Optimal Term Structure in a Monetary Economy with Incomplete Markets*

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Abstract

In a stochastic economy, the rebalancing of short and long term government debt positions can have real effects when markets are incomplete. This paper analyzes both stationary and dynamic policy rules for the term structure of interest rates. After proving the existence of a recursive representation of equilibrium, necessary conditions for Pareto efficiency are characterized. The necessary conditions are equivalent for both stationary and dynamic policy rules.

Keywords unconventional monetary policy – yield curve – asset span – incomplete markets – Pareto efficiency

JEL Classification **D52, E43, E44, E52**

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1 Introduction

In recent years, central banks have expanded the set of assets that they are willing to hold on their balance sheets. Conventional monetary policies, typically called open market operations, involve the buying and selling of short-term government bonds. This paper analyzes one class of unconventional monetary policies in which the central bank buys and sells government bonds of varying maturities. In a stochastic setting, the additional policy tools can be used by the central bank to increase welfare by expanding the risk-sharing potential for households in the economy.

The paper uses two classical ideas in general equilibrium to analyze a modern policy problem. First, with incomplete markets and a fixed asset structure, Pareto efficiency can be supported if the asset span contains the excess demand vectors for all households (Magill and Quinzii, 1996).[1] Second, with incomplete markets, monetary policy can have real effects by changing the asset span (Magill and Quinzii, 1992).

Though many applications of the policy-induced asset span can be considered, this paper focuses on the role of monetary policy in mitigating financial frictions caused by incomplete markets. Monetary policy is conducted by the central bank and operates through the purchase and sale of debt issued by the fiscal authority. Whereas conventional monetary policy only targets the short-term bond yields, the unconventional monetary policies analyzed in this paper target the entire term structure of interest rates. To study the welfare implications of unconventional monetary policies, this paper adopts the Pareto criterion as the benchmark. Even in the presence of incomplete markets, the broader tools available with term structure targeting, relative to open market operations, may be able to support a Pareto efficient allocation. The paper's main contribution is to characterize necessary conditions for Pareto efficiency.

In the model presented in the paper, money is valued via a cash-in-advance constraint as in Lucas and Stokey (1983). Heterogeneous households are infinite-lived and face idiosyncratic endowment risk. Households choose portfolios of the bonds of varying maturities. The bonds are nominally risk-free. Since household objective functions are specified in terms of real units of commodities, it is the real payouts of the bonds that matter. The real payouts are inversely related to the price levels supported by the monetary policy choice.

Monetary policy is chosen according to a vector of interest rate rules for the entire term structure. Two types of policy rules are considered: stationary and dynamic. A stationary policy rule is such that the interest rate targets only depend upon the realization of uncer-

[1] In a finite-horizon model with a single commodity, this asset span condition is satisfied under identical utility of the CARA or CRRA form when the endowment vector of each household is contained in the asset span.

tainty in the current period. A dynamic policy rule allows for the targets to depend on a longer history of variables.

When the number of assets equals the number of states of uncertainty (complete markets), Hoelle (2015) shows that the equilibrium allocation is Pareto efficient no matter which equilibrium policy rule is implemented. For this reason, I focus exclusively on settings with incomplete markets, where the number of assets is strictly less than the number of states of uncertainty.

Under incomplete markets, the necessary condition for Pareto efficiency imposes a lower bound on the number of assets such that the asset span contains the portfolio payouts for all households required for Pareto efficiency. This necessary condition is equivalent under both stationary and dynamic policy rules. If a dynamic policy rule can be found to support Pareto efficiency, then a stationary policy rule can be found to support the same equilibrium allocation. This is because any non-stationary elements of the dynamic policy rule have no real effects on the equilibrium under Pareto efficiency as the real side of the economy is characterized by a stationary real wealth distribution.

1.1 Literature Review

This paper considers a specific type of unconventional monetary policy in which, in addition to the short-term bonds typically bought and sold, the central bank buys and sells bonds with longer maturities. In reality, unconventional monetary policy can refer not only to the purchase and sale of bonds of all maturities, but additionally to the purchase of mortgage-backed securities and other real assets. Empirical works by Krishnamurthy and Vissing-Jorgensen (2011), Gagnon et al. (2011), Lenza et al. (2010), Kapetanios et al. (2012), and Baumeister and Benati (2013) have analyzed the effects of government purchases, including long term treasury purchases, lending to financial institutions, liquidity to financial markets, and mortgage-backed securities, on the yield curve.

The general equilibrium framework of this paper connects it most closely to the works of Peiris and Polemarchakis (2015), Magill and Quinzii (2014a,b), Adão et al. (2014), and Araújo et al. (2013).[2] In standard settings with exogenous probabilities, both Peiris and Polemarchakis (2013) and Adão et al. (2014) analyze determinacy in a complete markets setting. The indeterminacy in such a setting is only nominal, as shown in Hoelle (2015). In Adão et al. (2014), policy refers to the targets for the entire term structure. Such policy

[2] Classic papers in the real business cycle tradition that focus on monetary policy include Sargent and Wallace (1975), Kydland and Prescott (1977), Lucas and Stokey (1983), Barro and Gordon (1983), Chari et al. (1991), and Calvo and Guidotti (1993). In terms of general equilibrium models with incomplete markets and monetary policy, the initial papers were Magill and Quinzii (1992, 1996).

uniquely determines the equilibrium, notably the inflation rates. In Peiris and Polemarchakis (2015), policy refers to the total value of purchases made by the central bank, but not the composition of such purchases. In such a setting, nominal indeterminacy persists. It is only by targeting the composition of purchases that Peiris and Polemarchakis (2015) arrive at a result equivalent to Adão et al. (2014).

Magill and Quinzii (2014a) focus on how the targets on long-term bond yields suffice to ensure the uniqueness of the inflation rates, and do so in a model with endogenous probabilities. As in Adão et al. (2014), conventional monetary policy that only targets the short-term bond yields does not suffice. Magill and Quinzii (2014b) demonstrate how forward guidance, in which the short-term bond yield is fixed for a number of periods, replicates the result obtained when the entire term structure is targeted.

Araújo et al. (2013) is probably closest in spirit to the present paper as it focuses on the real effects of unconventional monetary policies via targeted changes in the collateral constraints.

With similar roots in the class of models with real frictions, New Monetarist models offer additional insights into unconventional monetary policy. Recent works by Williamson (2012) and Kiyotaki and Moore (2012) consider the effects of central bank purchases of liquid assets (conventional monetary policy), while Herrenbrueck (2016) considers the effects of central bank purchases of illiquid assets. These models extend the standard New Monetarist framework to allow for portfolio effects.

The New Keynesian class of models represent a complementary view of the effects of monetary policy as the models are characterized not by real frictions, but by nominal frictions.[3] Eggertsson and Woodford (2003) recommend that central banks commit to forward guidance, which are rules for how the short-term interest rates will be targeted after the economy departs from the zero lower bound constraint.[4] In contrast, McGough et al. (2005) demonstrate how targets on the long-term bond yields can avoid indeterminacy problems that arise near the zero lower bound when only the short-term bond yields are targeted.[5] Curdia and Woodford (2011) espouse policies that target the composition of asset purchases in lieu of policies that simply target the supply of reserves. Gertler and Karadi (2011) extend the business cycle models of Christiano et al. (2005) and Smets and Wouters (2007) by including credit intermediation frictions. Gertler and Karadi (2011) analyze the effects of

[3] A representative sample of the important papers using this class of models in a closed economy include Galí (1992), Sims (1992), Bernanke and Mihov (1998), Christiano et al. (1999), Taylor (1999), Clarida et al. (2000), Woodford (2003), Schmitt-Grohé and Uribe (2004a), and Uhlig (2005).

[4] Recent work by Dong (2014) shows that policies of forward guidance are not credible, and the time consistent policies look quite different than the proposed rules of Eggertsson and Woodford (2003).

[5] Woodford (2005) offers a critique that summarizes the main points from Eggertsson and Woodford (2003).

an increase of central bank credit intermediation to fill the void left by a decrease in private credit intermediation.

Related to the analysis of term structure policies, Andrés et al. (2004) apply the concept of imperfect substitutability between assets to argue that traditional policies of open market operations can have effects on the term structure. Chen et al. (2012) extends this concept of imperfect substitutability in a model similar to Christiano et al. (2005) and Smets and Wouters (2007) to simulate the effects of unconventional monetary policies, including the purchase of long-term bonds.

The remainder of the paper is organized as follows. Section 2 introduces the model. Section 3 introduces the recursive formulation of competitive equilibrium and verifies the existence of a Markov equilibrium. Section 4 analyzes the conditions for Pareto efficiency, under both stationary and dynamic policy rules. Section 5 provide numerical examples to illustrate the effects of policy on the yield curve. Section 6 concludes and the proofs are contained in Appendix A.

2 The Model

The model describes a closed economy with a single infinite-lived monetary authority (or central bank).

Time is discrete and infinite with time periods $t \in \{0, 1, ...\}$. The filtration of uncertainty follows a one-period Markov process with finite state space $\mathbf{S} = \{1, ..., S\}$, where $S > 1$. The realized state of uncertainty in any period t, denoted s_t, is a function only of the realized state in the previous period $t-1$, denoted s_{t-1}. This random process is characterized by a transition matrix $\Gamma \in \mathbb{R}^{S,S}$ whose elements are $\Gamma(s, s')$ for row s and column s'.

Define the history of realizations up to and including the realization s_t in period t as $s^t = (s_0, s_1, ..., s_t)$. For convenience, $\Gamma(s^t|s^\tau)$ for any $t > \tau$ refers to the probability that history s^t is realized conditional on the history s^τ. Additionally, let $s^{t+k} \succ s^t$ refer to the S^k histories $(s^t, \sigma_1, ..., \sigma_k)_{(\sigma_1,...,\sigma_k)\in\mathbf{S}^k}$ that are realized k periods from the date-event s^t.[6]

2.1 Households

In each date-event, a finite number of household types $h \in \mathbf{H} = \{1, ..., H\}$ trade and consume a single physical commodity. Heterogeneity requires $H > 1$.

Households receive the sequence of endowments $\{e^h(s^t)\}$. I assume that the endowments are stationary. Define the stationary endowment mapping as $\mathbf{e}^h : \mathbf{S} \to \mathbb{R}_{++}$ such that

[6] When $k = 0$, the set $s^{t+k} \succ s^t$ refers to the date-event s^t.

$e^h(s^t) = \mathbf{e}^h(s_t)$ for all date-events. Denote the aggregate endowment as $\mathbf{E} : \mathbf{S} \to \mathbb{R}_{++}$ such that $\mathbf{E}(s) = \sum_{h \in \mathbf{H}} \mathbf{e}^h(s) \ \forall s \in \mathbf{S}$. The model permits aggregate risk, i.e., $\mathbf{E}(s) \neq \mathbf{E}(\sigma)$ for some $s, \sigma \in \mathbf{S}$. Define $[\mathbf{e}] = \left[\mathbf{e}^h(s)\right]_{(s,h) \in \mathbf{S} \times \mathbf{H}}$ as the $S \times H$ endowment matrix. For technical reasons, the endowment vectors of households are assumed to be linearly independent.

