rho)$, then using arguments from above, $\exists z^h$ such that for (i) $\alpha(s) = \frac{\rho_L(s)}{\rho_L(0)}$ $\forall s > 0$, (ii) $p_l(s) = \frac{\rho_l(s)}{\rho_L(s)}$ $\forall (l,s) \in \mathbf{L} \times \mathbf{S}$, and (iii) $q = \alpha R$, $\left(x^h, z^h\right) \in B^h(p,q)$.

If $x^h \in \underset{x\in b^h(\rho)}{\arg\max}\ u^h(x)$, then since $\exists z^h$ such that $\left(x^h, z^h\right) \in B^h(p,q)$, is it also true that $\left(x^h, z^h\right) \in \underset{(x,z)\in B^h(p,q)}{\arg\max}\ u^h(x)$? Suppose not, that is, $\exists\, (y,\hat{z}) \in B^h(p,q)$ such that $u^h(y) > u^h\left(x^h\right)$. If $(y,\hat{z}) \in B^h(p,q)$, then $y \in b^h(\rho)$ (citing arguments from above). But this contradicts that $x^h \in \underset{x\in b^h(\rho)}{\arg\max}\ u^h(x)$.

For the third step, we use the previously stated fact that the market clearing conditions for assets are redundant given that the market clearing conditions for commodities hold. ■

Given Arrow's Equivalency Theorem, when the complete markets assumption holds, the First Basic Welfare Theorem is applicable. That is, if $J = S$, all general financial equilibrium allocations are Pareto optimal.

## 3.3 Incomplete Markets

Throughout this section, we assume that the No Arbitrage condition and the No Redundancy condition both hold. "Incomplete Markets" means that $J < S$ (fewer assets than states of uncertainty). The payout matrix $R$ is no longer invertible. How does this change things?

Consider any two payout matrices $R'$, $R''$ with the same column span: $\langle R' \rangle = \langle R'' \rangle$ (recall the definition of the span in (3.3)). These two matrices are then equivalent up to a change of basis. For our purposes in the general financial model, if a household $h$ has portfolio payouts $R'z^h$, then there exists $\hat{z}^h \in \mathbb{R}^J$ such that $R'z^h = R''\hat{z}^h$. Thus, given any payout matrix with full column rank $R'$ (No Redundancy condition holds), we consider the equivalent payout matrix $R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$ such that $R_1 \in \mathbb{R}^{J,J}$ has full rank.

With incomplete markets, given the equilibrium asset prices $q \in \mathbb{R}^J_{++}$, there no longer exists a unique vector of state prices. In fact, the set of state prices $\alpha : q = \alpha R$ is an $(S - J)-$dimensional space.

### 3.3.1 Pareto suboptimal

Consider the characterization of Pareto optimal allocations using either the problem $(PO_{h'})$ or the problem $(PO)$ from Chapter 2. A necessary condition for a Pareto optimal allocation is:

$$\forall h \in \mathbf{H}, \quad \lambda^1 = \kappa^h \lambda^h \text{ for some scalar } \kappa^h > 0.$$

Recall that $\lambda^h$ is the vector of Lagrange multipliers corresponding to the budget constraints for household $h$.

Let's add the constraint $\frac{\lambda^1(1)}{\lambda^1(0)} = \frac{\lambda^2(1)}{\lambda^2(0)}$ to the system of equations $\Phi$. Now the number of equations is one greater than the number of unknowns. Let's apply "A Different Application of Differential Topology" from Exercise 3 in Chapter 1 (similar to Exercises 8 and 9 in Chapter 2) to show that over a generic subset of endowments, the equation $\frac{\lambda^1(1)}{\lambda^1(0)} = \frac{\lambda^2(1)}{\lambda^2(0)}$ cannot hold. If the equation cannot hold, as it is a necessary condition for a Pareto optimal allocation, then the general financial equilibrium allocations are Pareto suboptimal (over a generic subset of endowments).

**Theorem 3.4** *Under Assumption R1-R4, if $J < S$, then over a generic subset of endowments, the general financial equilibrium allocations are Pareto suboptimal.*

**Proof.** Exercise 5. ■

Let's consider an example that illustrates why Theorem 3.4 only holds over a generic subset of household endowments (rather than over the entire set of household endowments). In the following example, we have incomplete markets and an equilibrium allocation that is Pareto optimal.

Let's introduce some machinery that will be useful in the example. Define the $S \times G$ price matrix (for states $s > 0$) as $P_{\backslash 0} = \begin{bmatrix} 0 & p(1) & 0 & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & 0 & p(S) \end{bmatrix}$ (this is the matrix $P$ with the first row removed). Define $\alpha \in \mathbb{R}^S_{++}$ an any vector of state prices satisfying $q = \alpha R$. The budget constraints in a general financial equilibrium are given by $P\left(e^h - x^h\right) + \begin{pmatrix} -q \\ R \end{pmatrix} z^h \geq 0$. This is equivalent to the following two conditions:

1. Single budget constraint
$$(1, \alpha) P\left(e^h - x^h\right) \geq 0.$$

2. Span condition
$$P_{\backslash 0}\left(x^h - e^h\right) \in \langle R \rangle .$$

The single budget constraint is satisfied by both Pareto optimal and Pareto suboptimal allocations. The key condition to determine Pareto optimality is the span condition.

**Example 3.1** *Let $L = 1$. All households have identical utility $u^h\left(x^h\right) = -\sum_{s \in \mathbf{S}} \left(x^h(s)\right)^{-\gamma}$ for some $\gamma > 0$. All households have endowments such that $\left(e^h(s)\right)_{s>0} \in \langle R \rangle$.*

*Since the utility is homothetic and identical across households, then there is a unique general financial equilibrium. Let's first find the Arrow-Debreu equilibrium and then show that this is also the general financial equilibrium. Proceeding in this manner, we will obtain the Pareto optimality as a byproduct.*

*You are asked to verify in Exercise 6 that the Arrow-Debreu equilibrium allocation is such that:*
$$x^h(s) = \theta^h \sum_{h \in \mathbf{H}} e^h(s) \quad \forall h \in \mathbf{H}$$
*for some household-specific fractions $\theta^h$.*

*Given that $L = 1$, the span condition is equivalent to $\left(x^h(s) - e^h(s)\right)_{s>0} \in \langle R \rangle$. Since $\left(e^h(s)\right)_{s>0} \in \langle R \rangle$ $\forall h \in \mathbf{H}$ and the space $\langle R \rangle$ is linear, the span condition reduces to $\left(x^h(s)\right)_{s>0} \in \langle R \rangle$ $\forall h \in \mathbf{H}$.*

*Do the household choices (in the Arrow-Debreu equilibrium) satisfy the span condition? Well, since $\left(e^h(s)\right)_{s>0} \in \langle R \rangle$ $\forall h \in \mathbf{H}$ (by assumption), then $\left(\sum_{h \in \mathbf{H}} e^h(s)\right)_{s>0} \in \langle R \rangle$. As $x^h(s) = \theta^h \sum_{h \in \mathbf{H}} e^h(s)$ and the space $\langle R \rangle$ is linear, then in fact $\left(x^h(s)\right)_{s>0} \in \langle R \rangle$ $\forall h \in \mathbf{H}$. Thus, the Arrow-Debreu equilibrium allocation satisfies the span condition and therefore* *satisfies the general financial equilibrium budget constraints. Consequently, the Arrow-Debreu equilibrium is also a general financial equilibrium.*

So we have a setting with incomplete markets (the example is valid for any $J < S$) and yet, the general financial equilibrium allocation is Pareto optimal. Why? The endowments $\left(e^h(s)\right)_{s>0} \in \langle R \rangle\ \forall h \in \mathbf{H}$. Whenever $J < S$, the space $\langle R \rangle$ is a measure zero subset of $\mathbb{R}^S$. Thus, the endowments in this example do **not** belong to a generic subset. We can see now why the result in Theorem 3.4 holds only over a generic subset of endowments.

### 3.3.2 Constrained Pareto suboptimal

With incomplete markets, it seems unfair to compare the general financial equilibrium allocations with Pareto optimal allocations. The set of Pareto optimal allocations implicitly allows the planner to make transfers between all households and all states of uncertainty. This is not possible in a general financial equilibrium. In a general financial equilibrium, the possible allocations are restricted by the fixed asset structure (transfers can only be made according to the payout matrix $R$). In this section, we introduce the concept of constrained Pareto optimality, where constrained Pareto optimal allocations are those that respect the fixed asset structure. We now have a fair comparison: general financial equilibrium allocations vs. constrained Pareto optimal allocations.

**Definition 3.2** *The allocation $\left(x^h\right)_{h\in\mathbf{H}}$ is constrained Pareto suboptimal if there exists equilibrium prices $\tilde{p}$ and a feasible allocation $\left(\tilde{x}^h\right)_{h\in\mathbf{H}} \in FA$ such that $\forall h \in \mathbf{H}$,*

$$\tilde{x}^h \in \underset{x:\tilde{P}_{\backslash 0}\left(x^h-e^h\right)\in\langle R\rangle}{\arg\max}\ u^h(x)$$

*and $\left(u^h\left(\tilde{x}^h\right)\right)_{h\in\mathbf{H}} > \left(u^h\left(x^h\right)\right)_{h\in\mathbf{H}}$.*[2]

We will see shortly that this definition is equivalent to a planner fixing the asset choices of households and making a transfer to each household in the initial state. The planner is choosing $\left(\tau^h\right)_{h\in\mathbf{H}}$ for $\tau^h \in \mathbb{R}^{J+1}$ such that $\sum_{h\in\mathbf{H}} \tau^h = 0$, where $\tau^h(1)$ is the transfer (in units of account) in state $s = 0$ and $\tau^h(j+1)$ for $j \in J$ is the holding of asset $j$. Given these

[2] Equilibrium prices $\tilde{p}$ simply means that $\sum_{h\in\mathbf{H}}\left(\tilde{x}^h - e^h\right) = 0$, which is a requirement of the set $FA$.

fixed transfers by the planner, the new budget constraints for each household become:

$$P\left(e^h - x^h\right) + \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \tau^h \geq 0.$$

The main result that we show in this section is that the general financial equilibrium allocations (under some conditions) are not constrained Pareto optimal. To motivate this, we first consider two examples. The examples dictate what conditions are required to obtain this main result.

In the first example, we consider any economy with only one physical commodity traded in each state $(L = 1)$.

**Example 3.2** *Consider any economy with* $L = 1$*. Suppose that the general financial equilibrium allocation* $\left(x^h\right)_{h\in\mathbf{H}}$ *is constrained Pareto suboptimal. Then there exists* $\left(\tilde{x}^h\right)_{h\in\mathbf{H}}$ *such that:*

1. $\tilde{x}^h \in \underset{x:\left(x^h(s)-e^h(s)\right)_{s>0}\in\langle R\rangle}{\arg\max}\ u^h(x) \quad \forall h \in \mathbf{H}$,
2. $\sum_{h\in\mathbf{H}} \left(\tilde{x}^h - e^h\right) = 0$, *and*
3. $\left(u^h\left(\tilde{x}^h\right)\right)_{h\in\mathbf{H}} > \left(u^h\left(x^h\right)\right)_{h\in\mathbf{H}}$.

*But then* $\left(x^h\right)_{h\in\mathbf{H}}$ *cannot be optimal choices for all households (as* $\tilde{x}^h$ *is chosen under the exact same budget constraints and some households strictly prefer* $\tilde{x}^h$*). Thus,* $\left(x^h\right)_{h\in\mathbf{H}}$ *cannot be a general financial equilibrium allocation. This finishes the argument that all general financial equilibrium allocations (when* $L = 1$*) are constrained Pareto optimal.*

Thus, we know that the main result (general financial equilibrium allocations are constrained Pareto suboptimal) requires $L > 1$.

In the second example, we consider an economy with $L > 1$ and a special form of utility (namely, identical and homothetic [specifically, Cobb-Douglas] utility).

**Example 3.3** *Suppose that the utility functions are* $u^h\left(x^h\right) = \sum_{s\in\mathbf{S}}\left(\sum_{l\in\mathbf{L}} \theta_l(s) \cdot \log\left(x_l^h(s)\right)\right)$ $\forall h \in \mathbf{H}$*. The Cobb-Douglas utility has some nice properties. For each household, the consumption is given by:*

$$x_l^h(s) = \alpha^h(s) \cdot \sum_{h\in\mathbf{H}} e_l^h(s)$$

*where $\alpha^h(s) \in (0,1)$ is determined according to a household's endowments and $\left(\alpha^h(s)\right)_{h\in\mathbf{H}} \in \Delta^{H-1}$ $\forall s \in \mathbf{S}$. The prices are given such that $p_l(s) = \frac{\theta_l(s)}{\theta_L(s)} \cdot \frac{\sum_{h\in\mathbf{H}} e^h_L(s)}{\sum_{h\in\mathbf{H}} e^h_l(s)}$ $\forall (l,s) \in \mathbf{L}\times\mathbf{S}$. You are asked to verify these previous two facts in Exercise 7.*

*In each state $s \in \mathbf{S}$ the household choices can be reduced to simply the fraction $\alpha^h(s) \in (0,1)$ for each $h \in \mathbf{H}$ (rather than the commodity vector $x^h(s) \in \mathbb{R}^L_+$). Suppose that the general financial equilibrium allocation $\left(x^h\right)_{h\in\mathbf{H}}$ is constrained Pareto suboptimal. Then there exists $\left(\tilde{x}^h\right)_{h\in\mathbf{H}}$ such that:*

1. $\tilde{x}^h \in \underset{x:\ \tilde{P}_{\backslash 0}\left(x^h(s)-e^h(s)\right)_{s>0} \in \langle R\rangle}{\arg\max}\ u^h(x) \qquad \forall h \in \mathbf{H}$,
2. $\sum_{h\in\mathbf{H}} \left(\tilde{x}^h - e^h\right) = 0$, *and*
3. $\left(u^h\left(\tilde{x}^h\right)\right)_{h\in\mathbf{H}} > \left(u^h\left(x^h\right)\right)_{h\in\mathbf{H}}$.

*Thus, $\exists \left(\tilde{\alpha}^h(s)\right)_{(h,s)\in\mathbf{H}\times\mathbf{S}}$ such that $\sum_{h\in\mathbf{H}} \tilde{\alpha}^h(s) = 1$ $\forall s \in \mathbf{S}$ and*

$$\left(\tilde{x}^h_l(s) = \tilde{\alpha}^h(s) \cdot \sum_{h\in\mathbf{H}} e^h_l(s)\right)_{(h,l,s)\in\mathbf{H}\times\mathbf{L}\times\mathbf{S}}$$

*are the maximizers of the household problem $(HP)$ and $\left(u^h\left(\tilde{x}^h\right)\right)_{h\in\mathbf{H}} > \left(u^h\left(x^h\right)\right)_{h\in\mathbf{H}}$. The commodity prices don't change with changes in $\left(\tilde{\alpha}^h(s)\right)_{(h,s)\in\mathbf{H}\times\mathbf{S}}$, so the choices $\left(\tilde{\alpha}^h(s)\right)_{(h,s)\in\mathbf{H}\times\mathbf{S}}$ lie in the budget sets for all households. This contradicts that $\left(x^h\right)_{h\in\mathbf{H}}$ is a general financial equilibrium allocation as some household is not optimally solving $(HP)$.*

With this second example, we know that the main result (general financial equilibrium allocations are constrained Pareto optimal) is not valid for all utility functions. Rather it is only valid for a generic subset of utility functions. The set of utility functions, unlike the set of endowments, belongs to an infinite-dimensional space. What does it mean to consider a full measure subset of an infinite-dimensional space? Well, this concept is not defined. Therefore, we need to redefine the set of utility functions as belonging to a finite-dimensional space. This is taken up in the proof in Section 3.5.

**Theorem 3.5** *Under Assumption R1-R4 (strengthening R2 such that $u^h$ is $C^3$), if $J < S$ and $L > 1$, then over a generic subset of endowments and utility functions, the general financial equilibrium allocations are constrained Pareto suboptimal.*

**Proof.** See Section 3.5. ∎

## 3.4 Real Assets

In the discussion so far in this chapter, the assets were numeraire assets, meaning that the assets paid out in the numeraire commodity $l = L$. In this section, the assets will be real assets, meaning that they will pay off in the vector of all commodities. By this specification, numeraire assets are a special case of real assets. With real assets, the asset $j$ in state $s > 0$ has the vector of payouts $y_j(s) \in \mathbb{R}^L_+$. This vector is often called the "yields." The yields vector is a column vector.

The value of asset $j$ payouts in state $s > 0$ is then given by $p(s) \cdot y_j(s) \geq 0$. The entire payout matrix is then the $S \times J$ matrix:

$$R(p) = \begin{bmatrix} p(1) \cdot y_1(1) & ... & p(1) \cdot y_J(1) \\ : & & : \\ p(S) \cdot y_1(S) & ... & p(S) \cdot y_J(S) \end{bmatrix}.$$

With numeraire assets, we can go ahead and assume that the payout matrix has full column rank (our No Redundancy condition). With real assets, we can no longer make that same assumption. The rank of the payout matrix $R(p)$ is endogenously determined as a function of the commodity prices $p$.

The definition of a general financial equilibrium remains the same as in Section 3.1.1. In the next section, I show that a financial equilibrium may not exist when real assets are considered. The section that follows will show how the approach developed by Duffie and Shafer (1985) is able to prove the generic existence of a general financial equilibrium with real assets.

### 3.4.1 Nonexistence

The following example was provided by Oliver Hart (1975). I present the Hart example in its original form. I do this not because I am too lazy to construct my own example, but because the original Hart example is beautiful in its simplicity.

In the example, Hart does not allow for consumption in the initial period. Such a modeling choice was fashionable at the time. This does not change our model at all, except that the utility function is now only defined over consumption in the states of uncertainty $s > 0$.