Assumption 1 $H \leq S$ and the endowment matrix $[\mathbf{e}]$ has full rank and is in general position. Specifically, this means that any H rows of $[\mathbf{e}]$ are linearly independent.

The consumption by household h in date-event s^t is denoted $c^h(s^t) \in \mathbb{R}_+$. The sequence of consumption for household h is denoted $\left\{c^h(s^t)\right\}$.

The household preferences are assumed to be identical and satisfy constant relative risk aversion:

$$\sum_{t=0}^{\infty} \beta^t \sum_{s^t} \Gamma\left(s^t | s_0\right) u\left(c^h\left(s^t\right)\right). \tag{1}$$

Assumption 2 The discount factor $\beta \in (0, 1)$ and $u(c) = \frac{c^{1-\rho}}{1-\rho}$ for $\rho > 0$ and $\rho \neq 1$ and $u(c) = \ln(c)$ for $\rho = 1$.

In each date-event s^t, the money supply is $M(s^t) > 0$ and the nominal price level is $p(s^t) > 0$.

Each household type contains a continuum of households, of equal mass. Households trade in perfectly competitive financial markets. The set of financial assets include a finite number J of nominally risk-free government bonds. The bonds are indexed by $j \in \mathbf{J} = \{1, ..., J\}$.

Asset 1 is a 1-period nominally risk-free bond, which I refer to as the short-term bond. The nominal payouts of a 1-period bond purchased in date-event s^t equal 1 for all date-events $s^{t+1} \succ s^t$ and 0 otherwise.

The set of long-term bonds is the set of $j-$period bonds, where $j \in \{2, ..., J\}$. A $j-$period bond is nominally risk-free, meaning that the nominal payouts of a $j-$period bond purchased in date-event s^t equal 1 for all date-events $s^{t+j} \succ s^t$ and 0 otherwise. The $j-$period nominally risk-free bond can be freely traded in all interim periods up to the maturation date.

The nominal asset price for a $j-$period bond issued in date-event s^t is denoted $q_j(s^t)$. The vector of all bond prices in date-event s^t is $q(s^t) = (q_j(s^t))_{j \in \mathbf{J}}$.

This paper focuses on economies with incomplete markets.

Assumption 3 $J < S$.

Each date-event is divided into two subperiods. In the initial subperiod, the money market and bond markets open. Denote $\hat{m}^h(s^t)$ as the money holding by household h by the close of the money market in date-event s^t. Denote $b_j^h(s^t) \in \mathbb{R}$ as the $j-$period nominal bond holding by household h by the close of the bond markets in date-event s^t. Each bond can either be held long or short by the household. Denote the entire portfolio as $b^h(s^t) = \left(b_j^h(s^t)\right)_{j\in\mathbf{J}} \in \mathbb{R}^J$.

Denote $\omega^h(s^t) \in \mathbb{R}$ as the nominal wealth held by household h for use in the date-event s^t. The initial period value $\omega^h(s_0)$ is a parameter of the model. The budget constraint, at the close of the money market and bond markets in date-event s^t, is given by:

$$\hat{m}^h\left(s^t\right) + \sum_{j\in\mathbf{J}} q_j\left(s^t\right) b_j^h\left(s^t\right) \leq \omega^h\left(s^t\right). \tag{2}$$

In the second subperiod of each date-event, the commodity market opens. The purchase of the commodity is subject to the cash-in-advance constraint:

$$p\left(s^t\right) c^h\left(s^t\right) \leq \hat{m}^h\left(s^t\right). \tag{3}$$

At the same time that consumption is being purchased on the commodity market, the households receive income from selling their endowment. Denote $m^h(s^t)$ as the money holding of household h by the close of the commodity market in date-event s^t :

$$m^h\left(s^t\right) = \hat{m}^h\left(s^t\right) + p\left(s^t\right)\mathbf{e}^h\left(s_t\right) - p\left(s^t\right) c^h\left(s^t\right). \tag{4}$$

Given the money definition (4), the cash-in-advance constraint (3) can be rewritten as:

$$m^h\left(s^t\right) \geq p\left(s^t\right)\mathbf{e}^h\left(s_t\right). \tag{5}$$

Entering into the date-events $s^{t+1} \succ s^t$, the nominal wealth available to household h is equal to the money holding plus the portfolio payout:

$$\omega^h\left(s^{t+1}\right) = m^h\left(s^t\right) + b_1^h\left(s^t\right) + \sum_{j\in\mathbf{J}\setminus\{1\}} q_{j-1}\left(s^{t+1}\right) b_j^h\left(s^t\right).$$

For simplicity, I define $q_0(s^t) = 1$ for all date-events. The wealth can then be expressed as:

$$\omega^h\left(s^{t+1}\right) = m^h\left(s^t\right) + \sum_{j\in\mathbf{J}} q_{j-1}\left(s^{t+1}\right) b_j^h\left(s^t\right). \tag{6}$$

Updating (2) using (4) and (6), the standard budget constraint is derived:

$$p\left(s^t\right)c^h\left(s^t\right)+m^h\left(s^t\right)+\sum_{j\in\mathbf{J}}q_j\left(s^t\right)b_j^h\left(s^t\right)\leq p\left(s^t\right)\mathbf{e}^h\left(s_t\right)+m^h\left(s^{t-1}\right)+\sum_{j\in\mathbf{J}}q_{j-1}\left(s^t\right)b_j^h\left(s^{t-1}\right). \tag{7}$$

Households are permitted to short-sell the nominal bonds, so I require the following implicit debt constraint:

$$\inf_{t,s^t}\left(\sum_{j\in\mathbf{J}}q_j\left(s^t\right)b_j^h\left(s^t\right)\right)>-\infty. \tag{8}$$

The household optimization problem is given by:

$$\begin{array}{ll} \underset{\left\{c^h(s^t),m^h(s^t),b^h(s^t)\right\}}{max} & \sum_{t=0}^{\infty}\beta^t\Gamma\left(s^t|s_0\right)u\left(c^h\left(s^t\right)\right) \\ subj.\ to & \text{budget constraint (7) } \forall t,s^t \\ & \text{cash-in-advance constraint (5) } \forall t,s^t \\ & \text{debt constraint (8)} \end{array}. \tag{9}$$

2.2 Monetary authority

The portfolio of debt positions for the monetary authority in date-event s^t is $B\left(s^t\right)=\left(B_j\left(s^t\right)\right)_{j\in\mathbf{J}}\in\mathbb{R}_+^J$, where $B_j\left(s^t\right)\geq 0$ refers to the amount of debt issued in terms of the $j-$period bond. In reality, the monetary authority does not issue debt, but buys or sells the debt issued by the fiscal authority. The fiscal authority's only role in this model is the debt choice, so for simplicity I allow the monetary authority to make this choice directly. The net debt position of the fiscal and monetary authority cannot be negative. A negative net debt position implies that the monetary authority is holding more debt than was issued by the fiscal authority.

The money supply issued by the monetary authority in date-event s^t is $M\left(s^t\right)>0$.

In the initial period s_0, the monetary authority has the nominal obligation $W\left(s_0\right)$.

The monetary authority has the following budget constraints, where the liabilities of the monetary authority are on the left-hand side of the equations and the assets of the monetary authority are on the right-hand side of the equations:

$$W\left(s_0\right)=M\left(s_0\right)+\sum_{j\in\mathbf{J}}q_j\left(s_0\right)B_j\left(s_0\right). \tag{10}$$

$$M\left(s^{t-1}\right)+\sum_{j\in\mathbf{J}}q_{j-1}\left(s^t\right)B_j\left(s^{t-1}\right)=M\left(s^t\right)+\sum_{j\in\mathbf{J}}q_j\left(s^t\right)B_j\left(s^t\right)\ \ \forall t,s^t.$$

2.3 Sequential competitive equilibrium

Definition 1 *A sequential competitive equilibrium (SCE) is the vector of household variables* $\left\{c^h\left(s^t\right), m^h\left(s^t\right), b^h\left(s^t\right)\right\}_{h\in\mathbf{H}}$*, the monetary authority variables* $\left\{B\left(s^t\right), M\left(s^t\right)\right\}$*, and the price variables* $\left\{p\left(s^t\right), q\left(s^t\right)\right\}$ *such that:*

1. *Given* $\left\{p\left(s^t\right), q\left(s^t\right)\right\}$ *and* $\omega^h\left(s_0\right)$*, each household chooses* $\left\{c^h\left(s^t\right), m^h\left(s^t\right), b^h\left(s^t\right)\right\}$ *to solve the household problem (9).*

2. *Given* $W\left(s_0\right)$*, the monetary authority variables* $\left\{B\left(s^t\right), M\left(s^t\right)\right\}$ *satisfy (10).*

3. *Markets clear:*

   (a) $\sum_{h\in\mathbf{H}} c^h\left(s^t\right) = \sum_{h\in\mathbf{H}} \mathbf{e}^h\left(s_t\right)\ \forall t, s^t.$

   (b) $\sum_{h\in\mathbf{H}} \omega^h\left(s_0\right) = W\left(s_0\right).$

   (c) $\sum_{h\in\mathbf{H}} m^h\left(s^t\right) = M\left(s^t\right)\ \forall t, s^t.$

   (d) $\sum_{h\in\mathbf{H}} b_j^h\left(s^t\right) = B_j\left(s^t\right)\quad \forall j \in \mathbf{J}$ *and* $\forall t, s^t.$

The equilibrium asset price $q_1\left(s^t\right) \leq 1$. Otherwise, the market clearing condition on the bond markets is not satisfied as households prefer to save using money holdings and not bond holdings. If $q_1\left(s^t\right) < 1$, the cash-in-advance constraints (5) will bind for all households. With binding cash-in-advance constraints (5), the market clearing condition for the money markets implies that the Quantity Theory of Money holds:

$$M\left(s^t\right) = p\left(s^t\right)\sum_{h\in\mathbf{H}} \mathbf{e}^h\left(s_t\right) = p\left(s^t\right)\mathbf{E}\left(s_t\right). \tag{11}$$

The Friedman rule in date-event s^t is such that $q_1\left(s^t\right) = 1$. Under the Friedman rule, money and the 1-period bond are perfect substitutes. Market clearing for both implies that the sum of the two is pinned down for all households and the monetary authority, but not the composition. The cash-in-advance constraints (5) need not bind under the Friedman rule. It is innocuous (i.e., without real effects) under the Friedman rule to set the household money holdings such that the cash-in-advance constraints (5) bind. This would allow the Quantity Theory of Money (11) to hold.

3 Markov Equilibrium

3.1 Constraints in real terms

Define the real debt positions for the monetary authority and the real bond positions for the households as $\hat{B}_j(s^t) = \frac{B_j(s^t)}{p(s^t)}$ and $\hat{b}_j^h(s^t) = \frac{b_j^h(s^t)}{p(s^t)}$, respectively. The portfolios are denoted $\hat{B}(s^t) = \left(\hat{B}_j(s^t)\right)_{j\in\mathbf{J}}$ and $\hat{b}^h(s^t) = \left(\hat{b}_j^h(s^t)\right)_{j\in\mathbf{J}}$, respectively. Market clearing in terms of nominal bond positions occurs if and only if market clearing in the real bond positions occurs. Additionally, define the inflation rate $\pi(s^t) = \frac{p(s^t)}{p(s^{t-1})}$.

The monetary authority constraints (10) in real terms, after using the Quantity Theory of Money (11), are given by:

$$\frac{1}{\pi(s^t)}\left(\mathbf{E}(s_{t-1}) + \sum_{j\in\mathbf{J}} q_{j-1}\left(s^t\right)\hat{B}_j\left(s^{t-1}\right)\right) = \mathbf{E}(s_t) + \sum_{j\in\mathbf{J}} q_j\left(s^t\right)\hat{B}_j\left(s^t\right). \tag{12}$$

The household problem will be recursive in terms of real wealth. Define the real wealth for household h entering date-event s^t as

$$\hat{\omega}^h(s^t) = \frac{\omega^h(s^t)}{p(s^t)} = \frac{1}{\pi(s^t)}\left(\mathbf{e}^h(s_{t-1}) + \sum_{j\in\mathbf{J}} q_{j-1}\left(s^t\right)\hat{b}_j^h\left(s^{t-1}\right)\right).$$

In equilibrium, the household budget constraints in equilibrium are given by:

$$c^h\left(s^t\right) + \sum_{j\in\mathbf{J}} q_j\left(s^t\right)\hat{b}_j^h\left(s^t\right) = \hat{\omega}^h(s^t). \tag{13}$$

The first order conditions with respect to bonds $\hat{b}_j^h(s^t)$ are given by:

$$q_j\left(s^t\right) = \beta\sum_{\sigma\in\mathbf{S}}\Gamma(s_t,\sigma)\left(\frac{c^h(s^t,\sigma)}{c^h(s^t)}\right)^{-\rho}\frac{q_{j-1}(s^t,\sigma)}{\pi(s^t,\sigma)}. \tag{14}$$

3.2 State space

For bonds traded in date-event s^t, I collect the payouts for all bonds $j\in\mathbf{J}$ in subsequent date-events $(s^t,\sigma)_{\sigma\in\mathbf{S}}$ in the real payout matrix $R(s^t)\in\mathbb{R}_+^{S,J}$, defined by

$$R\left(s^t\right) = \begin{bmatrix} \frac{1}{\pi(s^t,1)} & \frac{q_1(s^t,1)}{\pi(s^t,1)} & .. & \frac{q_{J-1}(s^t,1)}{\pi(s^t,1)} \\ : & : & : & : \\ \frac{1}{\pi(s^t,S)} & \frac{q_1(s^t,S)}{\pi(s^t,S)} & .. & \frac{q_{J-1}(s^t,S)}{\pi(s^t,S)} \end{bmatrix}.$$

It is convenient to write the real payout matrix as a product of two underlying matrices. Algebraically,

$$R\left(s^t\right) = \left(\Pi\left(s^t\right)\right)^{-1}\left(Q_0^{J-1}\left(s^t\right)\right),$$

where $\Pi\left(s^t\right) = \begin{bmatrix} \pi\left(s^t,1\right) & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & \pi\left(s^t,S\right) \end{bmatrix}$ is called the inflation matrix and $Q_0^{J-1}\left(s^t\right) = \begin{bmatrix} 1 & q_1(s^t,1) & .. & q_{J-1}(s^t,1) \\ : & : & : & : \\ 1 & q_1(s^t,S) & .. & q_{J-1}(s^t,S) \end{bmatrix}$ is called the bond price matrix. Each element in $Q_0^{J-1}\left(s^t\right)$ is inversely related to the interest rate for that particular bond and date-event. The matrix $Q_0^{J-1}\left(s^t\right)$ characterizes the term structure of interest rates. Since the inflation matrix $\Pi\left(s^t\right)$ has full rank, the real payout matrix $R\left(s^t\right)$ has full rank iff the bond matrix $Q_0^{J-1}\left(s^t\right)$ has full rank.

I restrict policy such that $Q_0^{J-1}\left(s^t\right)$ has full rank in all date-events. Hoelle (2014) shows that if the model includes a social welfare function for the monetary authority to maximize, the full rank condition must be satisfied unless a Pareto efficient allocation can be supported with a rank deficient bond price matrix.

Full rank condition $\quad Q_0^{J-1}\left(s^t\right)$ has full rank $\forall t, s^t$.

I utilize the recursive equilibrium concept of Markov equilibrium. The state space includes the aggregate shock realization in the current period, the wealth distribution, the vector of bond holdings for all households, and the bond prices. Define $\hat{\omega}(s^t) = \left(\hat{\omega}^h(s^t)\right)_{h\in\mathbf{H}\setminus\{H\}} \in \mathbb{R}^{H-1}$ as the real wealth distribution. Aggregate resource constraints uniquely determine the total real wealth $\sum_{h\in\mathbf{H}} \hat{\omega}^h(s^t)$, so only $(H-1)$ real wealth variables need be included in the state space. The bond prices must satisfy (14), implying that $q\left(s^t\right) \in \Delta_q$ for all date-events, where Δ_q is the compact set defined by:

$$\Delta_q = \left\{q\left(s^t\right) \in [0,1]^J : q_J\left(s^t\right) \leq ... \leq q_1\left(s^t\right)\right\}.$$

Definition 2 *The policy rule is the function* $\mathbf{T} : \mathbf{S} \times \mathbb{R}^{H-1} \rightarrow \Delta_q$ *such that* $q\left(s^t\right) = \mathbf{T}\left(s_t, \hat{\omega}\left(s^t\right)\right)$ *in all date-events.*

Define $\hat{b}\left(s^t\right) = \left(\hat{b}^h\left(s^t\right)\right)_{h\in\mathbf{H}} \in \mathbb{R}^{HJ}$ as the set of all household bond holdings. Given that $\hat{B}_j\left(s^t\right) = \sum_{h\in\mathbf{H}} \hat{b}_j^h\left(s^t\right) \geq 0$, the bond holdings must satisfy $\hat{b}\left(s^t\right) \in \Delta_b$ for all date-events, where Δ_b is the closed set defined by:

$$\Delta_b = \left\{ \hat{b}\left(s^t\right) \in \mathbb{R}^{HJ} : \sum_{h \in \mathbf{H}} \hat{b}_j^h\left(s^t\right) \geq 0 \right\}.$$

The state space is $\mathbf{S} \times \mathbb{R}^{H-1} \times \Delta_b \times \Delta_q$ with typical element $\left(s_t, \hat{\omega}(s^t), \hat{b}\left(s^t\right), q\left(s^t\right)\right)$. Given the equilibrium requirement of nonnegative consumption, define the equilibrium state space $\Omega \subseteq \mathbf{S} \times \mathbb{R}^{H-1} \times \Delta_b \times \Delta_q$ as the closed subset defined by:

$$\Omega = \left\{ \begin{array}{c} \left(s_t, \hat{\omega}(s^t), \hat{b}\left(s^t\right), q\left(s^t\right)\right) \in \mathbf{S} \times \mathbb{R}^{H-1} \times \Delta_b \times \Delta_q : \\ c^h\left(s^t\right) = \hat{\omega}^h\left(s^t\right) - \sum_{j \in \mathbf{J}} q_j\left(s^t\right) \hat{b}_j^h\left(s^t\right) \geq 0 \ \text{ for } h \in \mathbf{H} \backslash \{H\} \\ \sum_{h \in \mathbf{H} \backslash \{H\}} c^h\left(s^t\right) \leq \mathbf{E}\left(s\right) \end{array} \right\}.$$

3.3 Expectations correspondence

Define $\mathbf{Z}_\sigma \subseteq \mathbb{R}^{H-1} \times \Delta_b \times \Delta_q$ as the set containing the variables $\left(\hat{\omega}'\left(\sigma\right), \hat{b}'\left(\sigma\right), q'\left(\sigma\right)\right)$ such that (i) $c'^h(\sigma) = \hat{\omega}'^h(\sigma) - \sum_{j \in \mathbf{J}} q_j'(\sigma)\hat{b}_j'^h(\sigma) \geq 0$ for $h \in \mathbf{H} \backslash \{H\}$ and (ii) $\sum_{h \in \mathbf{H} \backslash \{H\}} c'^h(\sigma) \leq \mathbf{E}\left(\sigma\right)$. The expectations correspondence $g : \Omega \rightrightarrows \prod_{\sigma \in \mathbf{S}} \mathbf{Z}_\sigma$ is defined such that for

$$z = \left(\hat{\omega}, \hat{b}, q\right) \text{ and}$$
$$z'\left(\sigma\right) = \left(\hat{\omega}'\left(\sigma\right), \hat{b}'\left(\sigma\right), q'\left(\sigma\right)\right) \quad \forall \sigma \in \mathbf{S},$$

the vector of variables $\left(z'\left(1\right), ..., z'\left(S\right)\right) \in g\left(s, z\right)$ if the following conditions hold:

1. For all $\sigma \in \mathbf{S}$, $q'\left(\sigma\right) = \mathbf{T}\left(\sigma, \hat{\omega}'\left(\sigma\right)\right)$.

2. For all $j \in \mathbf{J}$, $\hat{B}_j = \sum_{h \in \mathbf{H}} \hat{b}_j^h$ and $\hat{B}_j'\left(\sigma\right) = \sum_{h \in \mathbf{H}} \hat{b}_j'^h\left(\sigma\right) \ \forall \sigma \in \mathbf{S}$.

3. For all $h \in \mathbf{H} \backslash \{H\}$, $c^h = \hat{\omega}^h - \sum_{j \in \mathbf{J}} q_j \hat{b}_j^h$ and $c'^h(\sigma) = \hat{\omega}'^h(\sigma) - \sum_{j \in \mathbf{J}} q_j'(\sigma)\hat{b}_j'^h(\sigma) \ \forall \sigma \in \mathbf{S}$.