**Example 3.4** *Consider an economy with two households ($H = 2$), two states of uncertainty* *in the final period ($S = 2$), and two physical commodities traded in each state ($L = 2$). Consumption only takes place in the final period, so the household is only endowed with commodities in the final period. The endowments are:*

$$\begin{array}{ll} e^1(1) = \left(\frac{5}{2}, \frac{50}{21}\right) & e^1(2) = \left(\frac{13}{21}, \frac{1}{2}\right) \\ e^2(1) = \left(\frac{1}{2}, \frac{13}{21}\right) & e^2(2) = \left(\frac{50}{21}, \frac{5}{2}\right) \end{array}.$$

*The utility functions are given by:*

$$\begin{aligned} u^1\left(x^1\right) &= 2^{1.5}\sqrt{x_1^1(1)} + \sqrt{x_2^1(1)} + 2^{1.5}\sqrt{x_1^1(2)} + \sqrt{x_2^1(2)} \\ u^2\left(x^2\right) &= \sqrt{x_1^2(1)} + 2^{1.5}\sqrt{x_2^2(1)} + \sqrt{x_1^2(2)} + 2^{1.5}\sqrt{x_2^2(2)} \end{aligned}$$

*There are two real assets that are traded in the initial period and pay out in the final period ($J = 2$). The asset yields are:*

$$\begin{array}{ll} y_1(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} & y_2(1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ y_1(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} & y_2(2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{array}.$$

*We will show that a general financial equilibrium does not exist for this economy by considering two cases.*

***Case I**: The payout matrix $R(p)$ has full column rank.*

*With a full rank payout matrix, using Arrow's Equivalency Theorem (Theorem 3.3), the general financial equilibrium allocation and commodity prices are in fact an Arrow-Debreu equilibrium. The Arrow-Debreu budget constraints are:*

$$\sum_{(l,s)\in\mathbf{L}\times\mathbf{S}} \rho_l(s)\, x_l^h(s) \leq \sum_{(l,s)\in\mathbf{L}\times\mathbf{S}} \rho_l(s)\, e_l^h(s)$$

*for both $h = 1, 2$. The equilibrium market clearing conditions are:*

$$x_l^1(s) + x_l^2(s) = e_l^1(s) + e_l^2(s) = 3 \quad \forall (l,s) \in \mathbf{L}\times\mathbf{S}.$$

*Uniqueness of an Arrow-Debreu equilibrium is guaranteed from the utility functions. It is* *easy to show (Exercise 8) that the Arrow-Debreu equilibrium is given by:*

$$\begin{array}{cc} x^1(1) = \left(\frac{8}{3}, \frac{1}{3}\right) & x^1(2) = \left(\frac{8}{3}, \frac{1}{3}\right) \\ x^2(1) = \left(\frac{1}{3}, \frac{8}{3}\right) & x^2(2) = \left(\frac{1}{3}, \frac{8}{3}\right) \\ \rho(1) = (1,1) & \rho(2) = (1,1) \end{array}.$$

*The corresponding general financial equilibrium allocation is the same, with the commodity prices given by $p(1) = (1,1)$ and $p(2) = (1,1)$. The payout matrix is given by:*

$$R(p) = \begin{bmatrix} p(1)\begin{pmatrix}1\\0\end{pmatrix} & p(1)\begin{pmatrix}0\\1\end{pmatrix} \\ p(2)\begin{pmatrix}1\\0\end{pmatrix} & p(2)\begin{pmatrix}0\\1\end{pmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

*This payout matrix does not have full column rank as required for* ***Case I****.*

***Case II****: The payout matrix $R(p)$ does not have full column rank.*

*The real assets are then redundant, so the general financial equilibrium is found as if only a single asset existed. The initial period budget constraint (recall, no consumption in this period) is given by $qz^h \leq 0$ for both $h = 1, 2$. To satisfy market clearing for the lone asset, then $z^h = 0$ for both $h = 1, 2$. Thus, the general financial equilibrium does not have any financial transfers, meaning that the allocation in state $s = 1$ is the Arrow-Debreu equilibrium allocation of that state considered in isolation (and likewise for state $s = 2$). Again, we know that there is a unique Arrow-Debreu equilibrium in both states. It is easy to show (Exercise 9) that the general financial equilibrium is given by:*

$$\begin{array}{cc} x^1(1) = \left(\frac{62}{21}, \frac{31}{21}\right) & x^1(2) = \left(\frac{32}{21}, \frac{1}{21}\right) \\ x^2(1) = \left(\frac{1}{21}, \frac{32}{21}\right) & x^2(2) = \left(\frac{31}{21}, \frac{62}{21}\right) \\ p(1) = (2,1) & p(2) = \left(\frac{1}{2}, 1\right) \end{array}.$$

*The payout matrix is given by:*

$$R(p) = \begin{bmatrix} p(1)\begin{pmatrix}1\\0\end{pmatrix} & p(1)\begin{pmatrix}0\\1\end{pmatrix} \\ p(2)\begin{pmatrix}1\\0\end{pmatrix} & p(2)\begin{pmatrix}0\\1\end{pmatrix} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ \frac{1}{2} & 1 \end{bmatrix}.$$

*This payout matrix does have full column rank, contradicting that we are in* ***Case II.***

### 3.4.2 Generic existence

Consider the Hart (1975) example above. Is there something special about the example? Well, the endowments and asset yields are specially chosen to arrive at contradictions in both cases. Do we expect similar results to hold for arbitrarily chosen endowments and asset yields? Well no, but how do we know that by restricting our attention to a generic subset of endowments and asset yields we can guarantee the existence of an equilibrium? We know this because Duffie and Shafer (1985) came along and introduced the method required to prove such a claim.

I state the result of Duffie and Shafer and then briefly sketch the proof method. The complete details of the proof have no place in a text like this, but the interested reader is directed toward the well-written primary source.

**Theorem 3.6** *Under Assumption R1-R3, over a generic subset of endowments and asset yields, a general financial equilibrium exists.*

The proof method proceeds as follows.

**Step 1**: Let $L^*$ be a $J-$dimensional linear subspace of $\mathbb{R}^S$ (think of this as the column span of a payout matrix). Define a pseudo-equilibrium as $\left(\left(x^h\right)_{h\in\mathbf{H}}, p, L^*\right)$ such that

1. $\forall h \in \mathbf{H}$, given $p$, $x^h$ is an optimal solution to the household problem $(HP)$

$$(HP) \qquad \begin{array}{ll} \text{maximize} & u^h\left(x^h\right) \\ \text{subject to} & x^h \in X^h \\ & P_{\backslash 0}\left(x^h - e^h\right) \in L^* \end{array}.$$

2. Markets clear

$$\sum\nolimits_{h\in\mathbf{H}} x_l^h\left(s\right) = \sum\nolimits_{h\in\mathbf{H}} e_l^h\ \left(s\right) \quad \forall(l,s) \in \mathbf{L}\times\mathbf{S}.$$

Show that a pseudo-equilibrium always exists.

**Step 2**: If $\left(\left(x^h\right)_{h\in\mathcal{H}}, p, L^*\right)$ is a pseudo-equilibrium, then we can show that, over a generic subset of endowments and asset yields, there exists $\left(\left(z^h\right)_{h\in\mathbf{H}}, q\right)$ such that $\left(\left(x^h, z^h\right)_{h\in\mathbf{H}}, p, q\right)$ is a general financial equilibrium with real assets. When we refer to a generic subset of asset yields, we are looking at a generic subset of a Grassmanian manifold. This is because the space of $J-$dimensional linear subspaces of $\mathbb{R}^S$ (the column span of a payout matrix) has the structure of a topological manifold called the "Grassmanian manifold."

## 3.5 Proof of Theorem 3.5

This section contains the rather long proof of Theorem 3.5. The original proof is due to Geanakoplos and Polemarchakis (1986), while the general method (which can be applied to all sorts of Pareto-improving policies) is due to Citanna, Kajii, and Villanacci (1998).

The equilibrium variables are $\xi = \left(\left(x^h, \lambda^h, z^h\right)_{h\in\mathbf{H}}, p, q\right)$ and the planner transfers were previously introduced as $\tau = \left(\tau^h\right)_{h\in\mathbf{H}}$. The principal task will be to show that the vector of household utility functions $U\left(\xi,\tau\right) = \left(u^1(x^1), .., u^H(x^H)\right)$ is a submersion (i.e., its derivative mapping is surjective).

Take as given a general financial equilibrium $\left(\left(x^h, \lambda^h, z^h\right)_{h\in\mathbf{H}}, p, q\right)$. Given parameters $\theta = \left(e^h, u^h\right)_{h\in\mathbf{H}}$, the variables $\hat{\xi} = \left(\left(\hat{x}^h, \hat{\lambda}^h\right)_{h\in\mathbf{H}}, \hat{p}\right)$ and transfers $\tau$ constitute a constrained feasible vector iff $\Gamma(\hat{\xi},\tau,\theta) = 0$, where

$$\Gamma(\hat{\xi},\tau,\theta) = \begin{pmatrix} FOCx \\ BC \\ MCx \\ MC\tau \end{pmatrix}^{3}$$

for $FOCx = \left[Du^h(\hat{x}^h) - \hat{\lambda}^h\hat{P}\right]^T_{h\in\mathbf{H}}$, $BC = \left(\hat{P}\left(e^h - \hat{x}^h\right) + \begin{pmatrix} 1 & 0 \\ 0 & R\end{pmatrix}\tau^h\right)_{h\in\mathbf{H}}$, $MCx = \left(\sum_{h\in\mathbf{H}}(e^h_{\backslash L}(s) - \hat{x}^h_{\backslash L}(s))\right)_{s\in\mathbf{S}}$, and $MC\tau = \sum_{h\in\mathbf{H}}\tau^h$. $\Gamma$ has $m = H\left(G+S+1\right) + G - (S+1) + J + 1$ equations.

Picking a vector of parameters $\bar{\theta} = \left(\bar{e}^h, \bar{u}^h\right)_{h\in\mathbf{H}}$ such that $\left(\bar{e}^h\right)_{h\in\mathbf{H}}$ belongs to a generic subset of $\mathbb{R}^{HG}_{++}$, then all resulting general financial equilibria are regular values of $\Phi$. In particular, this means that there exists an open set $\Theta'$ around $\bar{\theta}$ such that for any parameters $\theta\in\Theta'$, the resulting equilibria satisfy the rank condition of (3.1). Let $\left(\bar{x}^h\right)_{h\in\mathbf{H}}$ be one such equilibrium allocation given the parameters $\bar{\theta}$. Then, the set of allocations $\left(x^h\right)_{h\in\mathbf{H}}$ in a local neighborhood around $\left(\bar{x}^h\right)_{h\in\mathbf{H}}$ such that $\left(u^1(x^1), .., u^H(x^H)\right) >> \left(u^1(\bar{x}^1), .., u^H(\bar{x}^H)\right)$ is open.

[3] See the discussion in Section 3.3.2.

If for some planner transfers $\tau^* = \left(\tau^{h*}\right)_{h\in\mathbf{H}}$, the resulting constrained feasible allocation is Pareto superior, then all constrained feasible allocations for $\tau$ in an open neighborhood around $\tau^*$ are Pareto superior as well.

The analysis is conducted evaluating equations at $\tau = \overrightarrow{0}$. By definition, if $\Gamma(\hat{\xi}, \overrightarrow{0}, \theta) = 0$ and $\Phi(\xi, \theta) = 0$, then $\hat{\xi} = \left(\left(x^h, \lambda^h\right)_{h\in\mathbf{H}}, p\right)$.

Define the $(H+m) \times (m' + H(J+1))$ matrix $\Psi_0$ :

$$\Psi_0 = \begin{pmatrix} D_{x,\lambda,p}U(\hat{\xi}, \tau) & 0 \\ D_{x,\lambda,p}\Gamma(\hat{\xi}, \tau, \theta) & D_\tau \Gamma(\hat{\xi}, \tau, \theta) \end{pmatrix},$$

where $m' = H(G+S+1) + G - (S+1)$ is the number of variables in the vector $(x, \lambda, p) = \left(\left(x^h, \lambda^h\right)_{h\in\mathbf{H}}, p\right)$ and $H(J+1)$ is the number of planner transfers $\left(\tau^h\right)_{h\in\mathbf{H}}$. From Citanna, Kajii, and Villanacci (2002), if $\Psi_0$ has full row rank, $\exists \hat{\xi} \neq \left(\left(x^h, \lambda^h\right)_{h\in\mathbf{H}}, p\right)$ s.t. $\hat{\xi}$ satisfies $\Gamma(\hat{\xi}, \tau, \theta) = 0$ (for some $\tau$) and $U\left(\left(\hat{x}^h\right)_{h\in\mathbf{H}}\right) > U\left(\left(x^h\right)_{h\in\mathbf{H}}\right)$. For full row rank, I must ensure that there are fewer rows than columns, so I need the following inequality to hold:

$$H + m \leq m' + H(J+1). \tag{3.4}$$

(3.4) reduces to

$$H + J + 1 \leq H(J+1). \tag{3.5}$$

(3.5) is always satisfied as $H \geq 2$. The matrix $\Psi_0$ is square if $H + J + 1 = H(J+1)$, but if $H + J + 1 < H(J+1)$, then there are more columns than rows and I must remove some columns (it does not matter which) in order to obtain a square matrix $\Psi$. This matrix $\Psi$ does not have full rank iff $\exists \nu \in \mathbb{R}^{H+m}$ s.t. $\Phi'(\hat{\xi}, \overrightarrow{0}, \nu, \theta) = 0$ where

$$\Phi'(\hat{\xi}, \overrightarrow{0}, \nu, \theta) = \begin{pmatrix} \Psi^T \nu \\ \nu^T \nu/2 - 1 \end{pmatrix}.$$

For simplicity, I divide the vector $\nu^T$ into subvectors that each represent a certain equation in $\Psi$. Define $\nu^T = \left(\Delta u^T, \Delta x^T, \Delta\lambda^T, \Delta p^T, \Delta\tau^T\right) \in \mathbb{R}^{H+m}$ where each subvector corresponds sensibly to an equation (row) in $\Psi$ as follows:

$$\begin{array}{lcl} \Delta u^T & \Longleftrightarrow & D_{x,\lambda,p}U(\hat{\xi},\tau) \\ \Delta x^T & \Longleftrightarrow & FOCx \\ \Delta \lambda^T & \Longleftrightarrow & BC \\ \Delta p^T & \Longleftrightarrow & MCx \\ \Delta \tau^T & \Longleftrightarrow & MC\tau. \end{array}$$

A subset of the equations $\nu^T\Psi = 0$ are given by (corresponding to derivatives with respect to $\left((x^h, \lambda^h)_{h\in\mathcal{H}}\right)$ in that order):

$$\begin{array}{ll} \left(\Delta u^h Du^h(x^h) + \left(\Delta x^h\right)^T D^2u^h(x^h) - \left(\Delta\lambda^h\right)^T P - \Delta p^T\Lambda\right)_{h\in\mathbf{H}} = 0. & (3.6.a) \\ \left(-\left(\Delta x^h\right)^T P^T\right)_{h\in\mathbf{H}} = 0. & (3.6.b) \end{array} \tag{3.6}$$

where $\Lambda = \begin{bmatrix} \begin{pmatrix} I_{L-1} & \mid & 0 \end{pmatrix} & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & \begin{pmatrix} I_{L-1} & \mid & 0 \end{pmatrix} \end{bmatrix}$ as in Subsection 3.1.2.

The proof is complete if I can show that for a generic choice of $\theta \in \Theta$, there does not exist $(\xi, \nu)$ s.t.

$$\begin{array}{rcl} \Phi(\xi,\theta) & = & 0 \\ \Phi'((x,\lambda,p)\,,\overrightarrow{0},\nu,\theta) & = & 0. \end{array} \tag{3.7}$$

Counting equations and unknowns, (3.7) has $n$ equations in $\Phi$, $n$ variables $\xi$, $H+m+1$ equations in $\Phi'$, and only $H+m$ variables $\nu$. I must show that over a generic subset of parameters (exactly which generic subset will be discussed next), the derivative matrix $D_{x,\lambda,p,\nu}\begin{pmatrix}\Phi \\ \Phi'\end{pmatrix}$ has full row rank. I will reference the $(ND)$ condition of Citanna, Kajii, and Villanacci (1998), which is a sufficient condition for the full row rank of $D_{x,\lambda,p,\nu}\begin{pmatrix}\Phi \\ \Phi'\end{pmatrix}$.

The condition states that for $\tau = \overrightarrow{0}$ and $\hat{\xi} = \left( \left( x^h, \lambda^h \right)_{h \in \mathbf{H}}, p \right)$, the matrix

$$\left( \begin{pmatrix} \Psi^T \\ \nu^T \end{pmatrix} \quad D_\theta \Phi' \right) \text{ has full row rank,} \tag{3.8}$$

where $\theta$ are the parameters on which the genericity statement is made.

For simplicity, I break up the analysis into two cases: Case I: $\left(\Delta x^h\right)^T \neq 0 \ \forall h \in \mathbf{H}$ and Case II: $\left(\Delta x^h\right)^T = 0$ for some $h \in \mathbf{H}$. In Case I, I show that (3.8) holds over a generic subset of parameters. In Case II, I show that (3.7) will generically not have any solution.

### 3.5.1 Case I: $\left(\Delta x^h\right)^T \neq 0 \quad \forall h \in \mathbf{H}$

**Lemma 3.1** *For $\tau = \overrightarrow{0}$, then $D_u \Phi' = \begin{pmatrix} d\left(\dot{A}^h\right) \\ \overrightarrow{0} \end{pmatrix}$ where $d\left(\dot{A}^h\right)$ has full row rank and corresponds to the rows for derivatives with respect to $\left(x^h\right)_{h \in \mathbf{H}}$.*

**Proof.** The set $\mathcal{U}$ is infinite-dimensional and is endowed with the $C^3$ uniform convergence topology on compact sets. This means that a sequence of functions $\{u^\nu\}$ converges uniformly to $u$ iff $\{Du^\nu\}$, $\{D^2 u^\nu\}$, and $\{D^3 u^\nu\}$ uniformly converge to $Du$, $D^2 u$, and $D^3 u$, respectively. Additionally, any subspace of $\mathcal{U}$ is endowed with the subspace topology of the topology of $\mathcal{U}$. I will use the regularity result from Theorem 3.2 to define utility functions as locally belonging to the finite-dimensional subset $\mathcal{A} \subseteq \mathcal{U}$.

Using Theorem 3.2, pick a regular value $\bar{\theta}$. For that $\bar{\theta}$, there exist finitely many equilibria $\bar{\xi}_i$, $i = 1, ..., I$. Further, there exist open sets $\Sigma'$ and $A_i'^h$ s.t. $\bar{x}_i^h \in A_i'^h$, the sets $A_i'^h$ are disjoint across $i$, and $\forall \theta \in \Sigma'$, $\exists!$ equilibrium allocation $x_i^h \in A_i'^h$. Choose $A_i'^h$ such that the closure $clA_i'^h$ is compact and there exist disjoint open sets $\tilde{A}_i'^h$ s.t. $A_i'^h \subset clA_i'^h \subset \tilde{A}_i'^h$.

For each household, define a bump function $\delta^h : X^h \to [0, 1]$ with $I$ bumps as $\delta^h = 1$ on $A_i'^h$ and $\delta^h = 0$ on $(\tilde{A}_i'^h)^c$. Now, I define $u^h$ in terms of a $G \times G$ symmetric matrix $A^h$ by:

$$u^h(x^h; A^h) = \bar{u}^h(x^h) + \frac{1}{2}\delta^h(x^h) \sum\nolimits_i \left[ (x^h - \bar{x}_i^h)^T A^h (x^h - \bar{x}_i^h) \right].$$

Thus, the space of symmetric matrices (denote this as $\mathcal{A}$) is a finite dimensional subspace of $\mathcal{U}$. Since $\mathcal{A}$ has the subspace topology of $\mathcal{U}$, then $u^h(\cdot; A^\nu) \to u^h(\cdot; A)$ iff $A^\nu \to A$. This can be seen by taking derivatives and noting that the function $\bar{u}$ stays fixed at the regular value.

Taking derivatives with respect to $x^h \in A_i'^h$ yields:

$$\begin{aligned} D_x u^h(x^h; A^h) &= D\bar{u}^h(x^h) + A^h(x^h - \bar{x}_i^h) \\ D_{xx}^2 u^h(x^h; A^h) &= D^2\bar{u}^h(x^h) + A^h. \end{aligned}$$

$\mathcal{A}$ is a $G(G+1)/2$ dimensional space, so write $A^h$ as the vector

$$\left((A_{i,i}^h)_{i=1,..,G}, (A_{i,j}^h)_{i<j,i=1,..,G-1}\right).$$

Postmultiply $D_{xx}^2$ by $\Delta x^h$ :

$$D_{xx}^2 u^h(x^h; A^h)\Delta x^h = D^2\bar{u}^h(x^h)\Delta x^h + A^h\Delta x^h.$$

Taking derivatives with respect to the parameter $u^h$ is equivalent to taking derivatives with respect to $A^h$ :

$$D_u\left(D_{xx}^2 u^h(x^h; A^h)\Delta x^h\right) = D_A\left(A^h\Delta x^h\right)$$

$$= \begin{pmatrix} \Delta x_1^h & 0 & 0 & & & \\ 0 & ... & 0 & \Sigma(1) & ... & \Sigma(G-1) \\ 0 & 0 & \Delta x_G^h & & & \end{pmatrix} \in \mathbb{R}^{G,G(G+1)/2}$$

where the submatrix $\Sigma(i)$ is defined as

$$\Sigma(i) = \begin{pmatrix} 0 \in \mathbb{R}^{i-1,G-i} \\ \begin{pmatrix} \Delta x_{i+1}^h & ... & \Delta x_G^h \\ \Delta x_i^h & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & \Delta x_i^h \end{pmatrix} \end{pmatrix} \in \mathbb{R}^{G,G-i}.$$

Thus, since $\Delta x^h \neq 0$ (without loss of generality $\Delta x_1^h \neq 0$), then

$$rank D_{A^h}\left(D_{xx}^2 u^h(x^h; A^h)\Delta x^h\right) = G. \tag{3.9}$$

Out of all the rows $\Psi^T$, the utility function $u^h$ only appears in the row for derivatives with respect to $x^h$. This row in $\Psi^T$ for household $h$ is given by (as in (3.6.$a$)):

$$\begin{array}{ccccc} \underline{U(\xi,\hat{\xi})} & \underline{FOCx} & \underline{BC} & \underline{MCx} & \underline{MC\tau} \\ \left(Du^h(x^h)\right)^T & D^2u^h(x^h) & -P^T & -\Lambda^T & 0 \end{array}.$$

Thus, taking the derivative $D_{A^h}\Phi' = \begin{pmatrix} D_{A^h}\Psi^T\nu \\ 0 \end{pmatrix}$, the only nonzero terms correspond to the rows for derivatives with respect to $x^h$ :

$$D_{A^h}\left(\left(Du^h(x^h)\right)^T\Delta u^h + D^2u^h(x^h)\Delta x^h - P^T\Delta\lambda^h - \Lambda^T\Delta p\right)$$

$$= D_{A^h}\left(\left(Du^h(x^h;A^h)\right)^T\Delta u^h\right) + D_{A^h}\left(D^2u^h(x^h;A^h)\Delta x^h\right).$$

From the first derivative, $D_xu^h(x^h;A^h) = D\bar{u}(x^h) + A^h(x^h - \bar{x}_i^h) = D\bar{u}(x^h)$ evaluated at $\tau = \overrightarrow{0}$ (since $x^h = \bar{x}_i^h$). Thus $D_{A^h}(D_xu^h(x^h;A^h)\Delta u^h) = 0$. Using (3.9),

$$d\left(\dot{A}^h\right) = \begin{bmatrix} D_{A^1}\left(D^2u^1(x^1;A^1)\Delta x^1\right) & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & D_{A^H}\left(D^2u^H(x^H;A^H)\Delta x^H\right) \end{bmatrix}$$

is a full row rank matrix of size $HG \times HG(G+1)/2$. ■

The matrix $\left(\begin{pmatrix} \Psi^T \\ \nu^T \end{pmatrix} \quad D_A\Phi'\right)$ is given below (where the rows correspond to the equilibrium variables $\left((x^h,\lambda^h)_{h\in\mathbf{H}},p\right)$, planner transfers $(\tau)$, and vector $v^T$ in that order). Recall from (3.8) that I aim to show that this matrix has full row rank. I use the convention $c\left(M^h\right) = \begin{pmatrix} M^1 \\ : \\ M^H \end{pmatrix}$, $r\left(M^h\right) = \begin{pmatrix} M^1 & ... & M^H \end{pmatrix}$, and $d\left(M^h\right) = \begin{pmatrix} M^1 & 0 & 0 \\ 0 & ... & \\ 0 & 0 & M^H \end{pmatrix}$, where $c$ stands for column, $r$ stands for row, and $d$ stands for diagonal. For simplicity, define

$\Omega = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$. The matrix $\left( \begin{pmatrix} \Psi^T \\ \nu^T \end{pmatrix} \quad D_A\Phi' \right)$ is given by:

$$\begin{pmatrix} d\left(Du^h(x^h)^T\right) & d\left(D^2u^h\right) & d(-P^T) & c(-\Lambda^T) & 0 & d\left(\dot{A}^h\right) \\ 0 & d\left(-P\right) & 0 & 0 & 0 & 0 \\ 0 & r(-\left(\Lambda_2^h\right)^T) & r\left(\left(Z^h\right)^T\right) & 0 & 0 & 0 \\ 0 & 0 & d\left(\Omega^T\right) & 0 & c\left(I_{S+1}\right) & 0 \\ r(\Delta u^h) & r(\left(\Delta x^h\right)^T) & r(\left(\Delta\lambda^h\right)^T) & \Delta p^T & \Delta\tau^T & 0 \end{pmatrix}$$

where I define the following two submatrices:

$$\Lambda_2^h = \begin{bmatrix} \begin{pmatrix} \lambda^h(0)\, I_{L-1} \\ 0 \end{pmatrix} & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & \begin{pmatrix} \lambda^h(S)\, I_{L-1} \\ 0 \end{pmatrix} \end{bmatrix} \in \mathbb{R}^{G,G-(S+1)}.$$

$$Z^h = \begin{bmatrix} \left(e_{\setminus L}^h(0) - x_{\setminus L}^h(0)\right)^T & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & \left(e_{\setminus L}^h(S) - x_{\setminus L}^h(S)\right)^T \end{bmatrix} \in \mathbb{R}^{S+1,G-(S+1)}.$$

**Lemma 3.2** $\left(\left(\Delta u^h\right)_{h\in\mathbf{H}}, p\right) \neq 0.$

**Proof.** Suppose not, that is $\left(\Delta u^h, p\right) = 0$ for some $h$. From (3.6.$a$),

$$\left(\Delta x^h\right)^T D^2u^h(x^h) - \left(\Delta\lambda^h\right)^T P = 0. \tag{3.10}$$

Post-multiply (3.10) by $\Delta x^h$ and use (3.6.$b$) to yield:

$$\left(\Delta x^h\right)^T D^2u^h(x^h)\Delta x^h = 0. \tag{3.11}$$

From (3.11) and Assumption R2, then $\Delta x^h = 0$. But this contradicts that we are in Case I. ■

From Lemmas 3.1 and 3.2, the first and last row blocks of $\left( \begin{pmatrix} \Psi^T \\ \nu^T \end{pmatrix} \quad D_A\Phi' \right)$ are linearly independent from the others. By the definition of $\Lambda_2^h$, the submatrix $\begin{pmatrix} -P & 0 & 0 \\ 0 & .. & 0 \\ 0 & 0 & -P \\ -\left(\Lambda_2^1\right)^T & .. & -\left(\Lambda_2^H\right)^T \end{pmatrix}$ of size $[H(S+1)+G-(S+1)] \times HG$ is a full rank matrix. The submatrix $d\left(\Omega^T\right)$ also has full rank (as $R$ has full column rank). Thus, $\left( \begin{pmatrix} \Psi^T \\ \nu^T \end{pmatrix} \quad D_A\Phi' \right)$ has full rank and this concludes the proof under Case I.

### 3.5.2 Case II: $\left(\Delta x^h\right)^T = 0$ for some $h \in \mathbf{H}$

I will show that over a generic subset of endowments, (3.7) has no solution. Suppose $\exists h' \in \mathbf{H}$ such that $\left(\Delta x^{h'}\right)^T = 0$. From (3.6.$a$) and $\Phi$, I obtain

$$\begin{aligned} \Delta u^{h'} Du^{h'}(x^{h'}) - \left(\Delta\lambda^{h'}\right)^T P - \Delta p^T \Lambda &= 0 \\ Du^{h'}(x^{h'}) - \lambda^{h\prime} P &= 0 \end{aligned}$$

which together imply that $\Delta p^T = 0$ and $\left(\Delta\lambda^{h'}\right)^T = \Delta u^{h'} \lambda^{h\prime}$.

For all other $h \neq h'$, postmultiply $\Delta u^h Du^h(x^h)$ by $\Delta x^h$ and use the first order condition (consumption) in $\Phi$ and (3.6.$b$) to get $\Delta u^h Du^h(x^h)\Delta x^h = 0$. Next, postmultiply (3.6.$a$) by $\Delta x^h$ and use (3.6.$b$) to arrive at the equation:

$$\left(\Delta x^h\right)^T D^2 u^h(x^h)\Delta x^h = 0. \tag{3.12}$$

By Assumption R2, $\left(\Delta x^h\right)^T = 0 \ \forall h \in \mathbf{H}$. From what remains of (3.6.$a$) and $\Phi$, $\left(\Delta\lambda^h\right)^T = \Delta u^h \lambda^h \ \ \forall h \in \mathbf{H}$.

From the column for derivatives with respect to $\left(\tau^h\right)_{h\in\mathbf{H}}$,

$$\left(\Delta\lambda^h\right)^T \Omega + \Delta\tau^T = 0 \ \ \forall h \in \mathbf{H}. \tag{3.13}$$

As $\Omega = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$, then (3.13) implies $\Delta\lambda^h(0) = \Delta\lambda^1(0) \ \ \forall h \in \mathbf{H}$.

The terms $\left(\left(\Delta u^h, \left(\Delta\lambda^h\right)^T\right)_{h\in\mathbf{H}}, \Delta\tau^T\right)$ are the only nonzero elements of $\nu$. Further,

$\Delta u^h \lambda^h(0) = \Delta u^1 \lambda^1(0) \;\; \forall h \in \mathbf{H}$. This yields:

$$\Delta u^h = \Delta u^1 \frac{\lambda^1(0)}{\lambda^h(0)} \;\; \forall h \in \mathbf{H}. \tag{3.14}$$

The equation from $\nu^T \Psi = 0$ corresponding to derivatives with respect to $p(s)$ (for any $s > 0$) is given as follows (after replacing $\Delta\lambda^h(s)$ with $\Delta u^h \lambda^h(s)$):

$$\sum\nolimits_{h \in \mathbf{H}} \Delta u^h \lambda^h(s) \left(e^h_{\backslash L}(s) - x^h_{\backslash L}(s)\right)^T = 0. \tag{3.15}$$

For the analysis to hold at this point, I must use the assumption that $L > 1$ (as there are no commodity price variables when $L = 1$). In (3.15), only consider the first physical commodity $l = 1$ and use (3.14):

$$\Delta u^1 \lambda^1(0) \sum\nolimits_{h \in \mathbf{H}} \frac{\lambda^h(s)}{\lambda^h(0)} \left(e^h_1(s) - x^h_1(s)\right)^T = 0.$$

As with Theorem 3.4 and "A Different Application of Differential Topology" from Exercise 3 in Chapter 1, we can show that over a generic subset of endowments, the equation

$$\sum\nolimits_{h \in \mathbf{H}} \frac{\lambda^h(s)}{\lambda^h(0)} \left(e^h_1(s) - x^h_1(s)\right)^T \neq 0.$$

This implies that $\Delta u^1 = 0$, with (3.14) implying $\left(\Delta u^h\right)_{h \in \mathbf{H}} = 0$. Consequently, $\left(\Delta\lambda^h\right)^T = 0$ $\forall h \in \mathbf{H}$. From (3.13), $\Delta\tau^T = 0$. In conclusion, $\nu^T = 0$, which cannot satisfy $\Phi'$ as this system contains the equation $\nu^T \nu / 2 = 1$. This completes the argument that (3.7) cannot hold (generically), meaning that Case II is not possible (generically).

## 3.6 Exercises

1. Show that No Arbitrage is a necessary condition of equilibrium. That is, show that if No Arbitrage does not hold, then a general financial equilibrium does not exist.

2. Suppose that there are $S = 4$ states and $J = 2$ assets with payouts

$$R = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 0 & 1 \\ 3 & 2 \end{bmatrix}.$$

   Which of the following asset prices satisfy No Arbitrage (hint: Hens method): (i) $q = (0, 2)$, (ii) $q = (2, 1.75)$, and (iii) $q = (2, 1.25)$?

3. Suppose that there are $S = 4$ states and $J = 3$ assets with payouts

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ \frac{2}{3} & 1 & \frac{1}{3} \\ \frac{1}{2} & \frac{2}{3} & \frac{1}{2} \end{bmatrix}.$$

   Which of the following asset prices satisfy No Arbitrage: (i) $q = (1, 4, 2)$, (ii) $q = (2, 1, 1)$, and (iii) $q = (3, 2, 1)$?

4. Show that it is innocuous to assume No Redundancy. In other words, show that if $\left(\left(x^h, \lambda^h, z^h\right)_{h \in \mathbf{H}}, p, q\right)$ is a general financial equilibrium in which No Redundancy does not hold, then there exists $\left(\left(\hat{z}^h\right)_{h \in \mathbf{H}}, \hat{q}\right)$ such that $\left(\left(x^h, \lambda^h, \hat{z}^h\right)_{h \in \mathbf{H}}, p, \hat{q}\right)$ is a general financial equilibrium in which No Redundancy does hold.

5. Prove Theorem 3.4.

6. In Example 3.1, verify that the Arrow-Debreu equilibrium allocation is such that $\forall h \in \mathbf{H} : x^h(s) = \theta^h \sum_{h \in \mathbf{H}} e^h(s)$ for some $\theta^h$.

7. In Example 3.3, verify that the general financial equilibrium consumption is given by:

$$x_l^h(s) = \alpha^h(s) \cdot \sum_{h \in \mathbf{H}} e_l^h(s)$$

   for some $\alpha^h(s) \in (0, 1)$ and the prices are given such that $p_l(s) = \frac{\theta_l(s)}{\theta_L(s)}$ $\forall (l, s) \in \mathbf{L} \times \mathbf{S}$.

8. In Example 3.4, verify that the Arrow-Debreu equilibrium allocation and commodity

prices in Case I are given by:

$$\begin{array}{ll} x^1(1) = \left(\frac{8}{3}, \frac{1}{3}\right) & x^1(2) = \left(\frac{8}{3}, \frac{1}{3}\right) \\ x^2(1) = \left(\frac{1}{3}, \frac{8}{3}\right) & x^2(2) = \left(\frac{1}{3}, \frac{8}{3}\right) \\ \rho(1) = (1, 1) & \rho(2) = (1, 1) \end{array}.$$

9. In Example 3.4, verify that the general financial equilibrium allocation and commodity prices in Case II are given by:

$$\begin{array}{ll} x^1(1) = \left(\frac{62}{21}, \frac{31}{21}\right) & x^1(2) = \left(\frac{32}{21}, \frac{1}{21}\right) \\ x^2(1) = \left(\frac{1}{21}, \frac{32}{21}\right) & x^2(2) = \left(\frac{31}{21}, \frac{62}{21}\right) \\ p(1) = (2, 1) & p(2) = \left(\frac{1}{2}, 1\right) \end{array}.$$

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# Chapter 4

# Incomplete Markets and Money

In the previous chapter, we made the following price normalizations in all states $s \in \mathbf{S}$ : $p_L(s) = 1$. Does this choice matter? Would a different price normalization affect the equilibrium consumption? Consider the budget constraints in the previous chapter, when we have numeraire assets:

$$p(s)\left(e^h(s) - x^h(s)\right) + p_L(s)\sum_{j\in\mathcal{J}} r_j(s)\, z_j^h \geq 0 \text{ for } s > 0.$$

Thus, any other price normalization ($p_L(s) = k$ for some $k > 0$ or $p(s) \in \Delta^{L-1}$) does not change the budget constraint. Price normalizations are neutral; they have no real effects. The conclusion with numeraire assets (as well as with real assets, the generalization of numeraire assets) is that nominal indeterminacy does not imply real indeterminacy.

That conclusion does not hold when we look at a third type of assets: nominal assets. Nominal assets are those that pay off in the unit of account in all states $s > 0$. The unit of account can be thought of as the currency of the economy. Assets paying off in the unit of account change the wealth of households, but not through the promise of payment in a real commodity.

This chapter introduces a two-period general equilibrium model with uncertainty and nominal assets. In principle, we are free to make any price normalizations that we wish. Yet, in this context, it makes sense to include an institution that will pin down the price normalizations. That institution is a monetary exchange that issues money. The supplies of money will determine the nominal price levels.

In Section 4.1, I introduce the model. In Section 4.2, I consider an asset structure with complete markets. With complete markets, nominal indeterminacy does not imply real indeterminacy (this is the same result obtained for the numeraire/real asset case). In Section 4.3, I consider an asset structure with incomplete markets. With incomplete markets, nominal indeterminacy can imply real indeterminacy (the main result is Theorem 4.5). This is the problem (or the reality) with nominal assets.

## 4.1 The Model

The uncertainty of the dynamic model remains unchanged from Chapter 3. Additionally, the real side of the model remains the same. The household primitives are:

- $X^h = \mathbb{R}_+^G$ is the consumption set.
- $u^h : X^h \to \mathbb{R}$ is the utility function. We assume that the function is continuous, locally non-satiated, and quasi-concave.
- $e^h \in X^h$ is the endowment.

Each good $(l, s)$ has a market price $p_l(s)$. The vector of all prices is $p = (p_l(s))_{(l,s)\in\mathbf{L}\times\mathbf{S}} \in \mathbb{R}^G\backslash\{0\}$. The convention is that the price vector $p(s)$ is a row vector $\forall s \in \mathbf{S}$. I define the $(S+1) \times G$ price matrix $P = \begin{bmatrix} p(0) & 0 & 0 & 0 \\ 0 & p(1) & 0 & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & 0 & p(S) \end{bmatrix}$. The price normalizations will be determined as a function of the money supplies. The money supplies are $M_s > 0$ for all states $s \in \mathbf{S}$. The vector $M = (M_s)_{s\in\mathbf{S}}$ is a parameter of the model.

There are $J$ nominal assets that are traded in the initial period. Denote the price of an asset $j$ in the initial period as $q_j$. Denote all asset prices as $q = (q_j)_{j\in\mathbf{J}}$. These prices are variables in the equilibrium and will be determined to satisfy market clearing conditions.

In state $s > 0$, the asset $j$ has a fixed payout $r_j(s)$. The payouts are made in terms of the unit of account. The payouts are assumed to be nonnegative $r_j(s) \geq 0 \ \ \forall (j, s) \in \mathbf{J} \times \mathbf{S}\backslash\{0\}$, where $r_j = (r_j(s))_{s>0} > 0 \ \forall j \in \mathbf{J}$ . Let's collect the payouts of all assets in all states $s > 0$ into one payout matrix $R$. This payout matrix has $S$ rows and $J$ columns: $R = \begin{bmatrix} r_1 & ... & r_J \end{bmatrix} = \begin{bmatrix} r_1(1) & ... & r_J(1) \\ : & ... & : \\ r_1(S) & ... & r_J(S) \end{bmatrix}$.

Household $h$ chooses a portfolio $z^h \in \mathbb{R}^J$ in the initial period, where the position for a particular asset $j$ is denoted $z_j^h \in \mathbb{R}$.

The timing of the model is as follows. There are two distinct entities: households and the Monetary Exchange. There are three substages in the initial period. In the first substage, all households turn over their endowments to the Monetary Exchange. In return, the Monetary Exchange provides each household with $m^h(0) = p(0)e^h(0)$ units of account (view this as currency), so that $\sum_{h\in\mathbf{H}} m^h(0) = M_0$. In the second substage, households trade the nominal assets. Following this trade, each household has available the following amount of the unit of account: $\hat{m}^h(0) = m^h(0) - qz^h$. Given this amount of the unit of account, the households then enter the third substage of the initial period, in which they purchase commodities for consumption with the monetary resources that they have available:

$$p(0)\,x^h(0) \leq \hat{m}^h(0).$$

After the purchase of commodities, the money in the market $\sum_{h\in\mathbf{H}} \hat{m}^h(0) = M_0$ is returned to the Monetary Exchange.