4. For all $h \in \mathbf{H} \backslash \{H\}$ and all $\sigma \in \mathbf{S}$, the household wealth

$$\hat{\omega}'^h(\sigma) = \frac{1}{\pi'\left(\sigma\right)} \left(\mathbf{e}^h\left(s\right) + \sum_{j \in \mathbf{J}} q_{j-1}'\left(\sigma\right) \hat{b}_j^h \right). \tag{15}$$

5. For all $\sigma \in \mathbf{S}$, the monetary authority constraint (12):

$$\frac{1}{\pi'(\sigma)}\left(\mathbf{E}(s) + \sum_{j\in\mathbf{J}} q'_{j-1}(\sigma)\hat{B}_j\right) = \mathbf{E}(\sigma) + \sum_{j\in\mathbf{J}} q'_j(\sigma)\hat{B}'_j(\sigma). \tag{16}$$

6. For all $j \in \mathbf{J}$ and all $h \in \mathbf{H}$, the Euler equation (14):

$$q_j = \beta \sum_{\sigma\in\mathbf{S}} \Gamma(s,\sigma)\left(\frac{c'^h(\sigma)}{c^h}\right)^{-\rho} \frac{q'_{j-1}(\sigma)}{\pi'(\sigma)}. \tag{17}$$

3.4 Markov equilibrium definition

A Markov equilibrium is defined by a policy correspondence $\mathbf{V} : \mathbf{S} \times \mathbb{R}^{H-1} \rightrightarrows \Delta_b \times \Delta_q$ and a transition correspondence $\mathbf{F} : graph(\mathbf{V}) \rightrightarrows \prod_{\sigma\in\mathbf{S}} \mathbf{Z}_\sigma$ satisfying the following two properties:

1. For all $\left(s, \hat{\omega}, \hat{b}, q\right) \in graph(\mathbf{V})$ and all $\sigma \in \mathbf{S}$,

$$\mathbf{F}_\sigma\left(s, \hat{\omega}, \hat{b}, q\right) \subseteq g_\sigma\left(s, \hat{\omega}, \hat{b}, q\right). \tag{18}$$

2. For all $\left(s, \hat{\omega}, \hat{b}, q\right) \in graph(\mathbf{V})$ and all $\sigma \in \mathbf{S}$,

$$\left(\sigma, \mathbf{F}_\sigma\left(s, \hat{\omega}, \hat{b}, q\right)\right) \subseteq graph(\mathbf{V}). \tag{19}$$

Theorem 1 *Any Markov equilibrium in which the full rank condition is satisfied is equivalent to a sequential competitive equilibrium.*

Proof. See Section A.1. ■

Existence of a Markov equilibrium requires the following assumption on the policy rule.

Assumption 4 The policy rule $\mathbf{T} : \mathbf{S} \times \mathbb{R}^{H-1} \to \Delta_q$ is continuous.

Theorem 2 *If Assumption 4 is satisfied, then a Markov equilibrium exists.*

Proof. The proof of the existence of a Markov equilibrium proceeds by construction as in Duffie et al. (1994) and Kubler and Schmedders (2003). See Section A.2. ■

4 Necessary conditions for Pareto Efficiency

Under Assumption 2, the Pareto set is characterized by the consumption fractions $\left(\theta^h\right)_{h\in\mathbf{H}} \in \Delta^{H-1}$ such that $c^h\left(s^t\right) = \theta^h \mathbf{E}\left(s_t\right)$ for all h and for all date-events.

4.1 Stationary policy rules

Monetary authorities choose from the set of policy rules for which (i) a Markov equilibrium exists and (ii) the Markov equilibrium is equivalent to a sequential competitive equilibrium. Since continuity is sufficient for a Markov equilibrium to exist, monetary authorities are restricted to choose $\mathbf{T}$ from the set of continuous mappings for which the full rank condition is satisfied.

I denote a dynamic policy rule as any policy rule satisfying Assumption 4 and the full rank condition. A special case of a dynamic policy rule is a stationary policy rule.

Definition 3 *A stationary policy rule* $\mathbf{T} : \mathbf{S} \times \mathbb{R}^H \to \Delta_q$ *satisfies Assumption 4, the full rank condition, and specifies* $\left(\mathbf{q}_j\left(s\right)\right)_{(j,s)\in\mathbf{J}\times\mathbf{S}} \in \left(\Delta_q\right)^S$ *such that* $\left(\mathbf{q}_j\left(s_t\right)\right)_{j\in\mathbf{J}} = \mathbf{T}\left(s_t, \hat{\omega}(s^t)\right)$ *for all date-events.*

Lemma 1 *If the Markov equilibrium allocation is Pareto efficient and the policy rule is stationary, then the real wealth vectors* $\hat{\omega}^h\left(s^t\right)$ *are stationary, meaning that there exists* $\left(\hat{\boldsymbol{\omega}}^h\left(s\right)\right)_{(h,s)\in\mathbf{H}\times\mathbf{S}}$ *such that* $\hat{\omega}^h\left(s^t\right) = \hat{\boldsymbol{\omega}}^h\left(s_t\right)$ *for all* h *and all date-events.*

Proof. See Section A.3. ■

4.1.1 Case 1: $J < H$

Case 1 also requires $(H, J) \neq (2, 1)$.

Define the vector of stationary real household wealth as

$$\hat{\boldsymbol{\omega}}^h = \left(\hat{\boldsymbol{\omega}}^h\left(s\right)\right)_{s\in\mathbf{S}} \in \mathbb{R}^S.$$

With a stationary policy rule, the asset price matrix

$$Q_0^{J-1}\left(s^t\right) = \mathbf{Q}_0^{J-1} = \begin{bmatrix} 1 & \mathbf{q}_1\left(1\right) & .. & \mathbf{q}_{J-1}\left(1\right) \\ : & : & : & : \\ 1 & \mathbf{q}_1\left(S\right) & .. & \mathbf{q}_{J-1}\left(S\right) \end{bmatrix}.$$

The space Φ is referred to as the asset span. The asset payout matrix is $(\Pi(s^t))^{-1}\left(\mathbf{Q}_0^{J-1}\right)$ and the asset span is defined as

$$\left\langle (\Pi(s^t))^{-1}\left(\mathbf{Q}_0^{J-1}\right)\right\rangle = \left\{x \in \mathbb{R}^S : x = (\Pi(s^t))^{-1}\left(\mathbf{Q}_0^{J-1}\right) b \text{ for some } b \in \mathbb{R}^J\right\}.$$

The asset span $\left\langle (\Pi(s^t))^{-1}\left(\mathbf{Q}_0^{J-1}\right)\right\rangle$ is a linear subspace with dimension equal to $rank\left(\mathbf{Q}_0^{J-1}\right) = J$. The household budget constraints are satisfied provided that

$$\hat{\boldsymbol{\omega}}^h \in \left\langle (\Pi(s^t))^{-1}\left(\mathbf{Q}_0^{J-1}\right)\right\rangle \quad \forall h \in \mathbf{H}. \tag{20}$$

Lemma 2 *If $H > 2$, the stationary real wealth vectors have full rank, namely rank* $\left(\left(\hat{\boldsymbol{\omega}}^h\right)_{h\in\mathbf{H}}\right) = H$.

Proof. See Section A.4. ■

Under Case 1, there does not exist an asset span Φ such that $\hat{\boldsymbol{\omega}}^h \in \Phi \; \forall h \in \mathbf{H}$, which implies that a stationary policy rule cannot support a Pareto efficient Markov equilibrium allocation.

4.1.2 Case 2: $J \geq H$

Case 2 also includes the economy $(H, J) = (2, 1)$. Under Case 2, an asset span Φ can be mathematically found such that $\hat{\boldsymbol{\omega}}^h \in \Phi \; \forall h \in \mathbf{H}$, but the asset span Φ must be able to be supported by policy:

$$\Phi = \left\langle (\Pi(s^t))^{-1}\left(\mathbf{Q}_0^{J-1}\right)\right\rangle.$$

Employ the following transformation for the bond holding variables:

$$\begin{aligned} \tilde{b}_1^h\left(s^t\right) &= \mathbf{e}^h(s_t) + \hat{b}_1^h\left(s^t\right) \text{ and} \\ \tilde{b}_j^h\left(s^t\right) &= \hat{b}_j^h\left(s^t\right) \text{ for } j > 1. \end{aligned} \tag{21}$$

The debt positions can be similarly transformed:

$$\begin{aligned} \tilde{B}_1\left(s^t\right) &= \mathbf{E}(s_t) + \hat{B}_1\left(s^t\right) \text{ and} \\ \tilde{B}_j\left(s^t\right) &= \hat{B}_j\left(s^t\right) \text{ for } j > 1. \end{aligned} \tag{22}$$

Market clearing for the new variables holds if and only if market clearing for the original variables holds.

Lemma 3 *Under Case 2, a stationary policy rule must be supported by stationary inflation rates, meaning that there exists* $(\boldsymbol{\pi}(s))_{s\in\mathbf{S}}$ *such that* $\pi(s^t,\sigma)=\boldsymbol{\pi}(\sigma)$ *for all date-events.*

Proof. See Section A.5. ■

Since the real wealth vectors $\hat{\omega}^h(s^t)$ and the inflation rates $\pi(s^t)$ are stationary, $\tilde{b}^h_j(s^t)$ must be constant to satisfy the wealth equation (15):

$$\hat{\boldsymbol{\omega}}^h(\sigma)=\frac{1}{\boldsymbol{\pi}(\sigma)}\sum_{j\in\mathbf{J}}\mathbf{q}_{j-1}(\sigma)\,\tilde{b}^h_j(s^t).$$

Define the constant vectors $(\mathbf{b}^h_j)_{j\in\mathbf{S}}$ and $(\mathbf{B}_j)_{s\in\mathbf{S}}$ such that $\tilde{b}^h_j(s^t)=\mathbf{b}^h_j$ and $\tilde{B}_j(s^t)=\mathbf{B}_j$ for all h, for all j, and for all date-events.

There are two equivalent arguments to derive the generic necessary conditions for Pareto efficiency.

System of equations argument As a function of $(\mathbf{q}_1(s))_{s\in\mathbf{S}}$, the Euler equation for the short-term bond uniquely determines $(\boldsymbol{\pi}(s))_{s\in\mathbf{S}}$:

$$\mathbf{q}_1(s_t)=\beta\sum_{\sigma\in\mathbf{S}}\Gamma(s_t,\sigma)\left(\frac{\mathbf{E}(\sigma)}{\mathbf{E}(s_t)}\right)^{-\rho}\frac{1}{\boldsymbol{\pi}(\sigma)}. \tag{23}$$

The Euler equation for the longer-term bonds uniquely determine $(\mathbf{q}_j(s))_{s\in\mathbf{S}}$ for all $j>1$:

$$\mathbf{q}_j(s_t)=\beta\sum_{\sigma\in\mathbf{S}}\Gamma(s_t,\sigma)\left(\frac{\mathbf{E}(\sigma)}{\mathbf{E}(s_t)}\right)^{-\rho}\frac{\mathbf{q}_{j-1}(\sigma)}{\boldsymbol{\pi}(\sigma)}. \tag{24}$$

From Walras' Law, the monetary authority constraint (12) is equivalent to the sum of the household budget constraints (13). This implies a total of SH independent budget constraints:

$$\theta^h\mathbf{E}(s)-\mathbf{q}_1(s)\,\mathbf{e}^h(s)+\sum_{j\in\mathbf{J}}\mathbf{b}^h_j\left(\mathbf{q}_j(s)-\frac{\mathbf{q}_{j-1}(s)}{\boldsymbol{\pi}(s)}\right)=0. \tag{25}$$

The discounted present value budget constraints are used to determine the initial period price level $p(s_0)$ and the consumption fractions $(\theta^h)_{h\in\mathbf{H}}$. Under stationary policy rules and Pareto efficiency, the initial period discounted present value constraints for households are given by:

$$\frac{\omega^h(s_0)}{p(s_0)}=\sum_{k=0}^{\infty}\beta^kE_0\left[\left(\frac{\mathbf{E}(s_k)}{\mathbf{E}(s_0)}\right)^{-\rho}\left(\theta^h\mathbf{E}(s_k)-\mathbf{q}_1(s_k)\,\mathbf{e}^h(s_k)\right)\right].[7] \tag{26}$$

[7] The derivation is contained in the proof of Lemma 1.