A similar story unfolds in each state $s > 0$. Three substages take place. In the first, households sell their endowments to the Monetary Exchange and receive $m^h(s) = p(s)e^h(s)$ units of account, so that $\sum_{h\in\mathbf{H}} m^h(s) = M_s$. In the second substage, the payouts of the nominal assets are accounted for, so each household now has available the following amount of the unit of account: $\hat{m}^h(s) = m^h(s) + r(s)z^h$. In the third substage, households use this money to pay for the purchase of commodities:

$$p(s)\,x^h(s) \leq \hat{m}^h(s).$$

The money in the market from the sale of commodities, of size $\sum_{h\in\mathbf{H}} \hat{m}^h(s) = M_s$, is returned to the Monetary Exchange.

The story may seem involved, but it is a simple means to introduce a cash-in-advance constraint (also called a Clower constraint in recognition of Robert Clower's work on monetary models). In this setup, the velocity of money is set equal to 1 by definition (a single unit of account can only be used in one purchase). The monetary elements of the model are summarized succinctly in the definition of a financial equilibrium with money.

### 4.1.1 Existence

**Definition 4.1** *A financial equilibrium with money is* $\left(\left(x^h, z^h\right)_{h\in\mathbf{H}}, p, q\right)$ *such that*

1. $\forall h \in \mathbf{H}$, *given* $(p, q)$, $\left(x^h, z^h\right)$ *is an optimal solution to the household problem* $(HP)$

$$(HP) \qquad \begin{array}{ll} \textit{maximize} & u^h\left(x^h\right) \\ \textit{subject to} & x^h \in X^h \\ & z^h \in \mathbb{R}^J \\ & P\left(e^h - x^h\right) + \begin{pmatrix} -q \\ R \end{pmatrix} z^h \geq 0 \end{array}.$$

2. *Markets clear*

$$\begin{aligned} \sum\nolimits_{h\in\mathbf{H}} x_l^h(s) &= \sum\nolimits_{h\in\mathbf{H}} e_l^h\ (s) \quad \forall(l,s) \in \mathbf{L}\times\mathbf{S}. \\ \sum\nolimits_{h\in\mathbf{H}} z_j^h &= 0 \quad \forall j \in \mathcal{J}. \end{aligned}$$

3. *Monetary supply condition*

$$p(s)\sum\nolimits_{h\in\mathbf{H}} x^h(s) = M_s \quad \forall s \in \mathbf{S}.$$

It is important to verify that a financial equilibrium with money exists. The assumptions for this are given by:

Assumption E1: $X^h = \mathbb{R}_+^G \ \ \forall h \in \mathbf{H}$.

Assumption E2: $u^h : X^h \to \mathbb{R}$ is $C^1$, increasing, and quasi-concave $\forall h \in \mathbf{H}$.

Assumption E3: $e^h >> 0 \ \forall h \in \mathbf{H}$.

Assumption E4 (No Redundancy): The payout matrix $R$ has full column rank (that is, the assets are linearly independent).

Assumption E5: $M >> 0$.

**Theorem 4.1** *Under Assumptions E1-E5, a financial equilibrium with money* $\left(\left(x^h, z^h\right)_{h\in\mathbf{H}}, p, q\right)$ *exists.*

**Proof.** The proof is identical to the proof of Theorem 3.1, which is simply an extension of the first existence proof showing the existence of an Arrow-Debreu equilibrium (see Theorem 2.1 and the proof in Section 2.6). ■

## 4.2 Complete Markets

Under Assumption E4 (No Redundancy), complete markets means simply that $J = S$. Under this assumption, the Arrow's Equivalency Theorem (Theorem 3.3) is still valid. Thus, all financial equilibrium with money allocations are equivalent to Arrow-Debreu equilibrium allocations. Therefore, the allocations are Pareto optimal (First Basic Welfare Theorem; see Theorem 2.2). Additionally, all equilibria satisfy Finite Local Uniqueness (see Theorem 2.4).

This leads us to the following "immunity" result stating that nominal indeterminacy does not imply real indeterminacy.

**Theorem 4.2** *Assume E1-E5 and* $J = S$. *With the money supply* $M = (M_0, ..., M_S)$, *the financial equilibrium with money is* $\left(\left(x^h, z^h\right)_{h \in \mathbf{H}}, p, q\right)$. *Consider a change in money supply to* $M' = (M'_0, ..., M'_S)$. *Then the new financial equilibrium allocation is still* $\left(x^h\right)_{h \in \mathbf{H}}$.

This result can equivalently be called the "neutrality of monetary policy."

## 4.3 Incomplete Markets

This section shows that with incomplete markets, nominal indeterminacy can imply real indeterminacy. This result that I'll show can equivalently be called the "non-neutrality of monetary policy." Much of the material in this section is borrowed from Chapter 7 of the Magill and Quinzii (1996) text.

Let's define the purchasing power of money for all states $s \in \mathbf{S} : \nu_s = \frac{1}{\sum_{l \in \mathbf{L}} p_l(s)}$. Define $\nu = (\nu_s)_{s>0}$ and the diagonal matrix $[\nu] = \begin{bmatrix} \nu_1 & 0 & 0 \\ 0 & .. & 0 \\ 0 & 0 & \nu_S \end{bmatrix}$.

We scale the payout matrix by the purchasing power of money. The budget constraints for states $s > 0$ are given by:

$$[p]\left(x^h - e^h\right) = [\nu] Rz,$$

where I define $[p] = \begin{bmatrix} 0 & \frac{p(1)}{\sum_{l \in \mathbf{L}} p_l(1)} & 0 & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & 0 & \frac{p(S)}{\sum_{l \in \mathbf{L}} p_l(S)} \end{bmatrix} \in \mathbb{R}^{S,G}$. Recall that the span of the payout matrix is $\langle [\nu] R \rangle = \left\{x \in \mathbb{R}^S : x = [\nu] Rz \text{ for some } z \in \mathbb{R}^J\right\}$. The budget constraints are equivalently expressed by the following two conditions.

1. Single budget constraint
$$(1, \alpha) [p] \left(e^h - x^h\right) \geq 0$$
for any vector of state prices $\alpha \in \mathbb{R}^S_{++}$.

2. Span condition
$$[p] \left(x^h - e^h\right) \in \langle [\nu] R \rangle .$$

We make the following assumptions.

Assumption M1: $\begin{bmatrix} 1 \\ : \\ 1 \end{bmatrix} \in \langle R \rangle$.

Assumption M2: $J < S$.

Assumption M3: $J \leq H$.

**Theorem 4.3** *Under Assumptions E1-E5 and M1-M3, there is a subspace* $\Delta \subset \mathbb{R}^S$ *with* $\dim \Delta \geq S - J$ *such that if a change in the monetary policy from* $M$ *to* $M'$ *implies a change from* $\nu$ *to* $\nu'$ *with* $\nu' - \nu \in \Delta$, *then* $\langle [\nu] R \rangle \neq \langle [\nu'] R \rangle$.

Before we walk through the proof of this theorem, let's see what changes of monetary policy do not satisfy the theorem's conditions. For $M = (M_0, M_1, ..., M_S)$, consider the change to $M' = (M_0', M_1', ..., M_S')$ such that $M_s' = \beta M_s \ \forall s > 0$ and for some $\beta > 0$. Then $[\nu'] = \frac{1}{\beta} [\nu]$, so the span remains unchanged: $\langle [\nu] R \rangle = \langle [\nu'] R \rangle$. Thus, for this proportional monetary policy, there are no real effects (the policy is neutral).

Now, let's consider the proof of the theorem.

**Proof.** The span of the payout matrix $\langle [\nu] R \rangle$ has dimension equal to $J$ (from Assumption E4). Thus, the space that is orthogonal to $\langle [\nu] R \rangle$, denoted $\langle [\nu] R \rangle^\perp$, has dimension equal to $S - J$. Define $\Delta = \langle [\nu] R \rangle^\perp \setminus \{0\}$. From the statement of the theorem, $\nu' - \nu \in \Delta$. It must be that $\nu' \notin \langle [\nu] R \rangle$ (if not, then $\nu' - \nu \in \langle [\nu] R \rangle^\perp$ and $\nu' - \nu \in \langle [\nu] R \rangle$, a contradiction of $\nu' \neq \nu$). Under Assumption M1, $\nu' \in \langle [\nu'] R \rangle$ (as there exists $z = (1, 0, ..., 0)$ such that $\nu' = [\nu'] Rz$). If $\nu' \notin \langle [\nu] R \rangle$ and $\nu' \in \langle [\nu'] R \rangle$, it can only be the case that $\langle [\nu] R \rangle \neq \langle [\nu'] R \rangle$. ■

Theorem 4.3 is nice, but if we make one additional assumption, then we are able to prove a much stronger result. Specifically, the subspace $\Delta$ of possible policy changes can be shown to have dimension $S - 1$ (bigger than $S - J$).

Assumption M4: $R$ is in general position, meaning that any $J \times J$ submatrix has rank $J$.

**Theorem 4.4** *Under Assumptions E1-E5 and M1-M4, there is a subspace* $\Delta \subset \mathbb{R}^S$ *with* $\dim \Delta = S - 1$ *such that if a change in the monetary policy from* $M$ *to* $M'$ *implies a change from* $\nu$ *to* $\nu'$ *with* $\nu' - \nu \in \Delta$, *then* $\langle [\nu] R \rangle \neq \langle [\nu'] R \rangle$ .

The proof of Theorem 4.4 is more challenging than the proof of Theorem 4.3, but our efforts are rewarded with the knowledge that the asset span will change for any policy changes in a $(S-1)$ −dimensional subset $\Delta$.

**Proof.** Suppose that $\langle [\nu] R \rangle = \langle [\nu'] R \rangle$ . This implies that for any $j \in \mathbf{J}$, the vector $[\nu'] r_j$ is a linear combination of the vectors $[\nu] r_1, ..., [\nu] r_J$. We can express these linear combinations using the $J \times J$ matrix $C$ such that:

$$[\nu'] R = [\nu] RC. \tag{4.1}$$

(4.1) written for any state $s > 0$ is equivalent to the following form:

$$C^T [r(s)]^T = \frac{\nu'_s}{\nu_s} [r(s)]^T . \tag{4.2}$$

If we recall our lessons on eigenvalues, (4.2) implies that $\frac{\nu'_s}{\nu_s}$ is an eigenvalue, with associated eigenvector $[r(s)]^T$ , of the matrix $C^T$. Notice that the same can be said for any state $s > 0$.

Now this is where it gets interesting. Further recalling our lessons on eigenvalues (a good reference is Strang, 2006), if two eigenvalues are distinct, then their corresponding eigenvectors must be linearly independent. Assume that the eigenvalue for state $s$ is different from the eigenvalue for state $\sigma$ : $\frac{\nu'_s}{\nu_s} \neq \frac{\nu'_\sigma}{\nu_\sigma}$. Then this implies that $[r(s)]^T$ is linearly independent from $[r(\sigma)]^T$ . With $J < S$, all vectors $[r(s)]^T$ cannot be linearly independent from the vectors for all other states $\sigma \neq s$. Thus, we are able to find a collection of $J$ or fewer vectors $\left\{ [r(s)]^T \right\}_{s \in \mathbf{S}^*}$ such that the number of states $s \in \mathbf{S}^*$ is greater than the rank of the submatrix $[r(s)]_{s \in \mathbf{S}^*}$ . This is a contradiction of the general position assumption (Assumption M4).[1]

We conclude that $\forall s, \sigma > 0 : \frac{\nu'_s}{\nu_s} = \frac{\nu'_\sigma}{\nu_\sigma} = \beta > 0$. Thus, the only changes from $\nu$ to $\nu'$ that do not change the span are of the form $\nu'_s = \beta \nu_s \;\; \forall s > 0$. These changes lie in the

[1]Adding a row vector $r(s)$ to a matrix increases the rank by at most 1. Add states to the submatrix $[r(s)]_{s \in \mathbf{S}^*}$ until there are $J$ rows. The rank of the new submatrix is then at most $J - 1$. General position requires that a submatrix with any $J$ rows of $R$ has full rank.

one-dimensional subspace defined by the vector $\nu$ :

$$\nu' - \nu \in \langle \nu \rangle .$$

All other changes, those in the set $\nu' - \nu \in \langle \nu \rangle^{\perp}$ , where $\langle \nu \rangle^{\perp}$ has dimension $S-1$, will change the span: $\langle [\nu] R \rangle \neq \langle [\nu'] R \rangle$ . ■

The following lemma is crucial for proving the main result of this chapter (Theorem 4.5). You will get the opportunity to verify Lemma 4.1 through a series of exercises (Exercises 1-4) at the end of the chapter.

**Lemma 4.1** *Under Assumptions E1-E5 and M1-M3, over a generic subset of household endowments and money supplies, the vectors* $\left(a^h\right)_{h\in\mathbf{H}}$ *span* $\langle [\nu] R \rangle$ , *where* $a^h \in \mathbb{R}^S$ *is defined by:*

$$a^h = [p] \left(x^h - e^h\right) \quad \forall h \in \mathbf{H}.$$

Equivalently, this lemma states that the collection of $J$ vectors $\left(a^h\right)_{h\in\mathbf{J}}$ are linearly independent. This brings us to the main result of the chapter (which can be equivalently called the "non-neutrality of monetary policy").

**Theorem 4.5** *Under Assumptions E1-E5 and M1-M4, over a generic subset of household endowments and money supplies, there is a subspace* $\Delta \subset \mathbb{R}^S$ *with* $\dim \Delta = S-1$ *such that all changes in the monetary policy from* $M$ *to* $M'$ *that induce a change from* $\nu$ *to* $\nu'$ *with* $\nu' - \nu \in \Delta$, *are non-neutral. That is, the financial equilibrium with money allocation changes:* $x = \left(x^h\right)_{h\in\mathbf{H}} \neq x' = \left(x'^h\right)_{h\in\mathbf{H}}$ .

**Proof.** From Theorem 4.4, $\langle [\nu] R \rangle \neq \langle [\nu'] R \rangle$ . As the dimensions $\dim \langle [\nu] R \rangle = \dim \langle [\nu'] R \rangle = J$ (Assumption E4), then $\dim \left(\langle [\nu] R \rangle \cap \langle [\nu'] R \rangle\right) < J$. By Lemma 4.1, the vectors $\left(a^h\right)_{h\in\mathcal{H}}$ span $\langle [\nu] R \rangle$ , so some of the vectors cannot be elements of the set $\langle [\nu] R \rangle \cap \langle [\nu'] R \rangle$ . Since the vectors $a'^h = [p'] \left(x'^h - e^h\right) \in \langle [\nu'] R \rangle \;\; \forall h \in \mathcal{H}$ by definition, then $\left(a^h\right)_{h\in\mathcal{H}} \neq \left(a'^h\right)_{h\in\mathcal{H}}$ . By the definition of $a^h = [p] \left(x^h - e^h\right)$ and first order conditions $Du^h\left(x^h\right) - \lambda^h P = 0$, then this implies $\left(x^h\right)_{h\in\mathcal{H}} \neq \left(x'^h\right)_{h\in\mathcal{H}}$ , finishing the argument. ■

## 4.4 Exercises

Exercises 1-4 walk through all steps required to prove Lemma 4.1.

1. Show that the vectors $\left(a^h\right)_{h\in\mathbf{J}}$ are linearly independent iff the portfolio vectors $\left(z^h\right)_{h\in\mathbf{J}}$ are linearly independent.

2. Write down the system of equilibrium equations $\Phi : \Xi\times\Theta \to \mathbb{R}^n$ for $n = H\left(G+S+1+J\right)+G+J$ so that $\xi = \left(\left(x^h,\lambda^h,z^h\right)_{h\in\mathbf{H}}, p, q\right)$ is a financial equilibrium with money iff $\Phi\left(\xi\right)=0$. Here $\theta = \left(\left(e^h\right)_{h\in\mathbf{H}}, M\right) \in \Theta$.

3. Use the mathematical tools from Section 1.4 and the results in Theorems 2.4 and 3.2 to show that, for a generic selection of households endowments $e = \left(e^h\right)_{h\in\mathbf{H}}$ and money supplies $M = \left(M_0, M_1, ..., M_S\right)$, the financial equilibria with money satisfy Finite Local Uniqueness. This requires proving that (i) $\pi$ is proper and (ii) if $\Phi\left(\xi,\theta\right)=0$, then $rankD\Phi\left(\xi,\theta\right) = n$ (the derivative is taken with respect to the variables $\xi$ and the parameters $\theta = \left(\left(e^h\right)_{h\in\mathbf{H}}, M\right)$). The proof of (i) is straightforward, so only prove (ii).

4. Given the outcome of Exercise 3 above, show that $rankD\Phi^*\left(\xi^*,\theta\right) = n+J+1$, where $\Phi^*\left(\eta\right) = \begin{pmatrix}\Phi\left(\eta\right)\\ \mu\cdot\left(z^1,...,z^J\right)\\ \mu^T\mu/2-1\end{pmatrix}$ (again taking derivatives with respect to the variables $\xi^* = \left(\left(x^h,\lambda^h,z^h\right)_{h\in\mathbf{H}}, p, q, \mu\right) \in \mathbb{R}^{n+J}$ and the parameters $\theta = \left(\left(e^h\right)_{h\in\mathbf{H}}, M\right)$). The additional equations $\mu\cdot\left(z^1,...,z^J\right)$, where $\mu \in \mathbb{R}^J\backslash\{0\}$ (nonzero from $\mu^T\mu/2 = 1$), imply that the portfolios vectors $\left(z^h\right)_{h\in\mathbf{J}}$ are linearly dependent. Using "A Different Application of Differential Topology" from Exercise 3 in Chapter 1, we can then conclude that for a generic selection of $\theta = \left(\left(e^h\right)_{h\in\mathbf{H}}, M\right)$, the portfolio vectors $\left(z^h\right)_{h\in\mathbf{J}}$ are linearly independent.

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# Appendix A

# Solutions to the Exercises

## A.1 Mathematical Prerequisites

1. (Applying Farkas Lemma)

   Let $R$ be a $m \times n$ matrix and $\theta \in \mathbb{R}^n$. Use Farkas Lemma to prove the following implication.

$$\begin{aligned} \text{There does not exist } \theta \text{ such that } R\theta &> 0 \\ &\Downarrow \\ \exists \lambda &\in \mathbb{R}^m_{++} \text{ s.t. } \lambda^T R = 0. \end{aligned}$$

   (Note: The other implication is a simple one-line proof. Make sure that you are proving the correct implication [i.e., the multi-page behemoth of a proof]).

   **Solution:**

   Define $R = \begin{bmatrix} R_1 \\ : \\ R_m \end{bmatrix}$. Consider the 1st row of the matrix $R$. The following process will be repeated for all $m$ rows of $R$. The statement "There does not exist $\theta$ such that $R\theta > 0$" implies that $\nexists \theta$ such that $R_1\theta > 0$ and $R_i\theta \geq 0 \;\; \forall i = 2, ..., m$. Thus, $\nexists \theta$ such that $R_1\theta > 0 \geq -R_i\theta \;\; \forall i = 2, ..., m$. Define

$$Z(1) = \left\{ z \in \mathbb{R}^n : z = \left[ -R_2^T, ..., R_m^T \right] \cdot \lambda(1) \text{ for } \lambda(1) \in \mathbb{R}^{m-1}_+ \right\}.$$

With this definition, $\nexists\theta$ such that $\theta^T R_1^T > 0 \geq \theta^T z \ \ \forall z \in Z(1)$. Use the following table to connect the statement of Farkas Lemma to the proof at hand.

| Farkas Lemma | | This proof |
|---|---|---|
| $q^* \in \mathbb{R}^n \backslash \{0\}$ | $\Longleftrightarrow$ | $\theta \in \mathbb{R}^n \backslash \{0\}$ |
| $z^* \in \mathbb{R}^n$ | $\Longleftrightarrow$ | $R_1^T \in \mathbb{R}^n$ |
| $A \in \mathbb{R}^{n,m-1}$ | $\Longleftrightarrow$ | $\left[-R_2^T, ..., R_m^T\right]$ |
| $\alpha \in \mathbb{R}_+^{m-1}$ | $\Longleftrightarrow$ | $\lambda(1) \in \mathbb{R}_+^{m-1}$ |

As condition (ii) of Farkas Lemma does not hold, then condition (i) must. This means that $R_1^T \in Z(1)$, so $\exists\lambda(1) \in \mathbb{R}_+^{m-1}$ such that

$$R_1^T = \left[-R_2^T, ..., R_m^T\right] \cdot \lambda(1).$$

Rearranging and taking transposes, we have:

$$R_1 + \sum_{i=1}^{m-1} \lambda_i(1) \cdot R_{i+1} = 0.$$

Define $\tilde{\lambda}(1) \in \mathbb{R}_+^m$ such that $\tilde{\lambda}_1(1) = 1$ and $\tilde{\lambda}_i(1) = \lambda_{i-1}(1)$ for $i = 2, ..., m$. Then $\left(\tilde{\lambda}(1)\right)^T \cdot R = 0$.

Repeating the entire process for the remaining rows $i = 2, ..., m$, then we obtain $\tilde{\lambda}(2), ..., \tilde{\lambda}(m) \in \mathbb{R}_+^m$ such that $\tilde{\lambda}_i(i) = 1 \ \ \forall i = 2, ..., m$. In all cases, $\left(\tilde{\lambda}(i)\right)^T \cdot R = 0$ $\forall i = 2, ..., m$.

Define $\lambda \in \mathbb{R}_{++}^m$ as $\lambda = \sum_{i=1}^{m} \tilde{\lambda}(i)$. By construction, $\lambda^T \cdot R = 0$. This completes the argument.

2. (Applying Kakutani's Fixed Point Theorem)

   We will use the theory of correspondences to prove the existence of a Nash equilibrium. Consider a game with $I$ players. Each player $i$ has a finite number of actions $\left\{a_1^i, ...., a_{J_i}^i\right\}$. The set of strategies for each player $i$ is then the simplex $\Delta^{J_i - 1}$ of dimension $J_i - 1$. The strategies are simply the probabilities that each player assigns to each of the finite number of actions. For simplicity, denote the strategy set for each player as $S^i$ with element $s^i$. These sets are nonempty, compact, and convex.

If player $i's$ payoff value for the action profile $a = (a_1, ..., a_i, ..., a_I)$ is $p^i(a)$, then the objective function for each player is (using the convention $s = (s_1, ..., s_i, ..., s_I)$):

$$u^i(s) = \sum\nolimits_a p^i(a) \cdot \prod\nolimits_i s^i\left(a^i\right),$$

where $s^i\left(a^i\right)$ is the probability that player $i$ selects the action $a^i$. This objective function is quasi-concave in $s$ (as it's linear).

Denote $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_I)$ as the vector of strategies (not including the strategy for player $i$) and $\underset{j \neq i}{\times} S^j = S^1 \times ... \times S^{i-1} \times S^{i+1} \times ... \times S^I$ as the Cartesian product of the strategy sets (not including the strategy set for player $i$). Define $F^i : \underset{j \neq i}{\times} S^j \rightrightarrows S^i$ as the set of "feasible" strategies. Notice that $F^i\left(s_{-i}\right) = S^i \quad \forall s_{-i} \in \underset{j \neq i}{\times} S^j$.

Define $BR^i : \underset{j \neq i}{\times} S^j \rightrightarrows S^i$ as the best response correspondence for player $i$. Use the Berge's Maximum Theorem and Kakutani's Fixed Point Theorem to prove that the self-map $\left(BR^1, ..., BR^I\right) : \underset{i}{\times} S^i \rightrightarrows \underset{i}{\times} S^i$ has a fixed point. By definition, a fixed point is a Nash equilibrium.

**Solution:**

By definition, $S^i = \Delta^{J_i - 1}$ is compact, convex, and nonempty. These properties are maintained through the Cartesian product operator, meaning that $\underset{i}{\times} S^i$ is compact, convex, and nonempty.

The objective function is continuous and the strategy set is compact. Using the Extreme Value Theorem, the best response correspondence $BR^i$ is well-defined.

The objective function is quasi-concave (as it's linear) and the strategy set is convex. Consider any two $x, y \in BR^i\left(s_{-i}\right)$ for some $s_{-i} \in \underset{j \neq i}{\times} S^j$. By convexity of $S^i$, then $\theta x + (1-\theta)y \in S^i$. By quasi-concavity, $u^i\left(\theta x + (1-\theta)y\right) \geq u^i(x) = u^i(y)$. This implies that $\theta x + (1-\theta)y \in BR^i\left(s_{-i}\right)$, meaning that $BR^i$ is convex-valued.

Define the constraint correspondence $C^i : \underset{j \neq i}{\times} S^j \rightrightarrows S^i$. The correspondence is defined such that $C^i\left(s_{-i}\right) = S^i \quad \forall s_{-i} \in \underset{j \neq i}{\times} S^j$. Given this definition, the best response correspondence is formally defined as:

$$BR^i\left(s_{-i}\right) = \underset{s_i \in C^i(s_{-i})}{\arg\max}\ u^i(s).$$

The objective function $u^i(s)$ is continuous. Given that the codomain $S^i$ is compact and $C^i$ maps into the entire set $S^i$, the correspondence $C^i$ is both uhc and lhc. From Berge's Maximum Theorem, $BR^i$ is a uhc correspondence.

The vector of correspondences $\left(BR^1, ..., BR^I\right)$ has the same properties as each of its elements: well-defined, convex-valued, and uhc. Applying Kakutani's Fixed Point Theorem to the self-map $\left(BR^1, ..., BR^I\right) : \underset{i}{\times} S^i \rightrightarrows \underset{i}{\times} S^i$ guarantees the existence of a fixed point.

3. (A Different Application of Differential Topology)

   Let $\Xi \subset \mathbb{R}^J$ be the set of variables (with typical element $\xi$), $\Theta \subset \mathbb{R}^K$ be the set of parameters (with typical element $\theta$), $H = \Xi \times \Theta$ (so $\eta = (\xi, \theta)$), and $\Phi : H \rightarrow \mathbb{R}^L$ be the $C^1$ system of equations. Assume that $J < L$. Suppose that the projection $\pi$ is proper and the rank condition holds (that is, $rank D\Phi(\eta) = L$). Using the mathematical results from Section 1.4, what conclusions can be drawn? The conclusions will be of the form, "over a generic subset of parameters (the set of regular values $\Theta^r$), then...".

   **Solution:**

   With $J < L$, then all solutions $\eta \in M$ are critical points, as they must necessarily satisfy $rank D_\xi \Phi(\eta) < L$. Thus, $\pi(M)$ are critical values. Let's suppose that we are able to verify the two key properties of regularity: (i) $\pi$ is proper and (ii) $rank D\Phi(\eta) = L$ $\forall \eta \in M$. Then the Closedness Theorem and the Transversality Theorem allow us to conclude that the set of critical values is a closed subset of measure zero (equivalently, the set of regular values is an open subset of full measure). Thus, $\pi(M)$ is a closed subset of measure zero, where $\pi(M) = \{\theta : \exists \xi \text{ such that } \Phi(\xi, \theta) = 0\}$ by definition.

   In conclusion, over a generic subset of parameters (the set of regular values $\Theta^r = \Theta \backslash \pi(M)$), there does not exist a solution $\xi$ to the system of equations $\Phi(\xi, \theta) = 0$.

## A.2 Arrow-Debreu Model

1. Show that if $u^h$ is locally non-satiated $\forall h \in \mathcal{H}$, then the Arrow-Debreu equilibrium prices satisfy $p > 0$.

   **Solution:**

   To begin, I show that in any Arrow-Debreu equilibrium $\left(\left(x^h\right)_{h \in \mathbf{H}}, p\right)$, an optimal solution to the household problem $(HP)$ must satisfy $px^h = pe^h$ $\forall h \in \mathbf{H}$. Suppose otherwise, that $px^h < pe^h$ for some $h$. By local non-satiation, $\exists y^h \in N_\epsilon\left(x^h\right) \cap X^h$ such that $u^h\left(y^h\right) > u^h\left(x^h\right)$. The value for $\epsilon$ can be made small enough so that $py^h < pe^h$, contradicting that $x^h$ is an optimal solution to the household problem $(HP)$.

The initial price space is $p \in \mathbb{R}^G \backslash \{0\}$. To prove the claim, suppose that $p_g < 0$ for some $g \in \mathbf{G}$. Suppose that $x^h$ is the equilibrium consumption choice for household $h$. From the previous paragraph, $px^h = pe^h$. Define $\tilde{x}^h$ such that $\tilde{x}^h_g = x^h_g + \delta$ and $\tilde{x}^h_{g'} = x^h_{g'}$ $\forall g' \neq g$. By linearity, $p\tilde{x}^h < pe^h$. As $\tilde{x}^h$ is not optimal, it must be that $u^h\left(x^h\right) > u^h\left(\tilde{x}^h\right)$. By local non-satiation, $\exists y^h \in N_\epsilon\left(x^h\right) \cap X^h$ such that $u^h\left(y^h\right) > u^h\left(x^h\right)$. For $x^h$ to be optimal, it must be that $py^h > pe^h$. Define $\tilde{y}^h$ such that $\tilde{y}^h_g = y^h_g + \delta$ and $\tilde{y}^h_{g'} = y^h_{g'}$ $\forall g' \neq g$. There exists $\underline{\delta}$ such that $\forall \delta \geq \underline{\delta}$, $p\tilde{y}^h \leq pe^h$. For $x^h$ to be optimal, it must be that $u^h\left(x^h\right) \geq u^h\left(\tilde{y}^h\right)$. We have $u^h\left(y^h\right) > u^h\left(x^h\right) \geq u^h\left(\tilde{y}^h\right)$. Letting $(\epsilon, \underline{\delta}) \to 0$, we see that $u^h$ is strictly decreasing with $x^h_g$. This implies that the optimal solution to $(HP)$ lies on the boundary: $x^h \notin intX^h$. This violates local non-satiation as $\nexists z^h \in N_\epsilon\left(x^h\right) \cap X^h$ such that $u^h\left(z^h\right) > u^h\left(x^h\right)$.

2. Assume that $u^h$ is differentiable, differentiably strictly increasing, and concave $\forall h \in \mathbf{H}$. Using the results from the programming problems in this chapter, show that if $\left(\left(x^h\right)_{h\in\mathbf{H}}, p\right)$ is an Arrow-Debreu equilibrium, then the allocation $\left(x^h\right)_{h\in\mathbf{H}}$ is an optimal solution to the problem $(PO_{h'})$ for any $h'$. From Section 2.3, the allocation $\left(x^h\right)_{h\in\mathbf{H}}$ is then Pareto optimal. As this was the first proof of the First Basic Welfare Theorem, it is often called the "classical proof."

**Solution:**

The following are necessary and sufficient conditions for an Arrow-Debreu equilibrium:

$$\begin{array}{lll} Du^h(x^h) - \lambda^h p = 0 & \forall h \in \mathbf{H} & (FOC) \\ \sum_{h\in\mathbf{H}} \left(e^h_g - x^h_g\right) = 0 & \forall g \in \mathbf{G} & (MC) \end{array} .$$

The following are necessary and sufficient conditions for an optimal solution to $(PO_{h'})$ for household $h' = 1$ (without loss of generality). Let $\left(\tilde{\lambda}^h\right)_{h>1}$ be the Lagrange multipliers for the constraints $u^h(x^h) \geq u^h(x^{*h})$ and $\left(\mu_g\right)_{g\in\mathbf{G}}$ be the Lagrange multipliers for the constraints $\left(x^h\right)_{h\in\mathbf{H}} \in FA$, namely $\sum_{h\in\mathbf{H}} \left(e^h_g - x^h_g\right) = 0 \quad \forall g \in \mathbf{G}$ :

$$\begin{array}{ll} Du^1(x^1) - \mu = 0 & (FOC1) \\ \tilde{\lambda}^h Du^h(x^h) - \mu = 0 \;\; \forall h > 1 & (FOCh) \\ \sum_{h\in\mathbf{H}} \left(e^h_g - x^h_g\right) = 0 \quad \forall g \in \mathbf{G} & (FA) \end{array}.$$

To show that both systems of equations yield the same optimal allocation, define $\mu = \lambda^1 p >> 0$ and $\tilde{\lambda}^h = \frac{\lambda^1}{\lambda^h} > 0$.

3. Consider an economy with two households and two goods. Both utility functions are differentiable, strictly increasing, and strictly concave, but there is a missing market for the second good. That is, each household can consume no more than its initial endowment of the second good. Using an Edgeworth box, show that the equilibrium allocations are typically Pareto suboptimal.

**Solution:**

The Edgeworth box in Figure A.1 shows that the equilibrium allocation occurs at the endowment point. As the utility functions are strictly increasing, equilibrium prices must satisfy $p >> 0$. For any such prices, the only allocation that lies in both the budget set for $h = 1$ and the budget set for $h = 2$ is the endowment point $e$. This is an autarchic equilibrium.

The Edgeworth box in Figure A.2 shows that the set of Pareto optimal allocations (called the Pareto set) will typically not pass through the endowment point $e$. The Pareto set is defined (given the assumptions on the utility functions) as points where an indifference curve for $h = 1$ is tangent to an indifference curve for $h = 2$. Thus, the equilibrium allocation will be Pareto suboptimal.

4. Consider an economy with two households and two goods. The two households have utility functions

$$\begin{aligned} u^1\left(x^1\right) &= \frac{1}{2}\log\left(x^1_1 + x^2_1\right) + \frac{1}{2}\log\left(x^1_2\right) \\ u^2\left(x^2\right) &= \frac{1}{2}\log\left(x^2_1\right) + \frac{1}{2}\log\left(x^2_2\right) \end{aligned}$$

and endowments $e^1 = (1, 2)$ and $e^2 = (2, 1)$. Notice that household 1 cares about household 2's consumption of the first good ($x_1^2$). Using the programming problems from this chapter, characterize the Pareto optimal allocations and then characterize the equilibrium allocations. From this, can you claim that the First Basic Welfare Theorem holds? Use an Edgeworth box to illustrate your argument.

**Solution:**

The Pareto optimal allocations are $x^1 = (0, \theta)$ and $x^2 = (3, 3 - \theta)$ for any $\theta \in [0, 3]$. This is illustrated in the Edgeworth box below.

To find the Arrow-Debreu equilibrium, normalize $p_2 = 1$. The first order conditions for household $h = 2$ are given by:

$$\begin{aligned} \frac{0.5}{x_1^2} - \lambda^2 p_1 &= 0. \\ \frac{0.5}{x_2^2} - \lambda^2 &= 0. \end{aligned}$$

Solving for $x^2$ as a function of $\lambda^2$ and plugging into the budget constraint $p_1 (x_1^2 - 2) + (x_2^2 - 1) = 0$ yields:

$$\frac{0.5}{\lambda^2} + \frac{0.5}{\lambda^2} = 2p_1 + 1.$$

Thus, $\frac{1}{\lambda^2} = 2p_1 + 1$, so the demand functions are:

$$x_1^2 (p_1) = \frac{p_1 + 0.5}{p_1} \text{ and } x_2^2 (p_1) = p_1 + 0.5.$$

The first order conditions for household $h = 1$ are given by:

$$\begin{aligned} \frac{0.5}{x_1^1 + x_1^2} - \lambda^1 p_1 &= 0. \\ \frac{0.5}{x_2^1} - \lambda^1 &= 0. \end{aligned}$$

This means that $x_2^1 = \frac{0.5}{\lambda^1}$ and $x_1^1 + x_1^2 = \frac{0.5}{\lambda^1 p_1}$. By market clearing, this implies that $\frac{0.5}{\lambda^1 p_1} = 3$, so $\lambda^1 = \frac{1}{6p_1}$. This means that $x_2^1 = 3p_1$.

Market clearing requires that $x_2^1 + x_2^2 = 3$. From the demand functions, $3p_1 + p_1 + 0.5 = 3$, meaning that $p_1 = \frac{5}{8}$. This implies an equilibrium allocation given by:

$$\begin{aligned} x^1 &= (1.2, 1.875). \\ x^2 &= (1.8, 1.125). \end{aligned}$$

This allocation is plotted in the Edgeworth box in Figure A.3. The equilibrium allocation does not lie in the Pareto set, which is shown in red. This is a violation of the First Basic Welfare Theorem.

5. Consider an economy with two households and two goods. Prove that if the utility functions are Cobb-Douglas (of the form $u^h\left(x^h\right) = \alpha_1 \log\left(x_1^h\right) + \alpha_2 \log\left(x_2^h\right)$), then there is a unique Arrow-Debreu equilibrium. Does this property hold for an economy with more than two households and more than two goods?

   **Solution:**

   For the two good economy, we will solve for the demand functions. Normalize the price $p_2 = 1$. For any household $h$, the first order conditions are given by:

$$\begin{aligned} \frac{\alpha_1}{x_1^h} - \lambda^h p_1 &= 0. \\ \frac{\alpha_2}{x_2^h} - \lambda^h &= 0. \end{aligned}$$

   Solving for $x^h$ as a function of $\lambda^h$ and plugging into the budget constraint $p_1\left(x_1^h - e_1^h\right) + \left(x_2^h - e_2^h\right) = 0$ yields:

$$\frac{\alpha_1}{\lambda^h} + \frac{\alpha_2}{\lambda^h} = p_1 e_1^h + e_2^h.$$

   This means that $\frac{1}{\lambda^h} = \frac{1}{(\alpha_1+\alpha_2)}\left(p_1 e_1^h + e_2^h\right)$. The demand functions are given by:

$$x_1^h(p_1) = \frac{\alpha_1}{(\alpha_1+\alpha_2)}\left(\frac{p_1 e_1^h + e_2^h}{p_1}\right) \text{ and } x_2^h(p_1) = \frac{\alpha_2}{(\alpha_1+\alpha_2)}\left(p_1 e_1^h + e_2^h\right).$$

   Market clearing requires that $\sum_{h\in\mathbf{H}} x_1^h(p_1) = \sum_{h\in\mathbf{H}} e_1^h$. Given that the coefficients are household independent, then

$$\sum_{h\in\mathbf{H}} x_1^h(p_1) = \frac{\alpha_1}{p_1(\alpha_1+\alpha_2)} \cdot \left(p_1 \sum_{h\in\mathbf{H}} e_1^h + \sum_{h\in\mathbf{H}} e_2^h\right).$$

The market clearing condition $\sum_{h\in\mathbf{H}} x_1^h(p_1) = \sum_{h\in\mathbf{H}} e_1^h$ is then a linear function of $p_1$, meaning that there exists a unique price vector, and subsequently a unique equilibrium allocation. The market clearing condition allows us to write a closed form expression for the price $p_1$ :

$$\frac{\alpha_1}{p_1(\alpha_1+\alpha_2)} \cdot \left(p_1 \sum_{h\in\mathbf{H}} e_1^h + \sum_{h\in\mathbf{H}} e_2^h\right) = \sum_{h\in\mathcal{H}} e_1^h$$
$$\left(\frac{\alpha_1}{(\alpha_1+\alpha_2)} \cdot p_1 \sum_{h\in\mathbf{H}} e_1^h + \frac{\alpha_1}{(\alpha_1+\alpha_2)} \cdot \sum_{h\in\mathbf{H}} e_2^h\right) = p_1 \sum_{h\in\mathbf{H}} e_1^h \quad .$$
$$p_1 = \frac{\alpha_1 \cdot \sum_{h\in\mathbf{H}} e_2^h}{\alpha_2 \cdot \sum_{h\in\mathbf{H}} e_1^h}$$

If we repeat this procedure for an economy with an arbitrary number of goods and an arbitrary number of households, the demand functions are of the form:

$$x_g^h\left(p_{\setminus G}\right) = \frac{\alpha_g}{\sum_{g\in\mathbf{G}} \alpha_g} \left(\frac{pe^h}{p_g}\right) \text{ for } g < G \text{ and } x_G^h\left(p_{\setminus G}\right) = \frac{\alpha_G}{\sum_{g\in\mathbf{G}} \alpha_g} \left(pe^h\right).$$

Here we have normalized the price $p_G = 1$ and only consider the prices $p_{\setminus G} = (p_1, ..., p_{G-1})$. The market clearing conditions for the goods $g < G$ yield equations of the following form:

$$\frac{\alpha_g}{\sum_{g\in\mathbf{G}} \alpha_g} \left(p \sum_{h\in\mathbf{H}} e^h\right) = p_g \sum_{h\in\mathbf{H}} e_g^h \quad \forall g < G.$$

These linear equations ($G-1$ in total) have a unique solution for $p_{\setminus G}$. Thus, Cobb-Douglas utility always leads to a unique equilibrium allocation.