Define the stochastic discount factor matrix $\hat{\Gamma}$ as the $S \times S$ matrix with elements $\hat{\Gamma}(s,\sigma) = \beta\Gamma(s,\sigma)\left(\frac{\mathbf{E}(\sigma)}{\mathbf{E}(s)}\right)^{-\rho}$. The initial period discounted present value constraints for households can be expressed recursively as:

$$\frac{\omega^h(s_0)}{p(s_0)} = \left(I_S - \hat{\Gamma}\right)^{-1}_{(s_0)} \left(\theta^h \mathbf{E}(s) - \mathbf{q}_1(s)\,\mathbf{e}^h(s)\right)_{s\in\mathbf{S}}, \tag{27}$$

where I_S is the $S \times S$ identity matrix and $\left(I_S - \hat{\Gamma}\right)^{-1}_{(s_0)}$ is row s_0 of the matrix $\left(I_S - \hat{\Gamma}\right)^{-1}$.

From the recursive expression (27) for all households $h \in \mathbf{H}$, the short-term bond prices $(\mathbf{q}_1(s))_{s\in\mathbf{S}}$ uniquely determine the initial price level $p(s_0)$ and the consumption fractions $\left(\theta^h\right)_{h\in\mathbf{H}}$ as follows:

$$\begin{aligned} p(s_0) &= \frac{W(s_0)}{\left(I_S - \hat{\Gamma}\right)^{-1}_{(s_0)} \left(\mathbf{E}(s)\left(1 - \mathbf{q}_1(s)\right)\right)_{s\in\mathbf{S}}}. \\ \theta^h &= \frac{\left(I_S - \hat{\Gamma}\right)^{-1}_{(s_0)} \left(\omega^h(s_0)\,\mathbf{E}(s)\left(1 - \mathbf{q}_1(s)\right) + W(s_0)\,\mathbf{q}_1(s)\,\mathbf{e}^h(s)\right)_{s\in\mathbf{S}}}{\left(I_S - \hat{\Gamma}\right)^{-1}_{(s_0)} \left(W(s_0)\,\mathbf{E}(s)\right)_{s\in\mathbf{S}}}. \end{aligned} \tag{28}$$

Claim 1 *From the equilibrium equation for θ^h in (28), $\theta^h \in [0,1]$ $\forall h \in \mathbf{H}$.*

Proof. See Section A.6. ■

The following table displays equations and variables:

period 0 (27)	H equations	$p(s_0), \left(\theta^h\right)_{h\in\mathbf{H}\setminus\{H\}}$	H variables
Euler (23),(24)	SJ equations	$(\mathbf{q}_j(s))_{(j,s)\in\mathbf{J}\times\mathbf{S}}$	SJ variables
hh bc (25)	SH equations	$(\boldsymbol{\pi}(s))_{s\in\mathbf{S}}$	S variables
		$\left(\mathbf{b}^h_j\right)_{(h,j)\in\mathbf{H}\times\mathbf{J}}$	HJ variables
Total	$H + SJ + SH$	Total	$H + SJ + S + HJ$ variables

Table 1: Variables for stationary policy rules

The number of variables exceeds the number of equations when

$$SH \leq S + HJ.^8$$

[8] For the special case of $(H,J) = (2,1)$, the number of variables equals $S+1$ as the bond holdings $\mathbf{b}^2$ are redundant given $\mathbf{b}^1$ and the fact that $\hat{\boldsymbol{\omega}}^1 \in \Phi$ and $\hat{\boldsymbol{\omega}}^2 \in \Phi$, where Φ is 1-dimensional. The number of equations equals S as the budget constraints for household $h=2$ are redundant given the budget constraints of $h=1$. The number of variables exceeds the number of equations as $S+1 \geq S$ is satisfied for any value of S.

This is equivalent

$$J \geq \frac{S(H-1)}{H}.$$

The generic argument is completed in the proof of Lemma 4 in which I show that the equations are linearly independent (generically). In other words, if $J < \frac{S(H-1)}{H}$, then over a generic subset of household endowments, a solution to the equations does not exist.

Lemma 4 *If $H \leq J < \frac{S(H-1)}{H}$, then over a generic subset of household endowments, a stationary policy rule cannot support the Pareto efficient Markov equilibrium allocation.*

Proof. See Section A.7. ∎

Span argument The set of all asset spans is the Grassmanian manifold $Gr(J, S)$. The set $Gr(J,S)$ is $J(S-J)-$dimensional. If $H = J$, there exists a unique asset span $\Phi \in Gr(J,S)$ in this set such that $\hat{\boldsymbol{\omega}}^h \in \Phi\ \forall h \in \mathbf{H}$.

If $H < J$, then there exists a continuum of $\Phi \in Gr(J,S)$ such that $\hat{\boldsymbol{\omega}}^h \in \Phi\ \forall h \in \mathbf{H}$. The dimension of this continuum equals $(J-H)(S-J)$. In the $J(S-J)-$dimensional set of asset spans, any asset span in a $(J-H)(S-J)-$dimensional subset satisfies $\hat{\boldsymbol{\omega}}^h \in \Phi\ \forall h \in \mathbf{H}$. Thus, the number of policy variables must exceed $J(S-J) - (J-H)(S-J) = H(S-J)$.[9]

The variables under a stationary policy rule that change the asset span are $(\mathbf{q}_1(s))_{s \in \mathbf{S}}$. The vector $(\mathbf{q}_1(s))_{s \in \mathbf{S}}$ is $S-$dimensional. The number of policy variables exceeds $H(S-J)$ when $J \geq \frac{S(H-1)}{H}$.[10]

The generic argument is completed when I show that if $J < \frac{S(H-1)}{H}$, then over a generic subset of household endowments, there does not exist a vector $(\mathbf{q}_1(s))_{s \in \mathbf{S}}$ capable of supporting the Pareto efficient equilibrium asset span.

4.1.3 Necessary conditions for Pareto efficiency

The following span condition is a generic necessary condition for Pareto efficiency under stationary policy rules.

$$\text{Condition SC (span condition)} \qquad (H, J) = (2,1) \text{ or } J \geq \max\left\{H, \tfrac{S(H-1)}{H}\right\}.$$

Theorem 3 *Under Assumptions 1-4, Condition SC is a generic necessary condition for a stationary policy rule to support a Pareto efficient Markov equilibrium allocation.*

Proof. The proof is immediate given the results from Lemmas 2 and 4. ∎

[9] For the special case of $(H, J) = (2,1)$, a policy of dimension $J(S-J) = (S-1)$ is required.

[10] For the special case of $(H, J) = (2,1)$, the number of policy variables is S and this must exceed $S-1$ from the previous footnote, which is trivially satisfied.

4.2 Dynamic policy rules

The generic necessary condition is the same for the case of dynamic policy rules.

Theorem 4 *Under Assumptions 1-4, Condition SC is a generic necessary condition for a dynamic policy rule to support a Pareto efficient Markov equilibrium allocation.*

Proof. See Section A.8. ■

If no stationary policy rule is consistent with Pareto efficiency, then no dynamic policy rule is either. The reason is that Pareto efficiency imposes a stationarity property on the real wealth distribution. While dynamic policy rules seems to offer more policy tools, they are no better at satisfying this stationarity property than stationary policy rules.

4.3 Not sufficient for Pareto efficiency

Condition SC is sufficient for the existence of a solution to the equilibrium equations (27), (23), (24), and (25). Condition SC is not sufficient for the existence of a Pareto efficient equilibrium as an equilibrium additionally requires the following equilibrium bounds:

1. $\hat{B}_j\left(s^t\right) \geq 0\ \forall j$ and $\forall s^t$.

2. $q_1\left(s^t\right) \in [0, 1]\ \forall s^t$.[11]

General sufficient conditions are not available as it is not possible to impose conditions on the size of the economy to ensure that the equilibrium bounds are satisfied. The only way to determine if a Pareto efficient equilibrium exists for a given economy is to (i) find the equilibrium variable values that satisfy (27), (23), (24), and (25) and (ii) verify that they satisfy the above equilibrium bounds.

The following section provides an example and illustrates the conditions under which each of the equilibrium bounds is violated.

5 Example

5.1 Economy parameters

The economy contains $J = 2$ assets, $H = 2$ household types, and $S = 4$ states of uncertainty. The necessary condition $J \geq \max\left\{H, \frac{S(H-1)}{H}\right\}$ for $J = H = 2$ corresponds to $S \leq 4$.

[11] If $q_1\left(s^t\right) \in [0, 1]$, Euler equations imply that $q_j\left(s^t\right) \in \left[0, q_{j-1}\left(s^t\right)\right]$ for all $j \in \{2, ..., J\}$.

The specific parameter values are $\beta = 0.92$, relative risk aversion $\rho = 4$, and Markov probabilities given by $\Gamma = \begin{bmatrix} 0.4 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.4 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.4 \end{bmatrix}$. The household endowments are given in the following table:

	$\mathbf{e}^1(\sigma)$	$\mathbf{e}^2(\sigma)$	$\mathbf{E}(\sigma)$
State $\sigma = 1$	14	12	26
State $\sigma = 2$	12	15	27
State $\sigma = 3$	10	18	28
State $\sigma = 4$	8	21	29

Table 2: Endowments

The endowment distribution provides the economy with two useful properties. First, the economy contains aggregate risk, with aggregate endowments monotonic in σ. Second, the endowment vectors are negatively correlated (in this case, the vectors have perfect negative correlation) and this generates the strongest incentives for trade and the largest portfolio effects. The choice of endowment distribution has no effect on equilibrium existence.

The initial period is $s_0 = 1$ and the initial period wealth parameters are $\omega^1(s_0) = \omega^2(s_0) = 10$.

The final subsection demonstrates that the existence of a Pareto efficient equilibrium is robust to changes in the parameters $(\beta, \rho, \Gamma, \omega(s_0))$.

5.2 Equilibrium values

With only $H = 2$ household types, θ^1 uniquely characterizes the Pareto efficient allocation. The equilibrium consumption fraction is $\theta^1 = 0.434$. The initial price level is given by $p(s_0) = 0.307$. With $\omega^1(s_0) = \omega^2(s_0) = 10$, the initial real wealth $\hat{\boldsymbol{\omega}}^h(s_0) = \frac{\omega^h(s_0)}{p(s_0)}$ (recall $s_0 = 1$) is given by $\hat{\boldsymbol{\omega}}^1(s_0) = \hat{\boldsymbol{\omega}}^2(s_0) = 32.609$.