6. Show that the projection $\pi$ is proper. $\pi$ is the projection $\pi : \mathbb{R}_{++}^{H(G+1)+G-1} \times \mathbb{R}_{++}^{HG} \to \mathbb{R}_{++}^{HG}$ which maps $\left(\left(x^h, \lambda^h\right)_{h\in\mathbf{H}}, p, \left(e^h\right)_{h\in\mathbf{H}}\right) \mapsto \left(e^h\right)_{h\in\mathbf{H}}$ such that $\Phi\left(\left(x^h, \lambda^h\right)_{h\in\mathbf{H}}, p; \left(e^h\right)_{h\in\mathbf{H}}\right) = 0$.

**Solution:**

A projection is by definition a continuous mapping. Consider a compact set of endowments $\Theta'$. We have left to show that $\pi^{-1}(\Theta')$ is a compact set. As consumption is nonnegative and the endowments $\sum_{h\in\mathbf{H}} e_g^h$ are bounded $\forall g \in \mathbf{G}$, then the equilibrium consumption $\left(x^h\right)_{h\in\mathbf{H}}$ also belong to a compact set. From the first order conditions with respect to the good $G$, we have:

$$D_{x_G}u^h\left(x^h\right)-\lambda^h=0.$$

Recall that we have normalized the price $p_G=1$. As $D_{x_G}u^h\left(x^h\right)$ is bounded ($Du^h$ is continuous), then $\left(\lambda^h\right)_{h\in\mathcal{H}}$ belongs to a compact set. From the first order conditions with respect to all other goods $g<G$ :

$$D_{x_g}u^h\left(x^h\right)-\lambda^h p_g=0\quad\forall g<G.$$

As all terms are bounded, then $\left(p_g\right)_{g<G}$ belongs to a compact set.

Thus, the set $\pi^{-1}\left(\Theta'\right)$ is compact, finishing the argument that $\pi$ is proper.

7. Prove that for any endowments $\left(e^h\right)_{h\in\mathbf{H}}$ within an open set of allocations around a Pareto optimal allocation, the resulting Arrow-Debreu equilibrium is unique. To do this, you must verify that the matrix $D_{\left(x^h,\lambda^h\right)_{h\in\mathbf{H}},p}\Phi\left(\left(x^h,\lambda^h\right)_{h\in\mathbf{H}},p\right)$ has full rank (square matrix) given that the initial endowment is a Pareto optimal allocation $\left(e^h\right)_{h\in\mathbf{H}}=\left(x^{h*}\right)_{h\in\mathbf{H}}$. Notice that the derivative matrix only contains derivatives with respect to variables, not the endowment $e^1$.

**Solution:**

I will show that the matrix $M=D_{\left(x^h,\lambda^h\right)_{h\in\mathbf{H}},p}\Phi\left(\left(x^h,\lambda^h\right)_{h\in\mathbf{H}},p\right)$ has full row rank when $\left(e^h\right)_{h\in\mathbf{H}}=\left(x^{h*}\right)_{h\in\mathbf{H}}$, a Pareto optimal allocation. The FLU result then implies that there exists an open neighborhood around the Pareto optimal allocation such that for any endowment in this open neighborhood the equilibia are locally unique. This means that the number of equilibria for any endowment within this set must be equal to the number of equilibria at the endowment $\left(e^h\right)_{h\in\mathbf{H}}=\left(x^{h*}\right)_{h\in\mathbf{H}}$ (in other words, 1 equilibrium). This verifies uniqueness.

The rows of $M$ correspond to the equations in $\Phi\left(\left(x^h,\lambda^h\right)_{h\in\mathbf{H}},p\right)$, while the columns correspond to the variables that we are taking derivatives with respect to. The derivative matrix is given by:

$$M=\begin{bmatrix} ... & ... & 0 & 0 & 0 & 0 & : \\ ... & ... & 0 & 0 & 0 & 0 & : \\ 0 & 0 & D^2u^h\left(x^h\right) & -p^T & 0 & 0 & \begin{pmatrix} -\lambda^h I_{G-1} \\ 0 \end{pmatrix} \\ 0 & 0 & -p & 0 & 0 & 0 & \left(e^h_{\backslash G}-x^h_{\backslash G}\right)^T \\ 0 & 0 & 0 & 0 & ... & ... & : \\ 0 & 0 & 0 & 0 & ... & ... & : \\ ... & ... & \begin{pmatrix} -I_{G-1} & 0 \end{pmatrix} & 0 & ... & ... & 0 \end{bmatrix}.$$

To show that the matrix $M$ has full rank, we set $\nu^T M=0$ and must verify $\nu^T=0$, where $\nu\in\mathbb{R}^{H(G+1)+G-1}$ corresponds to equations in $\Phi$ :

$$\nu=\begin{pmatrix} : \\ \Delta x^h \\ \Delta\lambda^h \\ : \\ \Delta p \end{pmatrix}\begin{pmatrix} : \\ FOCh \\ BCh \\ : \\ MC \end{pmatrix}.$$

The equations $\nu^T M=0$ are given by:

$$\begin{array}{ll} : & \\ \left(\Delta x^h\right)^T D^2u^h\left(x^h\right)-\Delta\lambda^h p-\Delta p^T\left[\begin{array}{cc} I_{G-1} & 0 \end{array}\right]=0 & (A.1.a) \\ -\left(\Delta x^h\right)^T p^T=0 & (A.1.b) \\ : & \\ -\sum_{h\in\mathbf{H}}\lambda^h\left(\Delta x^h_{\backslash G}\right)^T+\sum_{h\in\mathbf{H}}\Delta\lambda^h\left(e^h_{\backslash G}-x^h_{\backslash G}\right)^T=0 & (A.1.c) \end{array}. \qquad (A.1)$$

As $\left(e^h\right)_{h\in\mathbf{H}}=\left(x^{h*}\right)_{h\in\mathbf{H}}$, then the term $\sum_{h\in\mathbf{H}}\Delta\lambda^h\left(e^h_{\backslash G}-x^h_{\backslash G}\right)^T=0$ in $(A.1.c)$. Let's show that $\nu^T=0$ in three steps:

(a) Postmultiply $(A.1.a)$ by $\Delta x^h$ :

$$\left(\Delta x^h\right)^T D^2u^h\left(x^h\right)\Delta x^h-\Delta\lambda^1 p\Delta x^h-\left(\Delta p_1,...,\Delta p_{G-1},0\right)\Delta x^h=0.$$

From $(A.1.b)$, $p\Delta x^h = 0$. This implies

$$\lambda^h \cdot \left(\Delta x^h\right)^T D^2 u^h \left(x^h\right) \Delta x^h = \lambda^h \cdot \sum_{g<G} \Delta p_g \Delta x_g^h, \tag{A.2}$$

where $\lambda^h > 0$ is multiplied by both sides.

(b) From $(A.1.c)$ :

$$\sum_{h \in \mathbf{H}} \lambda^h \cdot \left(\Delta x_1^h, ..., \Delta x_{G-1}^h\right) = 0.$$

(c) Add the equation $(A.2)$ over all households $h \in \mathcal{H}$. Using $(A.2)$, this implies

$$\sum_{h \in \mathbf{H}} \lambda^h \cdot \left(\Delta x^h\right)^T D^2 u^h \left(x^h\right) \Delta x^h = \sum_{g<G} \Delta p_g \left(\sum_{h \in \mathbf{H}} \lambda^h \cdot \Delta x_g^h\right) = 0. \tag{A.3}$$

As $u^h$ is differentiably strictly concave, the Hessian matrix $D^2 u^h \left(x^h\right)$ is negative definite. This means $\lambda^h \cdot \left(\Delta x^h\right)^T D^2 u^h \left(x^h\right) \Delta x^h \leq 0 \;\; \forall h \in \mathbf{H}$, with equality iff $\left(\Delta x^h\right)^T = 0$. $(A.3)$ implies $\left(\Delta x^h\right)_{h \in \mathbf{H}} = 0$. From $(A.1.a)$,

$$\Delta \lambda^h \left(p_1, ..., p_{G-1}, 1\right) + \left(\Delta p_1, ..., \Delta p_{G-1}, 0\right) = 0.$$

As $p >> 0$, then $\left(\Delta \lambda^h\right)_{h \in \mathbf{H}} = 0$. Consequently, $\Delta p^T = 0$.

Thus, $\nu^T = \left(..., \left(\Delta x^h\right)^T, \Delta \lambda^h, ..., \Delta p^T\right) = 0$, finishing the argument.

8. Using "A Different Application of Differential Topology" from Exercise 3 in Chapter 1, prove that over a generic subset of endowments, if $G > 1$, then $p_1 \neq p_2$.

**Solution:**

We utilize Assumptions R1-R3. Consider the system of equations $\Phi^* (\eta) = \begin{pmatrix} \Phi(\eta) \\ p_1 - p_2 \end{pmatrix}$.
As $L = J + 1 > J$, if I can show that $\pi$ is proper (same proof as Exercise 6) and $rankD\Phi^* (\eta) = L \;\; \forall \eta : \Phi^* (\eta) = 0$, then using the result from Exercise 3 in Chapter 1, over a generic subset of endowments, there does not exist a solution $\xi$ to the system of equations $\Phi^* (\xi, \theta) = 0$. This implies that any Arrow-Debreu equilibrium $\xi$ (defined such that $\Phi (\xi, \theta) = 0$) must be such that $p_1 - p_2 \neq 0$.

To show that $rankD\Phi^* (\eta) = L$, repeat the exact same three steps as in Section 2.5. Notice that these steps do not involve using the columns for the derivatives with respect to the price variables. This shows that the first $J$ equations of $\Phi^*$ are linearly independent. For the final equation of $\Phi^*$, use the column for the derivatives with respect to $p_1$ or $p_2$ to show that the final row is linearly independent from the rest. This verifies that the matrix $D\Phi^*(\eta)$ has full row rank $L$.

9. As a general result of the ideas in Exercise 3 above, show that over a generic subset of endowments, if there is a missing market for one of the goods, then the allocation $\left(x^h\right)_{h\in\mathbf{H}}$ is not Pareto optimal. To attack this problem, use "A Different Application of Differential Topology" from Exercise 3 in Chapter 1 and consider that a necessary condition for Pareto optimality is $\frac{D_g u^h\left(x^h\right)}{D_{g'}u^h\left(x^h\right)} = \frac{D_g u^{h'}\left(x^{h'}\right)}{D_{g'}u^{h'}\left(x^{h'}\right)}$ $\forall\left(h,h',g,g'\right)\in\mathbf{H}\times\mathbf{H}\times\mathbf{G}\times\mathbf{G}$.

**Solution:**

We utilize Assumptions R1-R3. Suppose wlog that the missing market is for good $g=1$. Then in equilibrium, $x_1^h = e_1^h$ $\forall h\in\mathbf{H}$. The equilibrium variables are $\xi = \left(\left(x^h,\mu^h,\lambda^h\right)_{h\in\mathbf{H}},p\right)$ and the equilibrium system of equations is

$$\Phi(\eta) = \begin{pmatrix} \begin{pmatrix} D_{x_1}u^h\left(x^h\right)-\mu^h-\lambda^h p_1 = 0 \\ \left(D_{x_g}u^h\left(x^h\right)-\lambda^h p_g = 0\right)_{1<g<G} \\ D_{x_G}u^h\left(x^h\right)-\lambda^h = 0 \\ e_1^h - x_1^h \\ p\left(e^h-x^h\right) \end{pmatrix}_{h\in\mathbf{H}} \\ \sum_{h\in\mathbf{H}}\left(e_{\setminus G}^h - x_{\setminus G}^h\right) \end{pmatrix}.$$

The number of equations in $\Phi$ equals the number of variables in $\xi$. From the first order conditions, $\frac{D_{x_1}u^h\left(x^h\right)}{D_{x_g}u^h\left(x^h\right)} = \frac{\mu^h+\lambda^h p_1}{\lambda^h p_g}$ $\forall g>1$. This means that $\frac{D_{x_1}u^h\left(x^h\right)}{D_{x_g}u^h\left(x^h\right)} = \frac{D_{x_1}u^{h'}\left(x^{h'}\right)}{D_{x_g}u^{h'}\left(x^{h'}\right)}$ iff $\frac{\mu^h}{\lambda^h} = \frac{\mu^{h'}}{\lambda^{h'}}$.

Consider the system of equations $\Phi^*(\eta) = \begin{pmatrix} \Phi(\eta) \\ \mu^1\lambda^2-\mu^2\lambda^1 \end{pmatrix}$. As $L=J+1>J$, if I can show that $\pi$ is proper (same proof as Exercise 6) and $rankD\Phi^*(\eta) = L$ $\forall\eta:\Phi^*(\eta)=0$, then using the result from Exercise 3 in Chapter 1, over a generic subset of endowments, there does not exist a solution $\xi$ to the system of equations $\Phi^*(\xi,\theta)=0$. This implies that any equilibrium $\xi$ (defined such that $\Phi(\xi,\theta)=0$) must be such that $\mu^1\lambda^2-\mu^2\lambda^1\neq 0$. Without $\frac{\mu^1}{\lambda^1}=\frac{\mu^2}{\lambda^2}$, the equilibrium allocation must be Pareto suboptimal.

The rows of $M = D\Phi^*(\eta)$ correspond to the equations in $\Phi^*$, while the columns correspond to the variables and parameters (endowment vectors $e^1$ and $e^2$) that we are taking derivatives with respect to. To show that the matrix $M$ has full rank, we set $\nu^T M = 0$ and must verify $\nu^T = 0$, where $\nu \in \mathbb{R}^{H(G+2)+G-1}$ corresponds to equations in $\Phi$ :

$$\nu = \begin{pmatrix} \vdots \\ \Delta x^h \\ \Delta \mu^h \\ \Delta \lambda^h \\ \vdots \\ \Delta p \\ \Delta EE \end{pmatrix} \begin{pmatrix} \vdots \\ FOCh \\ e_1^h - x_1^h \\ BCh \\ \vdots \\ MC \\ \mu^1\lambda^2 - \mu^2\lambda^1 \end{pmatrix}.$$

Given that $x_1^h = e_1^h$, the budget constraint is written only for the remaining $G-1$ goods.

From the columns corresponding to derivatives with respect to $(e^1, e^2)$, we obtain (as in Section 2.5) $(\Delta\mu^1, \Delta\lambda^1) = (\Delta\mu^2, \Delta\lambda^2) = 0$ and $\Delta p^T = 0$. For all households $h \in \mathbf{H}$, the columns corresponding to derivatives with respect to $x^h$ imply:

$$\left(\Delta x^h\right)^T D^2 u^h\left(x^h\right) + \left(\Delta\mu^h, \Delta\lambda^h\right) \begin{pmatrix} -1 & \overrightarrow{0} \\ & -p \end{pmatrix} = 0. \tag{A.4}$$

For the households $h > 2$, the columns corresponding to derivatives with respect to $\left(\mu^h, \lambda^h\right)$ imply:

$$\left(\Delta x^h\right)^T \begin{pmatrix} -1 & \overrightarrow{0} \\ & -p \end{pmatrix}^T = 0.$$

The previous two equations imply that $\left(\Delta x^h\right)^T D^2 u^h\left(x^h\right) \Delta x^h = 0 \;\; \forall h \in \mathbf{H}$, and thus $\left(\Delta x^h\right)^T = 0 \;\; \forall h \in \mathbf{H}$ using the assumption that $u^h$ is differentiably strictly concave. From (A.4), $\left(\Delta\mu^h, \Delta\lambda^h\right) = 0 \;\; \forall h \in \mathbf{H}$.

For the column for the derivatives with respect to $\mu^1$, then $\Delta EE = 0$ (as $\lambda^2 > 0$).

Thus, $\nu^T = 0$, finishing the argument.

# A.3 General Financial Model

1. Show that No Arbitrage is a necessary condition of equilibrium. That is, show that if No Arbitrage does not hold, then a general financial equilibrium does not exist.

   **Solution:**

   Assume that No Arbitrage does not hold. Then $\exists z^h$ such that $\begin{pmatrix} -q \\ R \end{pmatrix} z^h > 0$. Such a portfolio $z^h \in \mathbb{R}^J$ and there always exists a portfolio $\tilde{z}^h = \kappa \cdot z^h$ for $\kappa > 1$ such that $\begin{pmatrix} -q \\ R \end{pmatrix} \tilde{z}^h > \begin{pmatrix} -q \\ R \end{pmatrix} z^h$. As the asset set $\mathbb{R}^J$ is unbounded, then there will never exist a solution to the household problem $(HP)$.

2. Suppose that there are $S = 4$ states and $J = 2$ assets with payouts

$$R = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 0 & 1 \\ 3 & 2 \end{bmatrix}.$$

   Which of the following asset prices satisfy No Arbitrage (hint: Hens method): (i) $q = (0, 2)$, (ii) $q = (2, 1.75)$, and (iii) $q = (2, 1.25)$?

   **Solution:**

   To solve this problem, let's use the Hens method for two assets. I plot the points $(r_1(1), r_2(1)) = (2, 3)$, $(r_1(2), r_2(2)) = (1, 1)$, $(r_1(3), r_2(3)) = (0, 1)$, and $(r_1(4), r_2(4)) = (3, 2)$ in Figure A.4. The x-axis corresponds to asset $j = 1$ and the y-axis to asset $j = 2$. The prices $q$ satisfy No Arbitrage iff they lie in the interior of the cone generated by the 4 pairs of plotted points.

   As indicated in Figure A.4, the asset prices $q = (0, 2)$ lie on the boundary of the cone, the asset prices $q = (2, 1.75)$ lie in the interior of the cone, and the asset prices $q = (2, 1.25)$ lie outside the cone. Thus, the only prices that satisfy No Arbitrage are $q = (2, 1.75)$.

3. Suppose that there are $S = 4$ states and $J = 3$ assets with payouts

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ \frac{2}{3} & 1 & \frac{1}{3} \\ \frac{1}{2} & \frac{2}{3} & \frac{1}{2} \end{bmatrix}.$$

Which of the following asset prices satisfy No Arbitrage: (i) $q = (1, 4, 2)$, (ii) $q = (2, 1, 1)$, and (iii) $q = (3, 2, 1)$?