The optimal policy choice supports the welfare-maximizing Pareto efficient allocation in equilibrium. The policy choice is such that the inflation rates are given in Table 3.

	$\boldsymbol{\pi}(\sigma)$	Inflation Rate
State $\sigma = 1$	1.305	30.5%
State $\sigma = 2$	1.210	21.0%
State $\sigma = 3$	1.125	12.5%
State $\sigma = 4$	1.050	5.0%

Table 3: Stochastic Inflation Rates

The stationary asset prices, which are the policy choices of the monetary authority, are given in Table 4.

	$\mathbf{q}_1(\sigma)$	$\mathbf{q}_2(\sigma)$
State $\sigma = 1$	0.648	0.491
State $\sigma = 2$	0.741	0.578
State $\sigma = 3$	0.845	0.676
State $\sigma = 4$	0.959	0.787

Table 4: Asset Prices

It is straightforward to verify that the Euler equations for assets $j \in \{1, 2\}$ and states $\sigma \in \{1, 2, 3, 4\}$ are satisfied given the inflation rates and asset prices in Tables 3 and 4:

$$\mathbf{q}_j(\sigma) = \beta \sum_{\psi \in \mathbf{S}} \Gamma(\sigma, \psi) \left(\frac{\mathbf{E}(\psi)}{\mathbf{E}(\sigma)} \right)^{-\rho} \frac{\mathbf{q}_{j-1}(\psi)}{\boldsymbol{\pi}(\psi)} \quad \forall j \in \{1, 2\}, \forall \sigma \in \{1, 2, 3, 4\}.$$

Given the initial price level $p(s_0)$ and the stochastic inflation rates $(\boldsymbol{\pi}(\sigma))_{\sigma \in \mathbf{S}}$, it is straightforward to determine the time series for prices $\{p(s^t)\}$. Nevertheless, it is easier to express household wealth as real wealth $\hat{\omega}^h(s^t) = \frac{\omega^h(s^t)}{p(s^t)}$ and portfolio choices as the real bond holdings $\hat{b}_j^h(s^t) = \frac{b_j^h(s^t)}{p(s^t)}$. By definition, the real wealth distribution is given by:

$$\hat{\omega}^h\left(s^t, \sigma\right) = \frac{1}{\boldsymbol{\pi}(\sigma)} \left(\mathbf{e}^h(s) + \hat{b}_1^h\left(s^t\right) + \mathbf{q}_1(\sigma)\, \hat{b}_2^h\left(s^t\right) \right) \quad \forall h \in \{1, 2\}, \forall \sigma \in \{1, 2, 3, 4\}.$$

Per the equilibrium construction, $\mathbf{e}^h(s) + \hat{b}_1^h(s^t)$ is constant and $\hat{b}_2^h(s^t)$ is constant across all date-events. This implies that both real wealth and real bond holdings have a stationary distribution, meaning that there exists $(\hat{\boldsymbol{\omega}}^h(\sigma))_{\sigma \in \mathbf{S}}$ such that $\hat{\omega}^h(s^t, \sigma) = \hat{\boldsymbol{\omega}}^h(\sigma)$ and there exists $\left(\hat{\mathbf{b}}_j^h(\sigma)\right)_{\sigma \in \mathbf{S}}$ such that $\hat{b}_j^h(s^t, \sigma) = \hat{\mathbf{b}}_j^h(\sigma)$ for all date-events. The real wealth distribution for households is given in Table 5:

	$\hat{\boldsymbol{\omega}}^1(\sigma)$	$\hat{\boldsymbol{\omega}}^2(\sigma)$
State $\sigma = 1$	32.609	32.609
State $\sigma = 2$	38.222	33.130
State $\sigma = 3$	44.742	33.201
State $\sigma = 4$	52.280	32.729

Table 5: Wealth Distribution

Household $h = 2$ has higher endowments (on average), which results in higher consumption than household $h = 1$. The wealth distribution is not directly tied to consumption. Its values are determined to match the initial wealth positions (equal for both households) and support the equilibrium allocation. In other words, although household $h = 1$ receives higher portfolio payouts (as represented by the real wealth values), it also has larger portfolio expenditures (compare Tables 6.A and Tables 6.B below). The real effect is lower consumption for household $h = 1$ relative to $h = 2$.

The equilibrium budget constraints are given by:

$$c^h\left(s^t, \sigma\right) + \mathbf{q}_1(\sigma)\,\hat{\mathbf{b}}_1^h(\sigma) + \mathbf{q}_2(\sigma)\,\hat{\mathbf{b}}_2^h(\sigma) \leq \hat{\boldsymbol{\omega}}^h(\sigma) \;\; \forall h \in \{1,2\}, \forall \sigma \in \{1,2,3,4\}.$$

In equilibrium, $c^h(s^t, \sigma) = \theta^h \mathbf{E}(\sigma)$ $\forall h \in \{1, 2\}$ and $\forall \sigma \in \{1, 2, 3, 4\}$. Tables 6.A and 6.B illustrate that the budget constraints hold for both households in all states:

	$\hat{\mathbf{b}}_1^1(\sigma)$	$\hat{\mathbf{b}}_2^1(\sigma)$	$\theta^1\mathbf{E}(\sigma)$	$\mathbf{q}_1(\sigma)\,\hat{\mathbf{b}}_1^1(\sigma)$	$\mathbf{q}_2(\sigma)\,\hat{\mathbf{b}}_2^1(\sigma)$	$\hat{\boldsymbol{\omega}}^1(\sigma)$
State $\sigma = 1$	2.92	39.57	11.29	1.89	19.43	32.61
State $\sigma = 2$	4.92	39.57	11.73	3.64	22.85	38.22
State $\sigma = 3$	6.92	39.57	12.16	5.84	26.74	44.74
State $\sigma = 4$	8.92	39.57	12.60	8.55	31.13	52.28

Table 6.A: Household $h = 1$ budget constraint

	$\hat{\mathbf{b}}_1^2(\sigma)$	$\hat{\mathbf{b}}_2^2(\sigma)$	$\theta^2\mathbf{E}(\sigma)$	$\mathbf{q}_1(\sigma)\,\hat{\mathbf{b}}_1^2(\sigma)$	$\mathbf{q}_2(\sigma)\,\hat{\mathbf{b}}_2^2(\sigma)$	$\hat{\boldsymbol{\omega}}^2(\sigma)$
State $\sigma = 1$	47.55	-26.26	14.71	30.80	-12.90	32.61
State $\sigma = 2$	44.55	-26.26	15.27	33.02	-15.16	33.13
State $\sigma = 3$	41.55	-26.26	15.84	35.11	-17.75	33.20
State $\sigma = 4$	38.55	-26.26	16.40	36.99	-20.66	32.73

Table 6.B: Household $h = 2$ budget constraint

Walras' Law indicates that the real monetary authority debt positions consistent with market clearing must satisfy the monetary authority constraints $\forall \sigma \in \{1, 2, 3, 4\}$. As with household bond holdings, there exists $\left(\hat{\mathbf{B}}_j(\sigma)\right)_{\sigma \in \mathbf{S}}$ such that $\hat{B}_j(s^t, \sigma) = \hat{\mathbf{B}}_j(\sigma)$ for all date-events. The debt positions are nonnegative:

	$\hat{\mathbf{B}}_1(\sigma)$	$\hat{\mathbf{B}}_2(\sigma)$
State $\sigma = 1$	50.47	13.30
State $\sigma = 2$	49.47	13.30
State $\sigma = 3$	48.47	13.30
State $\sigma = 4$	47.47	13.30

Table 7: Monetary authority debt positions

5.3 Robustness

The example is robust to changes in economic parameters. Recall that the economic parameters are $\Gamma = \begin{bmatrix} 0.4 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.4 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.4 \end{bmatrix}$, $\beta = 0.92$, $\rho = 4$, and $\omega^1(s_0) = \omega^2(s_0) = 10$. Define the transition matrix in terms of a single persistence parameter ϕ such that $\Gamma = \begin{bmatrix} \phi & \frac{1-\phi}{3} & \frac{1-\phi}{3} & \frac{1-\phi}{3} \\ \frac{1-\phi}{3} & \phi & \frac{1-\phi}{3} & \frac{1-\phi}{3} \\ \frac{1-\phi}{3} & \frac{1-\phi}{3} & \phi & \frac{1-\phi}{3} \\ \frac{1-\phi}{3} & \frac{1-\phi}{3} & \frac{1-\phi}{3} & \phi \end{bmatrix}$. Table 8 below captures the ceterus paribus robustness analysis concerning the existence of a Pareto efficient equilibrium. For example, the row for ϕ displays the range of values for ϕ over which a Pareto efficient equilibrium exists, while holding fixed all parameters at the example values except ϕ. Table 8 also provides the reason for non-existence outside this range (the reason for non-existence for values below the lower bound on the far left and the reason for non-existence for values above the upper bound on the far right). Similar analysis occurs for the remaining 3 rows.

Parameters	**Reason**	**Lower**	**Example**	**Upper**	**Reason**
ϕ	N/A	0	0.4	0.55	$\mathbf{q}_1(4) > 1$
β	exceeds error bounds	0.70	0.92	0.932	$\mathbf{q}_1(4) > 1$
ρ	$\hat{\mathbf{B}}_2(\sigma) < 0$ all σ	2.4	4	4.6	$\mathbf{q}_1(4) > 1$
$\omega^1(s_0)$	exceeds error bounds	3.9	10	14.5	$\mathbf{q}_1(4) > 1$

Table 8: Robustness analysis over parameters $(\phi, \beta, \rho, \omega^1(s_0))$

The error "exceeds error bounds" means that the computational algorithm was unable to converge to a solution satisfying the error bounds of the algorithm. Due to the 'near' collinearity in the payout matrix required for a solution, the algorithm is unable to find a solution. Such a solution would not be practical anyway as it prescribes bond positions exceeding 10ˆ4. Despite the limitations of the model at the lower and upper bounds, it should be clear from Table 8 that the example considered above is robust to changes in economic parameters.

6 Conclusion

This paper has analyzed the real effects of policy that targets the term structure of interest rates. In a setting of incomplete markets, it is possible for policy to be chosen to support

Pareto efficiency and the necessary conditions for Pareto efficiency are characterized. The necessary conditions are equivalent for both stationary and dynamic policy rules and require that the number of financial markets must be sufficient to span the vector of portfolio payouts for all households.

With this theoretical foundation, a violation of the necessary condition leads to Pareto inefficient allocations for all policy rules (generically). Future work will analyze the welfare implications of dynamic policy rules relative to stationary policy rules when the necessary condition is violated.

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A Proofs

A.1 Proof of Claim 1 (Equivalence)

To show that Markov equilibria satisfy the SCE definition, the Euler equation (17) must be necessary and sufficient for household optimality. Necessary is immediate. Sufficiency follows as in Proposition 3.2 from Duffie et al. (1994) since the full rank condition is satisfied.

A.2 Proof of Theorem 2 (Existence)

One necessary condition for a SCE is that $\hat{b}(s^t)$ lies in a compact set for all date-events. Denote the compact set for bonds as $\Delta_b^* \subseteq \Delta_b$. The set Δ_q is already compact.

Define the initial correspondence $\mathbf{V}^0 : \mathbf{S} \times \mathbb{R}^{H-1} \rightrightarrows \Delta_b \times \Delta_q$ such that $\mathbf{V}^0(s, \hat{\omega}) = \Delta_b^* \times \Delta_q$ for all $(s, \hat{\omega}) \in \mathbf{S} \times \mathbb{R}^{H-1}$. Define the operator G that maps from the correspondence $\mathbf{V}^n$: $\mathbf{S} \times \mathbb{R}^{H-1} \rightrightarrows \Delta_b \times \Delta_q$ to a new correspondence $\mathbf{V}^{n+1} : \mathbf{S} \times \mathbb{R}^{H-1} \rightrightarrows \Delta_b \times \Delta_q$ as follows:

$$\mathbf{V}^{n+1}(s, \hat{\omega}) = \left\{ \begin{array}{l} \left(\hat{b}, q\right) \in \Delta_b \times \Delta_q : \exists \left(\hat{\omega}'(\sigma), q'(\sigma), \hat{b}'(\sigma)\right)_{\sigma \in \mathbf{S}} \text{ such that} \\ \text{(i) } \left(\hat{b}'(\sigma), q'(\sigma)\right) \in \mathbf{V}^n(\sigma, \hat{\omega}'(\sigma)) \text{ for all } \sigma \in \mathbf{S} \\ \text{(ii) } \left(\hat{b}, (\hat{\omega}'(\sigma))_{\sigma \in \mathbf{S}}\right) \text{ satisfy (15) and (17)} \end{array} \right\}.$$

In words, given the correspondence $\mathbf{V}^n : \mathbf{S} \times \mathbb{R}^{H-1} \rightrightarrows \Delta_b \times \Delta_q$, the solution to two nonlinear systems of equations determines the image of the new correspondence $\mathbf{V}^{n+1}$: $\mathbf{S} \times \mathbb{R}^{H-1} \rightrightarrows \Delta_b \times \Delta_q$.

The first nonlinear system of equations takes as given $(\hat{\omega}'(\sigma))_{\sigma \in \mathbf{S}}$. Therefore, for all $\sigma \in \mathbf{S}$, $\left(\hat{b}'(\sigma), q'(\sigma)\right)$ is determined by:

$$\left(\hat{b}'(\sigma), q'(\sigma)\right) \in \mathbf{V}^n(\sigma, \hat{\omega}'(\sigma)).$$

Given $\left(\hat{\omega}'(\sigma), \hat{b}'(\sigma), q'(\sigma)\right)$, the household budget constraint determines a unique value for $c'^h(\sigma)$ (for all households):

$$c'^h(\sigma) = \hat{\omega}'^h(\sigma) - \sum_{j \in \mathbf{J}} q_j'(\sigma) \hat{b}_j'^h(\sigma).$$

From the policy rule, $q = \mathbf{T}(s, \hat{\omega})$. There exists a unique vector $\left(\hat{b}, (\pi'(\sigma))_{\sigma \in \mathbf{S}}\right)$ such that the following equations are satisfied: (i) the Euler equations (17) for all J assets and all H households given the consumption equation $c^h = \hat{\omega}^h - \sum_{j \in \mathbf{J}} q_j \hat{b}_j^h$, and (ii) the monetary authority constraints (16) for all $\sigma \in \mathbf{S}$ given the market clearing conditions $\hat{B}_j = \sum_{h \in \mathbf{H}} \hat{b}_j^h$ and $\hat{B}_j'(\sigma) = \sum_{h \in \mathbf{H}} \hat{b}_j'^h(\sigma)$.

To verify uniqueness, since the asset prices $(q'(\sigma))_{\sigma \in \mathbf{S}}$ and the consumption $\left(c'^h(\sigma)\right)_{\sigma \in \mathbf{S}}$ vectors are fixed, the right-hand side of the Euler equation (17) is a strictly decreasing and continuous function of $\hat{b}$.[12] With HJ Euler equations and HJ variables $\hat{b}$, a unique solution is guaranteed.

The second nonlinear system of equations takes as given $\left(\hat{b}, (\pi'(\sigma), q'(\sigma))_{\sigma \in \mathbf{S}}\right)$. There exists a unique real wealth vector $\hat{\omega}'^h(\sigma)$ for all $\sigma \in \mathbf{S}$ and all $h \in \mathbf{H}$ from (15).

Define $\mathbf{V}^* : \mathbf{S} \times \mathbb{R}^{H-1} \rightrightarrows \Delta_b \times \Delta_q$ such that

$$\mathbf{V}^*(s, \hat{\omega}) = \bigcap_{n=0}^{\infty} \mathbf{V}^n(s, \hat{\omega}) \quad \text{for all } (s, \hat{\omega}) \in \mathbf{S} \times \mathbb{R}^{H-1}.$$

Lemma 5 *If the policy rule satisfies Assumption 4, then a Markov equilibrium exists.*

Proof. The argument follows as in Theorem 1 of Kubler and Schmedders (2003). Since a SCE exists for all parameters, then $\mathbf{V}^n$ is a well-defined correspondence for all $n \geq 0$. Since $\mathbf{T}$ is continuous, the graph of g is a closed subset of $\Omega \times \prod_{\sigma \in \mathbf{S}} \mathbf{Z}_\sigma$. Since the graph of g is a closed subset of $\Omega \times \prod_{\sigma \in \mathbf{S}} \mathbf{Z}_\sigma$, then $\mathbf{V}^n(s, \hat{\omega})$ is closed for all $n \geq 0$ and all $(s, \hat{\omega}) \in \mathbf{S} \times \mathbb{R}^{H-1}$. The images are nested, by construction: $\mathbf{V}^{n+1}(s, \hat{\omega}) \subseteq \mathbf{V}^n(s, \hat{\omega})$ for all $n \geq 0$ and all $(s, \hat{\omega}) \in \mathbf{S} \times \mathbb{R}^{H-1}$. The infinite intersection $\bigcap_{n=0}^{\infty} \mathbf{V}^n(s, \hat{\omega})$ of nested, closed, and nonempty sets is itself nonempty, and this holds for all $(s, \hat{\omega}) \in \mathbf{S} \times \mathbb{R}^{H-1}$. Thus, the policy correspondence $\mathbf{V}^* : \mathbf{S} \times \mathbb{R}^{H-1} \rightrightarrows \Delta_b \times \Delta_q$ is well-defined.

Additionally, the policy correspondence $\mathbf{V}^* : \mathbf{S} \times \mathbb{R}^{H-1} \rightrightarrows \Delta_b \times \Delta_q$ is upper hemicontinuous. The initial correspondence is upper hemicontinuous (as it is compact) and since the images of subsequent correspondences are closed and nested subsets of the image of the initial correspondence, then upper hemicontinuity is preserved. ■

[12] An increase in $\hat{b}_j^h$ means an increase in $\hat{B}_j$ from the market clearing condition. An increase in $\hat{B}_j$ means an increase in $\pi'(\sigma)$ from the monetary authority constraint (16) (for all states $\sigma \in \mathbf{S}$). An increase in $\hat{b}_j^h$ means a decrease in c^h (as q_j is positive) and a decrease in $\left(\frac{1}{c^h}\right)^{-\rho}$. Thus, the right-hand side of the Euler equation (17) is a strictly decreasing function.

A.3 Proof of Lemma 1

The household budget constraints for date-event s^t is given by:

$$c^h\left(s^t\right) - q_1\left(s^t\right)\mathbf{e}^h\left(s_t\right) + \sum_{j\in\mathbf{J}} q_j\left(s^t\right)\tilde{b}_j^h\left(s^t\right) = \frac{1}{\pi\left(s^t\right)}\sum_{j\in\mathbf{J}} q_{j-1}\left(s^t\right)\tilde{b}_j^h\left(s^{t-1}\right). \tag{29}$$

The exact same constraint can be written for date-event $\left(s^t,\sigma\right)$:

$$\frac{1}{\pi\left(s^t,\sigma\right)}\sum_{j\in\mathbf{J}} q_{j-1}\left(s^t,\sigma\right)\tilde{b}_j^h\left(s^t\right) = c^h\left(s^t,\sigma\right) - q_1\left(s^t,\sigma\right)\mathbf{e}^h\left(\sigma\right) + \sum_{j\in\mathbf{J}} q_j\left(s^t,\sigma\right)\tilde{b}_j^h\left(s^t,\sigma\right). \tag{30}$$

Multiply both sides of (30) by $\beta\left(\frac{c^h\left(s^t,\sigma\right)}{c^h\left(s^t\right)}\right)^{-\rho}$, take the conditional expectation, and cite the Euler equation (14):

$$\sum_{j\in\mathbf{J}} q_j(s^t)\tilde{b}_j^h\left(s^t\right) = E_t\left[\beta\left(\frac{c^h\left(s^t,\sigma\right)}{c^h\left(s^t\right)}\right)^{-\rho}\left(c^h\left(s^t,\sigma\right) - q_1\left(s^t,\sigma\right)\mathbf{e}^h\left(\sigma\right) + \sum_{j\in\mathbf{J}} q_j\left(s^t,\sigma\right)\tilde{b}_j^h\left(s^t,\sigma\right)\right)\right]. \tag{31}$$

The real household wealth vectors are defined as:

$$\hat{\omega}^h\left(s^t\right) = \frac{1}{\pi\left(s^t\right)}\sum_{j\in\mathbf{J}} q_{j-1}\left(s^t\right)\tilde{b}_j^h\left(s^{t-1}\right).$$

Inserting the new expression (31) back into the date-event s^t budget constraint (29) and iterating forward yields the discounted present value equation

$$\hat{\omega}^h\left(s^t\right) = \sum_{k=0}^{\infty}\sum_{s^{t+k}\succ s^t}\beta^k E_t\left[\left(\frac{c^h\left(s^{t+k}\right)}{c^h\left(s^t\right)}\right)^{-\rho}\left(c^h\left(s^{t+k}\right) - q_1\left(s^{t+k}\right)\mathbf{e}^h\left(s_{t+k}\right)\right)\right], \tag{32}$$

after citing the transversality condition. The discounted present value equation (32) must hold in all date-events s^t.

Under Pareto efficiency, there exist fractions $\left(\theta^h\right)_{h\in\mathbf{H}}$ such that $c^h\left(s^{t+k}\right) = \theta^h\mathbf{E}\left(s_{t+k}\right)$ $\forall h,k$. Under a stationary policy rule, $q_1\left(s^{t+k}\right) = \mathbf{q}_1\left(s_{t+k}\right)$ for all date-events. Under both Pareto efficiency and a stationary policy rule, the discounted present value equation (32) is given by:

$$\hat{\omega}^h\left(s^t\right) = \sum_{k=0}^{\infty}\sum_{s^{t+k}\succ s^t}\beta^k E_t\left[\left(\frac{\mathbf{E}\left(s_{t+k}\right)}{\mathbf{E}\left(s_t\right)}\right)^{-\rho}\left(\theta^h\mathbf{E}\left(s_{t+k}\right) - \mathbf{q}_1\left(s_{t+k}\right)\mathbf{e}^h\left(s_{t+k}\right)\right)\right]. \tag{33}$$

The right-hand side of the discounted present value equation (33) in date-event s^t only depends on the realization s_t (and this holds for all date-events). Therefore, the real wealth variables $\hat{\omega}^h\left(s^t\right)$ are stationary.