**Solution:**

We need to find a linear operation for the matrix $\begin{pmatrix} -q \\ R \end{pmatrix}$ such that the new payout matrix has the risk-free payout of 1 in all states. To do this, simply replace Column 1 of $\begin{pmatrix} -q \\ R \end{pmatrix}$ with the sum of Columns 1 and 3:

$$\begin{pmatrix} -q^* \\ R^* \end{pmatrix} = \begin{bmatrix} -q_1 - q_3 & -q_2 & -q_3 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & \frac{1}{3} \\ 1 & \frac{2}{3} & \frac{1}{2} \end{bmatrix}.$$

We are now ready to plot the points $(r_2^*(1), r_3^*(1)) = (0, 0)$, $(r_2^*(2), r_3^*(2)) = (1, 1)$, $(r_2^*(3), r_3^*(3)) = \left(1, \frac{1}{3}\right)$, and $(r_2^*(4), r_3^*(4)) = \left(\frac{2}{3}, \frac{1}{2}\right)$. Figure A.5 shows these plotted points and the convex hull that they generate. The prices $q$ satisfy No Arbitrage iff $\left(\frac{q_2^*}{q_1^*}, \frac{q_3^*}{q_1^*}\right) = \left(\frac{q_2}{q_1+q_3}, \frac{q_3}{q_1+q_3}\right)$ lies in the interior of the convex hull. For the three asset prices in the question, we have (i) $\left(\frac{q_2}{q_1+q_3}, \frac{q_3}{q_1+q_3}\right) = \left(\frac{4}{3}, \frac{2}{3}\right)$, (ii) $\left(\frac{q_2}{q_1+q_3}, \frac{q_3}{q_1+q_3}\right) = \left(\frac{1}{3}, \frac{1}{3}\right)$, and (iii) $\left(\frac{q_2}{q_1+q_3}, \frac{q_3}{q_1+q_3}\right) = \left(\frac{1}{2}, \frac{1}{4}\right)$.

As indicated in Figure A.5, the pair $\left(\frac{4}{3}, \frac{2}{3}\right)$ lies outside the convex hull, the pair $\left(\frac{1}{3}, \frac{1}{3}\right)$ lies on the boundary of the convex hull, and the pair $\left(\frac{1}{2}, \frac{1}{4}\right)$ lies in the interior of the convex hull. Thus, the only prices that satisfy No Arbitrage are $q = (3, 2, 1)$.

4. Show that it is innocuous to assume No Redundancy. In other words, show that if $\left(\left(x^h, \lambda^h, z^h\right)_{h \in \mathbf{H}}, p, q\right)$ is a general financial equilibrium in which No Redundancy does not hold, then there exists $\left(\left(\hat{z}^h\right)_{h \in \mathbf{H}}, \hat{q}\right)$ such that $\left(\left(x^h, \lambda^h, \hat{z}^h\right)_{h \in \mathbf{H}}, p, \hat{q}\right)$ is a general

financial equilibrium in which No Redundancy does hold.

**Solution:**

Consider a payout matrix $R$ that does not satisfy No Redundancy. For this payout matrix, the general financial equilibrium is $\left(\left(x^h, \lambda^h, z^h\right)_{h \in \mathbf{H}}, p, q\right)$. From the first order conditions with respect to $z^h$, then $\lambda^h \begin{pmatrix} -q \\ R \end{pmatrix} = 0 \;\; \forall h \in \mathbf{H}$. As $R$ does not satisfy No Redundancy, then $rankR = J^* < J$. Without loss of generality, assume that the first $J^*$ columns of $R$ are linearly independent, so we can write $R = \begin{bmatrix} R^* & | & R^{**} \end{bmatrix}$ such that $R^*$ is an $S \times J^*$ submatrix with $rankR^* = J^*$. I can define asset holdings $\hat{z}^h = \left(\hat{z}_j^h\right)_{j=1,\ldots,J^*}$ such that $\begin{pmatrix} -q \\ R \end{pmatrix} z^h = \begin{pmatrix} -\hat{q} \\ R^* \end{pmatrix} \hat{z}^h$. Each of the columns in $R^{**}$ is a linear combination of the columns in $R^*$, meaning that $r_j = R^* \theta$ for some $\theta = \begin{pmatrix} \theta_1 \\ : \\ \theta_{j^*} \end{pmatrix} \in \mathbb{R}^{j^*}$ and $\forall j > j^*$, where $r_j = \begin{bmatrix} r_j(1) \\ : \\ r_j(S) \end{bmatrix}$. Thus, define the new asset holdings as:

$$\hat{z}_j^h = z_j^h + \theta_j \sum_{k > j^*} z_k^h.$$

Since $\begin{pmatrix} -q \\ R \end{pmatrix} z^h = \begin{pmatrix} -\hat{q} \\ R^* \end{pmatrix} \hat{z}^h$, the real variables $\left(\left(x^h, \lambda^h\right)_{h \in \mathbf{H}}, p\right)$ remain unchanged. The new asset prices satisfy $\hat{q}_j = q_j \;\; \forall j \leq j^*$ as $\lambda^h \begin{pmatrix} -\hat{q} \\ R^* \end{pmatrix} = 0 \;\; \forall h \in \mathbf{H}$.

Thus, for the payout matrix $R^*$ (satisfying No Redundancy as $R^*$ has full column rank), $\left(\left(x^h, \lambda^h, \hat{z}^h\right)_{h \in \mathbf{H}}, p, \hat{q}\right)$ is a general financial equilibrium.

5. Prove Theorem 3.4.

**Solution:**

This proof will use "A Different Application of Differential Topology" from Exercise 3 in Chapter 1. That is, we will write down a system of equations that must be satisfied by a general financial equilibrium that is also Pareto optimal. This system of equations will satisfy $J < L$ (more equations than variables). If we can then show that (i) $\pi$ is proper and (ii) $rankD\Phi(\eta) = L \ \ \forall \eta \in M$, then we can apply "A Different Application of Differential Topology" conclusion from Exercise 3 to conclude that over a generic subset of endowments, there does not exist variables $\xi = \left( \left(x^h, \lambda^h, z^h\right)_{h \in \mathbf{H}}, p, q \right)$ such that $\left( \left(x^h, \lambda^h, z^h\right)_{h \in \mathbf{H}}, p, q \right)$ is a general financial equilibrium and $\left(x^h\right)_{h \in \mathbf{H}}$ is a Pareto optimal allocation.

The condition "$\pi$ is proper" can be verified in the same fashion as Exercise 6 in Chapter 2 (the details are left to the intrepid reader).

As in Exercise 9 in Chapter 2, a necessary and sufficient condition for Pareto optimality is $\frac{D_{(l,s)}u^h\left(x^h\right)}{D_{(l',s')}u^h\left(x^h\right)} = \frac{D_{(l,s)}u^{h'}\left(x^{h'}\right)}{D_{(l',s')}u^{h'}\left(x^{h'}\right)}$ for any pair of households $(h, h')$ and any commodity-state pairs $(l, s)$. Using the first order condition with respect to consumption, this condition is equivalent to $\frac{\left(\lambda^h(1), \ldots, \lambda^h(S)\right)}{\lambda^h(0)} = \frac{\left(\lambda^{h'}(1), \ldots, \lambda^{h'}(S)\right)}{\lambda^{h'}(0)}$ for any pair of households $(h, h')$, that is, the Lagrange multipliers are proportional.

Define the system of equations $\Phi : \underset{h \in \mathbf{H}}{\times} \left(\mathbb{R}^{G+S+1}_{++} \times \mathbb{R}^J\right) \times \mathbb{R}^{G-(S+1)}_{++} \times \mathbb{R}^J_{++} \to \mathbb{R}^{n+1}$ as:

$$\Phi\left( \left(x^h, \lambda^h, z^h\right)_{h \in \mathbf{H}}, p, q \right) = \left( \begin{array}{c} \left( \begin{array}{c} \left[Du^h\left(x^h\right) - \lambda^h P\right]^T \\ P\left(e^h - x^h\right) + \left( \begin{array}{c} -q \\ R \end{array} \right) z^h \\ \left[ \lambda^h \left( \begin{array}{c} -q \\ R \end{array} \right) \right]^T \end{array} \right)_{h \in \mathbf{H}} \\ \sum_{h \in \mathbf{H}} \left( e^h_{\backslash G} - x^h_{\backslash G} \right) \\ \sum_{h \in \mathbf{H}} z^h \\ \lambda^1(1)\lambda^2(0) - \lambda^1(0)\lambda^2(1) \end{array} \right),$$

where $n = H(G + S + 1 + J) + G - (S+1) + J$. The additional equation $\lambda^1(1)\lambda^2(0) - \lambda^1(0)\lambda^2(1) = 0$ is a necessary condition for Pareto optimality.

I will show that $rankD\Phi\left( \left(x^h, \lambda^h, z^h\right)_{h \in \mathbf{H}}, p, q \right) = n+1$, taking derivatives with respect to the variables $\left(x^h, \lambda^h, z^h\right)_{h \in \mathbf{H}}$ and the endowment $e^1$. By assumption, $J < S$. This means that $R$ has full column rank $J$ (by No Redundancy) and $\exists s > 0$ such that the submatrix $R(\backslash s)$ defined by removing the row for state $s$ also has full column rank $J$. Without loss of generality, let $s = 1$. This means that $R(\backslash 1) = \left[ \begin{array}{c} r(2) \\ : \\ r(S) \end{array} \right]$ has full column rank $J$.

Premultiply the derivative matrix $D\Phi\left(\left(x^h,\lambda^h,z^h\right)_{h\in\mathbf{H}},p,q\right)$ by

$$\nu^T=\left(...,\left(\left(\Delta x^h\right)^T,\left(\Delta\lambda^h\right)^T,\left(\Delta z^h\right)^T\right),...,\Delta p^T,\Delta q^T,\Delta EE\right),$$

where $\Delta EE\in\mathbb{R}$ corresponds to the extra equation $\lambda^1(1)\lambda^2(0)-\lambda^1(0)\lambda^2(1)=0$. The proof proceeds as in the proof of Theorem 3.2, with one exception. In Step 3, the columns corresponding to the derivatives with respect to $\lambda^1$ are given by:

$$\begin{aligned}\left(\Delta z^1\right)^T\Psi^T+\Delta EE\left(-\lambda^2(1),\lambda^2(0),0,...,0\right) &= 0 \quad\text{(A.5)}\\ \left(\Delta z^1\right)^T\begin{pmatrix}-q^T & r(1)^T & R(\backslash 1)^T\end{pmatrix}+\Delta EE\left(-\lambda^2(1),\lambda^2(0),0,...,0\right) &= 0\end{aligned}$$

As $\Psi=\begin{pmatrix}-q\\ r(1)\\ R(\backslash 1)\end{pmatrix}$ and $R(\backslash 1)$ has full column rank, then (A.5) implies that $\left(\Delta z^1\right)^T=0$ and $\Delta EE=0$ (as both $\lambda^2(0)>0$ and $\lambda^2(1)>0$). The rest of the proof remains unchanged, ultimately resulting in $\nu^T=0$. Thus $D\Phi\left(\left(x^h,\lambda^h,z^h\right)_{h\in\mathbf{H}},p,q\right)$ has full row rank $n+1$, finishing the argument.

6. In Example 3.1, verify that the Arrow-Debreu equilibrium allocation is such that $\forall h\in\mathbf{H}:x^h(s)=\theta^h\sum_{h\in\mathbf{H}}e^h(s)$ for some $\theta^h$.

   **Solution:**

   With utility function utility $u^h\left(x^h\right)=-\sum_{s\in\mathbf{S}}\left(x^h(s)\right)^{-\gamma}$, the first order conditions with respect to consumption are given by:

$$\gamma\left(x^h(s)\right)^{-(\gamma+1)}-\lambda^h\rho(s)=0\quad\forall s\in\mathbf{S}. \tag{A.6}$$

   Recall that in an Arrow-Debreu equilibrium, there is a single budget constraint and thus a single Lagrange multiplier $\lambda^h>0$. I denote the Arrow-Debreu prices as $(\rho(s))_{s\in\mathbf{S}}$ such that $\rho(0)=1$. Equation (A.6) yields:

$$x^h(s)=\left(\frac{\lambda^h}{\gamma}\right)^{-\frac{1}{\gamma+1}}\rho(s)^{-\frac{1}{\gamma+1}}. \tag{A.7}$$

From the Arrow-Debreu budget constraint:

$$\sum_{s\in\mathbf{S}} \rho(s)x^h(s) = \sum_{s\in\mathbf{S}} \rho(s)e^h(s).$$

Using the expression for $x^h(s)$ from (A.7), the budget constraint becomes:

$$\left(\frac{\lambda^h}{\gamma}\right)^{-\frac{1}{\gamma+1}} \sum_{s\in\mathbf{S}} \rho(s)^{\frac{\gamma}{\gamma+1}} = \sum_{s\in\mathbf{S}} \rho(s)e^h(s).$$

This implies that

$$\left(\frac{\lambda^h}{\gamma}\right)^{-\frac{1}{\gamma+1}} = \frac{\sum_{s\in\mathbf{S}} \rho(s)e^h(s)}{\sum_{s\in\mathbf{S}} \rho(s)^{\frac{\gamma}{\gamma+1}}} \quad \forall h \in \mathbf{H}. \tag{A.8}$$

The market clearing condition $\sum_{h\in\mathbf{H}} x^h(s) = \sum_{h\in\mathbf{H}} e^h(s)$ for any state $s \in \mathcal{S}$ implies

$$\frac{\rho(s)^{-\frac{1}{\gamma+1}}}{\sum_{s\in\mathbf{S}} \rho(s)^{\frac{\gamma}{\gamma+1}}} \sum_{s\in\mathbf{S}} \rho(s) \sum_{h\in\mathbf{H}} e^h(s) = \sum_{h\in\mathbf{H}} e^h(s) \quad \forall s \in \mathbf{S}.$$

Solving for $\rho(s)^{-\frac{1}{\gamma+1}}$ yields

$$\rho(s)^{-\frac{1}{\gamma+1}} = \frac{\left(\sum_{s\in\mathbf{S}} \rho(s)^{\frac{\gamma}{\gamma+1}}\right)\left(\sum_{h\in\mathbf{H}} e^h(s)\right)}{\sum_{s\in\mathbf{S}} \rho(s) \sum_{h\in\mathbf{H}} e^h(s)} \quad \forall s \in \mathbf{S}. \tag{A.9}$$

Define $\theta^h = \frac{\sum_{s\in\mathbf{S}} \rho(s)e^h(s)}{\sum_{s\in\mathbf{S}} \rho(s) \sum_{h\in\mathbf{H}} e^h(s)}$ $\forall h \in \mathbf{H}$. Notice that $\sum_{h\in\mathbf{H}} \theta^h = 1$, by definition. Insert (A.8) and (A.9) into the expression for $x^h(s)$ from (A.7) yields:

$$x^h(s) = \theta^h \left(\sum_{h\in\mathbf{H}} e^h(s)\right).$$

This completes the argument.

7. In Example 3.3, verify that the general financial equilibrium consumption is given by:

$$x_l^h(s) = \alpha^h(s) \cdot \sum\nolimits_{h\in\mathbf{H}} e_l^h(s)$$

for some $\alpha^h(s) \in (0,1)$ and the prices are given such that $p_l(s) = \frac{\theta_l(s)}{\theta_L(s)}$ $\forall (l,s) \in \mathbf{L}\times\mathbf{S}$.

**Solution:**

With utility function utility $u^h(x^h) = \sum\nolimits_{s\in\mathbf{S}} \left(\sum\nolimits_{l\in\mathbf{L}} \theta_l(s)\cdot\log\left(x_l^h(s)\right)\right)$, the first order conditions with respect to consumption are given by:

$$\frac{\theta_l(s)}{x_l^h(s)} - \lambda^h(s)p_l(s) = 0 \quad \forall (l,s) \in \mathbf{L}\times\mathbf{S}.$$

Solving for $x_l^h(s)$ yields

$$x_l^h(s) = \frac{\theta_l(s)}{\lambda^h(s)p_l(s)} \quad \forall (l,s) \in \mathbf{L}\times\mathbf{S}. \tag{A.10}$$

The budget constraint for any household in any state $s \in \mathcal{S}$ is given by:

$$\sum_{l\in\mathbf{L}} p_l(s)x_l^h(s) = \sum_{l\in\mathbf{L}} p_l(s)e_l^h(s).$$

Using the expression for $x_l^h(s)$ from (A.10) yields:

$$\frac{1}{\lambda^h(s)}\sum_{l\in\mathbf{L}} \theta_l(s) = \sum_{l\in\mathbf{L}} p_l(s)e_l^h(s).$$

Thus, the expression for $x_l^h(s)$ can be updated as:

$$x_l^h(s) = \left(\frac{\theta_l(s)}{\sum\limits_{l\in\mathbf{L}}\theta_l(s)}\right)\frac{\sum\limits_{l\in\mathbf{L}} p_l(s)e_l^h(s)}{p_l(s)} \quad \forall (l,s) \in \mathbf{L}\times\mathbf{S}. \tag{A.11}$$

Consider the market clearing conditions $\sum\limits_{h\in\mathbf{H}} x_l^h(s) = \sum\limits_{h\in\mathbf{H}} e_l^h(s)$ for any $(l,s) \in \mathbf{L}\times\mathbf{S}$ :

$$\frac{1}{p_l(s)}\left(\frac{\theta_l(s)}{\sum\limits_{l\in\mathbf{L}}\theta_l(s)}\right)\sum_{l\in\mathbf{L}} p_l(s)\sum_{h\in\mathbf{H}} e_l^h(s) = \sum_{h\in\mathbf{H}} e_l^h(s).$$

This implies that $p_l(s)$ can be expressed as:

$$p_l(s) = \left( \frac{\theta_l(s)}{\sum\limits_{l \in \mathbf{L}} \theta_l(s)} \right) \frac{\sum\limits_{l \in \mathbf{L}} p_l(s) \sum\limits_{h \in \mathbf{H}} e_l^h(s)}{\sum\limits_{h \in \mathbf{H}} e_l^h(s)}. \tag{A.12}$$

Given the price normalization $p_L(s) = 1 \ \ \forall s \in \mathcal{S}$, then

$$\frac{1}{\theta_L(s)} = \left( \frac{1}{\sum\limits_{l \in \mathbf{L}} \theta_l(s)} \right) \frac{\sum\limits_{l \in \mathbf{L}} p_l(s) \sum\limits_{h \in \mathbf{H}} e_l^h(s)}{\sum\limits_{h \in \mathbf{H}} e_L^h(s)},$$

by definition. For any $(l, s) \in \mathbf{L} \times \mathbf{S}$, equation (A.12) verifies that

$$p_l(s) = \frac{\theta_l(s)}{\theta_L(s)} \cdot \frac{\sum_{h \in \mathbf{H}} e_L^h(s)}{\sum_{h \in \mathbf{H}} e_l^h(s)}.$$

Using the expression (A.12) for $p_l(s)$, then the expression (A.11) for $x_l^h(s)$ can be updated as:

$$x_l^h(s) = \frac{\sum\limits_{l \in \mathbf{L}} p_l(s) e_l^h(s)}{\sum\limits_{l \in \mathbf{L}} p_l(s) \sum\limits_{h \in \mathbf{H}} e_l^h(s)} \left( \sum_{h \in \mathbf{H}} e_l^h(s) \right) \quad \forall (l, s) \in \mathbf{L} \times \mathbf{S}.$$

Define $\alpha^h(s) = \frac{\sum\limits_{l \in \mathbf{L}} p_l(s) e_l^h(s)}{\sum\limits_{l \in \mathbf{L}} p_l(s) \sum\limits_{h \in \mathbf{H}} e_l^h(s)}$ and notice that $\sum\limits_{h \in \mathbf{H}} \alpha^h(s) = 1 \ \ \forall s \in \mathbf{S}$, by definition. This completes the second part of the claim that $x_l^h(s) = \alpha^h(s) \cdot \sum_{h \in \mathbf{H}} e_l^h(s) \ \ \forall (l, s) \in \mathbf{L} \times \mathbf{S}$.

8. In Example 3.4, verify that the Arrow-Debreu equilibrium allocation and commodity prices in Case I are given by:

$$\begin{array}{ll} x^1(1) = \left(\frac{8}{3}, \frac{1}{3}\right) & x^1(2) = \left(\frac{8}{3}, \frac{1}{3}\right) \\ x^2(1) = \left(\frac{1}{3}, \frac{8}{3}\right) & x^2(2) = \left(\frac{1}{3}, \frac{8}{3}\right) \\ \rho(1) = (1, 1) & \rho(2) = (1, 1) \end{array}.$$

**Solution:**

To solve for the Arrow-Debreu equilibrium, I make the price normalization $\rho_2(2) = 1$. The equilibrium variables are $(x^1(1), x^1(2), x^2(1), x^2(2))$ (8 consumption variables), $(\lambda^1, \lambda^2)$ (2 Lagrange multipliers for the 2 budget constraints, 1 for each household), and $(\rho_1(1), \rho_2(1), \rho_1(2))$ (3 price variables). The equations are the first order conditions with respect to consumption (8 of these), the 2 budget constraints $\rho(1) \cdot x^h(1) + \rho(2) \cdot x^h(2) = \rho(1) \cdot e^h(1) + \rho(2) \cdot e^h(2)$ for $h = 1, 2$, and the 3 market clearing conditions $x_1^1(1) + x_1^2(1) = 3$, $x_2^1(1) + x_2^2(1) = 3$, and $x_1^1(2) + x_1^2(2) = 3$.

The first order conditions for household $h = 1$ are given by:

$$\begin{array}{ll} 2^{0.5}\left(x_1^1(1)\right)^{-0.5} - \lambda^1 \rho_1(1) = 0 & 2^{-1}\left(x_2^1(1)\right)^{-0.5} - \lambda^1 \rho_2(1) = 0 \\ 2^{0.5}\left(x_1^1(2)\right)^{-0.5} - \lambda^1 \rho_1(2) = 0 & 2^{-1}\left(x_2^1(2)\right)^{-0.5} - \lambda^1 = 0 \end{array}.$$

The expressions for consumption are given by:

$$\begin{array}{ll} x_1^1(1) = \frac{2}{\left(\lambda^1 \rho_1(1)\right)^2} & x_2^1(1) = \frac{1}{4\left(\lambda^1 \rho_2(1)\right)^2} \\ x_1^1(2) = \frac{2}{\left(\lambda^1 \rho_1(2)\right)^2} & x_2^1(2) = \frac{1}{4\left(\lambda^1\right)^2} \end{array}. \tag{A.13}$$

Inserting these expressions into the budget constraint yields:

$$\left(\frac{1}{\lambda^1}\right)^2 \left(\frac{2}{\rho_1(1)} + \frac{1}{4\rho_2(1)} + \frac{2}{\rho_1(2)} + \frac{1}{4}\right) = \frac{5}{2}\rho_1(1) + \frac{50}{21}\rho_2(1) + \frac{13}{21}\rho_1(2) + \frac{1}{2}.$$

The expressions for consumption are then given by:

$$\begin{aligned} x_1^1(1) &= \frac{2}{\rho_1(1)^2}\left(\frac{\frac{5}{2}\rho_1(1) + \frac{50}{21}\rho_2(1) + \frac{13}{21}\rho_1(2) + \frac{1}{2}}{\frac{2}{\rho_1(1)} + \frac{1}{4\rho_2(1)} + \frac{2}{\rho_1(2)} + \frac{1}{4}}\right) \\ x_2^1(1) &= \frac{1}{4\left(\rho_2(1)\right)^2}\left(\frac{\frac{5}{2}\rho_1(1) + \frac{50}{21}\rho_2(1) + \frac{13}{21}\rho_1(2) + \frac{1}{2}}{\frac{2}{\rho_1(1)} + \frac{1}{4\rho_2(1)} + \frac{2}{\rho_1(2)} + \frac{1}{4}}\right) \\ x_1^1(2) &= \frac{2}{\rho_1(2)^2}\left(\frac{\frac{5}{2}\rho_1(1) + \frac{50}{21}\rho_2(1) + \frac{13}{21}\rho_1(2) + \frac{1}{2}}{\frac{2}{\rho_1(1)} + \frac{1}{4\rho_2(1)} + \frac{2}{\rho_1(2)} + \frac{1}{4}}\right) \\ x_2^1(2) &= \frac{1}{4}\left(\frac{\frac{5}{2}\rho_1(1) + \frac{50}{21}\rho_2(1) + \frac{13}{21}\rho_1(2) + \frac{1}{2}}{\frac{2}{\rho_1(1)} + \frac{1}{4\rho_2(1)} + \frac{2}{\rho_1(2)} + \frac{1}{4}}\right) \end{aligned} \tag{A.14}$$

Let's make use of the symmetry assumed in this economy. The commodity $(l, s) = (1, 1)$

for household $h = 1$ is identical to the commodity $(l, s) = (2, 2)$ for household $h = 2$. Additionally, the commodity $(l, s) = (1, 2)$ for household $h = 1$ is identical to the commodity $(l, s) = (2, 1)$ for household $h = 2$. This means that $\rho(1) = \rho(2) = (\rho, 1)$ and we only need to consider the market clearing condition for the commodity $(l, s) = (1, 1)$.

The market clearing condition $x_1^1(1) + x_1^2(1) = 3$ is then given by:

$$\frac{2}{\rho^2}\left(\frac{\frac{5}{2}\rho + \frac{50}{21} + \frac{13}{21}\rho + \frac{1}{2}}{\frac{2}{\rho} + \frac{1}{4} + \frac{2}{\rho} + \frac{1}{4}}\right) + \frac{1}{2\rho^2}\left(\frac{\frac{1}{2}\rho + \frac{13}{21} + \frac{50}{21}\rho + \frac{5}{2}}{\frac{1}{4\rho} + 2 + \frac{1}{4\rho} + 2}\right) = 3.$$

Distributing the 2 in the first term and the $\frac{1}{2}$ in the second term yields:

$$\frac{1}{\rho^2}\left(\frac{\frac{5}{2}\rho + \frac{50}{21} + \frac{13}{21}\rho + \frac{1}{2}}{\frac{1}{\rho} + \frac{1}{8} + \frac{1}{\rho} + \frac{1}{8}}\right) + \frac{1}{\rho^2}\left(\frac{\frac{1}{2}\rho + \frac{13}{21} + \frac{50}{21}\rho + \frac{5}{2}}{\frac{1}{2\rho} + 4 + \frac{1}{2\rho} + 4}\right) = 3. \tag{A.15}$$

The value of $\rho = 1$ is a solution to (A.15). With $\rho = 1$, then $\left(\frac{1}{\lambda^1}\right)^2 = \frac{6}{4.5} = \frac{4}{3}$ and from (A.13):

$$\begin{array}{ll} x_1^1(1) = \frac{8}{3} & x_2^1(1) = \frac{1}{3} \\ x_1^1(2) = \frac{8}{3} & x_2^1(2) = \frac{1}{3} \end{array}.$$

The consumptions for household $h = 2$ are found using the market clearing conditions.

9. In Example 3.4, verify that the general financial equilibrium allocation and commodity prices in Case II are given by:

$$\begin{array}{ll} x^1(1) = \left(\frac{62}{21}, \frac{31}{21}\right) & x^1(2) = \left(\frac{32}{21}, \frac{1}{21}\right) \\ x^2(1) = \left(\frac{1}{21}, \frac{32}{21}\right) & x^2(2) = \left(\frac{31}{21}, \frac{62}{21}\right) \\ p(1) = (2, 1) & p(2) = \left(\frac{1}{2}, 1\right) \end{array}.$$

**Solution:**

To solve for the general financial equilibrium, I consider each state $s = 1, 2$ as its own Arrow-Debreu economy.

In state $s = 1$, I make the price normalization $p_2(1) = 1$. The equilibrium variables are $(x^1(1), x^2(1))$ (4 consumption variables), $(\lambda^1, \lambda^2)$ (2 Lagrange multipliers for the 2 budget constraints, 1 for each household), and $p_1(1)$ (1 price variable). The equations are the first order conditions with respect to consumption (4 of these), the 2 budget constraints $p(1) \cdot x^h(1) = p(1) \cdot e^h(1)$ for $h = 1, 2$, and 1 market clearing condition $x_1^1(1) + x_1^2(1) = 3$.

The first order conditions for household $h = 1$ are given by:

$$2^{0.5}\left(x_1^1(1)\right)^{-0.5} - \lambda^1 p_1(1) = 0 \qquad 2^{-1}\left(x_2^1(1)\right)^{-0.5} - \lambda^1 = 0 \ .$$

The expressions for consumption are given by:

$$x_1^1(1) = \tfrac{2}{\left(\lambda^1 p_1(1)\right)^2} \qquad x_2^1(1) = \tfrac{1}{4\left(\lambda^1\right)^2} \ . \tag{A.16}$$

Inserting these expressions into the budget constraint yields:

$$\left(\frac{1}{\lambda^1}\right)^2\left(\frac{2}{p_1(1)} + \frac{1}{4}\right) = \frac{5}{2}p_1(1) + \frac{50}{21}.$$

The expressions for consumption are then given by:

$$\begin{aligned} x_1^1(1) &= \frac{2}{p_1(1)^2}\left(\frac{\frac{5}{2}p_1(1) + \frac{50}{21}}{\frac{2}{p_1(1)} + \frac{1}{4}}\right) \\ x_2^1(1) &= \frac{1}{4}\left(\frac{\frac{5}{2}p_1(1) + \frac{50}{21}}{\frac{2}{p_1(1)} + \frac{1}{4}}\right) \end{aligned}$$

Similarly, the expressions for consumption of household $h = 2$ can be found as:

$$\begin{aligned} x_1^2(1) &= \frac{1}{4p_1(1)^2}\left(\frac{\frac{1}{2}p_1(1) + \frac{13}{21}}{\frac{1}{4p_1(1)} + 2}\right) \\ x_2^1(1) &= 2\left(\frac{\frac{1}{2}p_1(1) + \frac{13}{21}}{\frac{1}{4p_1(1)} + 2}\right) \end{aligned}$$

The market clearing condition $x_1^1(1) + x_1^2(1) = 3$ is then given by:

$$\frac{2}{p_1(1)^2}\left(\frac{\frac{5}{2}p_1(1) + \frac{50}{21}}{\frac{2}{p_1(1)} + \frac{1}{4}}\right) + \frac{1}{4p_1(1)^2}\left(\frac{\frac{1}{2}p_1(1) + \frac{13}{21}}{\frac{1}{4p_1(1)} + 2}\right) = 3.$$

The value $p_1(1) = 2$ is the solution to this equation. With $p_1(1) = 2$, then $\left(\frac{1}{\lambda^1}\right)^2 = \frac{124}{21}$

and from (A.16):

$$x_1^1(1) = \frac{124}{21}\frac{2}{(2)^2} = \frac{62}{21} \qquad x_2^1(1) = \frac{124}{21}\frac{1}{4} = \frac{31}{21} \ .$$

The consumptions for household $h = 2$ are found using the market clearing conditions.

For state $s = 2$, the equilibrium variables are equal to those from state $s = 1$, with the commodity $l = 1$ switched with commodity $l = 2$. For instance, the relative price is now $\frac{p_1(2)}{p_2(2)} = \frac{p_2(1)}{p_1(1)} = \frac{1}{2}$.

## A.4 Incomplete Markets and Money

Exercises 1-4 walk through all steps required to prove Lemma 4.1.

1. Show that the vectors $\left(a^h\right)_{h\in\mathbf{J}}$ are linearly independent iff the portfolio vectors $\left(z^h\right)_{h\in\mathbf{J}}$ are linearly independent.

   **Solution:**

   The vectors $\left(a^h\right)_{h\in\mathbf{J}}$ are linearly independent provided that the matrix $\left(a^1, ..., a^J\right)$ has full column rank. By definition, $a^h = [\nu]\, Rz^h \in \mathbb{R}^S$, so $\left(a^1, ..., a^J\right) = [\nu]\, R\left(z^1, ..., z^J\right)$. The matrix $[\nu]\, R \in \mathbb{R}^S$ has full column rank by definition. Thus, $[\nu]\, R\left(z^1, ..., z^J\right)$ has full rank iff $\left(z^1, ..., z^J\right)$ has full rank.

2. Write down the system of equilibrium equations $\Phi : \Xi\times\Theta \to \mathbb{R}^n$ for $n = H\,(G + S + 1 + J) + G + J$ so that $\xi = \left(\left(x^h, \lambda^h, z^h\right)_{h\in\mathbf{H}}, p, q\right)$ is a financial equilibrium with money iff $\Phi(\xi) = 0$. Here $\theta = \left(\left(e^h\right)_{h\in\mathbf{H}}, M\right) \in \Theta$.

   **Solution:**

   The price vectors are now $p(s) = (p_1(s), ..., p_L(s)) \in \mathbb{R}^L_{++}$. There are no price normalizations.

Define the system of equations $\Phi : \underset{h\in\mathbf{H}}{\times}\left(\mathbb{R}^{G+S+1}_{++}\times\mathbb{R}^{J}\right)\times\mathbb{R}^{G}_{++}\times\mathbb{R}^{J}_{++}\to\mathbb{R}^{n}$ as:

$$\Phi\left(\left(x^h,\lambda^h,z^h\right)_{h\in\mathbf{H}},p,q\right)=\begin{pmatrix}\begin{pmatrix}\left[Du^h\left(x^h\right)-\lambda^h P\right]^T\\ P\left(e^h-x^h\right)+\begin{pmatrix}-q\\R\end{pmatrix}z^h\\ \left[\lambda^h\begin{pmatrix}-q\\R\end{pmatrix}\right]^T\end{pmatrix}_{h\in\mathbf{H}}\\ \sum_{h\in\mathbf{H}}\left(e^h-x^h\right)\\ \sum_{h\in\mathbf{H}}z^h\\ \left(p\left(s\right)\sum_{h\in\mathbf{H}}x^h\left(s\right)-M_s\right)_{s\in\mathbf{S}}\end{pmatrix}.$$

3. Use the mathematical tools from Section 1.4 and the results in Theorems 2.4 and 3.2 to show that, for a generic selection of households endowments $e=\left(e^h\right)_{h\in\mathbf{H}}$ and money supplies $M=\left(M_0,M_1,...,M_S\right)$, the financial equilibria with money satisfy Finite Local Uniqueness. This requires proving that (i) $\pi$ is proper and (ii) if $\Phi\left(\xi,\theta\right)=0$, then $rankD\Phi\left(\xi,\theta\right)=n$ (the derivative is taken with respect to the variables $\xi$ and the parameters $\theta=\left(\left(e^h\right)_{h\in\mathbf{H}},M\right)$). The proof of (i) is straightforward, so only prove (ii).

   **Solution:**

   We are tasked with showing $rankD\Phi\left(\xi,\theta\right)=n$, i.e., that $D\Phi\left(\xi,\theta\right)$ has full row rank. The easiest way is to first consider the columns of $D\Phi\left(\xi,\theta\right)$ corresponding to the derivatives with respect to the money supplies $M=\left(M_0,M_1,...,M_S\right)$. These parameters only appear in one equation each, $p\left(s\right)\sum_{h\in\mathbf{H}}x^h\left(s\right)-M_s=0$. Thus, these final $S+1$ equations are linearly independent from all other equations.

   To show that the remaining equations are linearly independent, proceed using the exact same steps as in the proof of Theorem 3.2.

4. Given the outcome of Exercise 3 above, show that $rankD\Phi^*\left(\xi^*,\theta\right)=n+J+1$, where $\Phi^*\left(\eta\right)=\begin{pmatrix}\Phi\left(\eta\right)\\ \mu\cdot\left(z^1,...,z^J\right)\\ \mu^T\mu/2-1\end{pmatrix}$ (again taking derivatives with respect to the variables $\xi^*=\left(\left(x^h,\lambda^h,z^h\right)_{h\in\mathbf{H}},p,q,\mu\right)\in\mathbb{R}^{n+J}$ and the parameters $\theta=\left(\left(e^h\right)_{h\in\mathbf{H}},M\right)$). The additional equations $\mu\cdot\left(z^1,...,z^J\right)$, where $\mu\in\mathbb{R}^J\backslash\{0\}$ (nonzero from $\mu^T\mu/2=1$), imply that the portfolios vectors $\left(z^h\right)_{h\in\mathbf{J}}$ are linearly dependent. Using "A Different

Application of Differential Topology" from Exercise 3 in Chapter 1, we can then conclude that for a generic selection of $\theta = \left(\left(e^h\right)_{h\in\mathbf{H}}, M\right)$, the portfolio vectors $\left(z^h\right)_{h\in\mathbf{J}}$ are linearly independent.

**Solution:**

We are tasked with showing $rankD\Phi^*(\xi^*,\theta) = n+J+1$, i.e., that $D\Phi^*(\xi^*,\theta)$ has full row rank. The easiest way is to first consider the columns of $D\Phi^*(\xi^*,\theta)$ corresponding to the derivatives with respect to the money supplies $M = (M_0, M_1, ..., M_S)$. These parameters only appear in one equation each, $p(s)\sum_{h\in\mathbf{H}} x^h(s) - M_s = 0$. Thus, these final $S+1$ equations are linearly independent from all other equations.

Set $\nu^T D\Phi^*(\xi^*,\theta) = 0$, where $\nu \in \mathbb{R}^{n+J+1}$ corresponds to equations in $\Phi^*$ :

$$\nu = \begin{pmatrix} \vdots \\ \Delta x^h \\ \Delta\lambda^h \\ \Delta z^h \\ \vdots \\ \Delta p \\ \Delta q \\ \Delta M \\ \Delta\mu \\ \Delta EE \end{pmatrix} \begin{pmatrix} \vdots \\ FOCxh \\ BCh \\ FOCzh \\ \vdots \\ MCx \\ MCz \\ \left(p(s)\sum_{h\in\mathbf{H}} x^h(s) - M_s\right)_{s\in\mathbf{S}} \\ \mu\cdot\left(z^1,...,z^J\right) \\ \mu^T\mu/2 - 1 \end{pmatrix}.$$

Full row rank is verified upon showing that $\nu = 0$. The preceding paragraph specifies that $\Delta M = 0$.

Recall Steps 1-3 in the proof of Theorem 3.2. These steps are used to show that $\left((\Delta x^1)^T, \left(\Delta\lambda^1\right)^T, (\Delta z^1)^T, \Delta p^T, \Delta q^T\right) = 0$, by using the columns of $D\Phi^*(\xi^*,\theta)$ corresponding to the derivatives with respect to $\left(x^1, \lambda^1, z^1, e^1\right)$. For households $h > 1$, using the columns corresponding to derivatives with respect to $\left(x^h, \lambda^h, e^h\right)$, the same Steps 1-3 can be used to show that $\left(\left(\Delta x^h\right)^T, \left(\Delta\lambda^h\right)^T, \left(\Delta z^h\right)^T\right) = 0 \quad \forall h > 1$.

The vector $\mu \neq 0$, so wlog suppose that $\mu_1 \neq 0$. Then the columns corresponding to derivatives with respect to $\left(z_1^1, ..., z_1^J\right)$ yield:

$$\Delta\mu_j \cdot \mu_1 = 0 \quad \forall j = 1, ...J.$$

Thus, $\Delta\mu^T = 0$. Derivatives with respect to $\mu$ then imply that $\Delta EE \cdot \mu = 0$, so $\Delta EE = 0$ (as $\mu \neq 0$).

All told, $\nu = 0$, finishing the proof that $rankD\Phi^* (\xi^*, \theta) = n + J + 1$.