A.4 Proof of Lemma 2

I first consider the special case of $(H, J) = (2, 1)$. By (15), the total real wealth $\left(\sum_{h \in \mathbf{H}} \hat{\omega}^h\left(s^t, \sigma\right)\right)_{\sigma \in \mathbf{S}} \propto \left(\mathbf{E}(\sigma) + \sum_{j \in \mathbf{J}} \mathbf{q}_j\left(\sigma\right) \sum_{h \in \mathbf{H}} \hat{b}_j^h\left(s^t, \sigma\right)\right)_{\sigma \in \mathbf{S}}$. By (16), $\left(\frac{1}{\pi(s^t,\sigma)}\right)_{\sigma \in \mathbf{S}} = \left(\frac{\mathbf{E}(\sigma) + \sum_{j \in \mathbf{J}} \mathbf{q}_j(\sigma) \sum_{h \in \mathbf{H}} \hat{b}_j^h\left(s^t, \sigma\right)}{\mathbf{E}(s) + \sum_{j \in \mathbf{J}} \mathbf{q}_{j-1}(\sigma) \sum_{h \in \mathbf{H}} \hat{b}_j^h\left(s^t\right)}\right)_{\sigma \in \mathbf{S}}$. For the special case of $J = 1$, $\left(\sum_{h \in \mathbf{H}} \hat{\omega}^h\left(s^t, \sigma\right)\right)_{\sigma \in \mathbf{S}} = \left(\frac{1}{\pi(s^t,\sigma)}\right)_{\sigma \in \mathbf{S}}$ and therefore $\left(\sum_{h \in \mathbf{H}} \hat{\omega}^h\left(s^t, \sigma\right)\right)_{\sigma \in \mathbf{S}} \in \left\langle\left(\Pi\left(s^t\right)\right)^{-1}\left(\mathbf{Q}_0^{J-1}\right)\right\rangle$.[13] For stationary wealth vectors and $H = 2$:

$$\hat{\boldsymbol{\omega}}^1 \in \left\langle\left(\Pi\left(s^t\right)\right)^{-1}\left(\mathbf{Q}_0^{J-1}\right)\right\rangle \Longrightarrow \hat{\boldsymbol{\omega}}^2 \in \left\langle\left(\Pi\left(s^t\right)\right)^{-1}\left(\mathbf{Q}_0^{J-1}\right)\right\rangle.$$

In equilibrium, $rank\left(\hat{\boldsymbol{\omega}}^1, \hat{\boldsymbol{\omega}}^2\right) = 1$, so only $J = 1$ asset is required such that

$$\hat{\boldsymbol{\omega}}^h \in \left\langle\left(\Pi\left(s^t\right)\right)^{-1}\left(\mathbf{Q}_0^{J-1}\right)\right\rangle \;\forall h \in \mathbf{H}.$$

Recall the discounted present value equation (33):

$$\hat{\boldsymbol{\omega}}^h\left(s_t\right) = \sum_{k=0}^{\infty} \sum_{s^{t+k} \succ s^t} \beta^k E_t\left[\left(\frac{\mathbf{E}\left(s_{t+k}\right)}{\mathbf{E}\left(s_t\right)}\right)^{-\rho}\left(\theta^h \mathbf{E}\left(s_{t+k}\right) - \mathbf{q}_1\left(s_{t+k}\right) \mathbf{e}^h\left(s_{t+k}\right)\right)\right].$$

This can be expressed recursively as

$$\hat{\boldsymbol{\omega}}^h = \left(I_S - \hat{\Gamma}\right)^{-1}\left(\theta^h \mathbf{E}\left(s\right) - \mathbf{q}_1\left(s\right) \mathbf{e}^h\left(s\right)\right)_{s \in \mathbf{S}},$$

where $\hat{\Gamma}$ is the $S \times S$ matrix with elements $\hat{\Gamma}\left(s, \sigma\right) = \beta \Gamma\left(s, \sigma\right)\left(\frac{\mathbf{E}(\sigma)}{\mathbf{E}(s)}\right)^{-\rho}$.

[13] For the general case with $J > 1$, $\left(\sum_{h \in \mathbf{H}} \hat{\omega}^h\left(s^t, \sigma\right)\right)_{\sigma \in \mathbf{S}} \in \left\langle\left(\Pi\left(s^t\right)\right)^{-1}\left(\mathbf{Q}_0^{J-1}\right)\right\rangle$ only holds if $\left(\hat{\omega}^h\left(s^t, \sigma\right)\right)_{\sigma \in \mathbf{S}} \in \left\langle\left(\Pi\left(s^t\right)\right)^{-1}\left(\mathbf{Q}_0^{J-1}\right)\right\rangle$ for all $h \in \mathbf{H}$, which requires that $\left\langle\left(\Pi\left(s^t\right)\right)^{-1}\left(\mathbf{Q}_0^{J-1}\right)\right\rangle$ has at least H dimensions (from Lemma 2).

Define $\mathbf{z}^h = \left(\theta^h \mathbf{E}(s) - \mathbf{q}_1(s)\,\mathbf{e}^h(s)\right)_{s\in\mathbf{S}}$. Since $\left(I_S - \hat{\Gamma}\right)^{-1}$ has full rank,

$$rank\left(\left(\hat{\boldsymbol{\omega}}^h\right)_{h\in\mathbf{H}}\right) = rank\left(\left(\mathbf{z}^h\right)_{h\in\mathbf{H}}\right).$$

I claim that $rank\left(\left(\mathbf{z}^h\right)_{h\in\mathbf{H}}\right) = H$.

A.4.1 Case 1: $(\mathbf{q}_1(s))_{s\in\mathbf{S}} \neq \overrightarrow{1}$

Consider $\left(\left(\mathbf{z}^h\right)_{h\in\mathbf{H}}\right) v = 0$ for any vector $v \in \mathbb{R}^H$. This implies that

$$\left(\mathbf{E}(s) \sum_{h\in\mathbf{H}} v^h\theta^h - \mathbf{q}_1(s) \sum_{h\in\mathbf{H}} v^h \mathbf{e}^h(s)\right)_{s\in\mathbf{S}} = 0. \quad (34)$$

Subcase 1(a): If $\sum_{h\in\mathbf{H}} v^h\theta^h = 0$, Assumption 1 implies $v = 0$ and $rank\left(\left(\mathbf{z}^h\right)_{h\in\mathbf{H}}\right) = H$.

Subcase 1(b): Consider $\sum_{h\in\mathbf{H}} v^h\theta^h \neq 0$. By definition, $\mathbf{E}(s) = \sum_{h\in\mathbf{H}} \mathbf{e}^h(s)$. Define $\mu^h = \frac{v^h}{\sum_{h\in\mathbf{H}} v^h\theta^h}$. The system of equations (34) is given by:

$$\left(\sum_{h\in\mathbf{H}} \mathbf{e}^h(s) - \mathbf{q}_1(s) \sum_{h\in\mathbf{H}} \mu^h \mathbf{e}^h(s)\right)_{s\in\mathbf{S}} = 0.$$

By Assumption 1, there exists $\mathbf{S}^* \subseteq \mathbf{S}$ with $\#\mathbf{S}^* = H$ satisfying both $(\mathbf{q}_1(s))_{s\in\mathbf{S}^*} \neq \overrightarrow{1}$ and $[\mathbf{e}]_{(h,s)\in\mathbf{H}\times\mathbf{S}^*}$ is an invertible matrix. Consider only the states $s \in \mathbf{S}^*$. In matrix notation:

$$[\mathbf{e}]_{(h,s)\in\mathbf{H}\times\mathbf{S}^*} \overrightarrow{1} = diag\left((\mathbf{q}_1(s))_{s\in\mathbf{S}^*}\right) [\mathbf{e}]_{(h,s)\in\mathbf{H}\times\mathbf{S}^*} \left(\mu^h\right)_{h\in\mathbf{H}}.$$

Since $[\mathbf{e}]_{(h,s)\in\mathbf{H}\times\mathbf{S}^*}$ is invertible and $diag\left((\mathbf{q}_1(s))_{s\in\mathbf{S}^*}\right)$ is a diagonal matrix, the unknown coefficients $\left(\mu^h\right)_{h\in\mathbf{H}}$ must satisfy:

$$\left(\mu^h\right)_{h\in\mathbf{H}} = diag\left((\mathbf{q}_1(s))_{s\in\mathbf{S}^*}\right)^{-1} \overrightarrow{1} = \left(\frac{1}{\mathbf{q}_1(s)}\right)_{s\in\mathbf{S}^*}.$$

Since $(\mathbf{q}_1(s))_{s\in\mathbf{S}^*} \neq \overrightarrow{1}$, then $\mu^h \geq 1\ \forall h \in \mathbf{H}$, with strict inequality for at least one h. By definition, this means $v^h \geq \sum_{h\in\mathbf{H}} v^h\theta^h\ \forall h \in \mathbf{H}$, with strict inequality for at least one h. Multiplying both sides θ^h and summing:

$$\sum_{h\in\mathbf{H}} v^h\theta^h > \left(\sum_{h\in\mathbf{H}} \theta^h\right) \sum_{h\in\mathbf{H}} v^h\theta^h.$$

This contracts that $\sum_{h\in\mathbf{H}} \theta^h = 1$.

There does not exist $v \in \mathbb{R}^H$ such that (34) is satisfied. Subcase 1(b) is not possible.

A.4.2 Case 2: $(\mathbf{q}_1(s))_{s\in\mathbf{S}} = \overrightarrow{1}$

Suppose, in order to obtain a contradiction, that $(\mathbf{q}_1(s))_{s\in\mathbf{S}} = \overrightarrow{1}$. The Euler equations applied iteratively implies $(\mathbf{q}_j(s))_{s\in\mathbf{S}} = \overrightarrow{1}$ $\forall j > 1$. All together, $rank\left(\mathbf{Q}_0^{J-1}\right) = 1$ under the Friedman rule.

I claim $rank\left(\hat{\boldsymbol{\omega}}^h, \hat{\boldsymbol{\omega}}^{h'}\right) = 2$ for any h, h'. Set $\left(\mathbf{z}^h, \mathbf{z}^{h'}\right)\begin{pmatrix} v^h \\ v^{h'} \end{pmatrix} = 0$ for any vector $\begin{pmatrix} v^h \\ v^{h'} \end{pmatrix} \in \mathbb{R}^2$. The system of equations (34) for h, h' is:

$$\left(\sum_{k\in\mathbf{H}} \mathbf{e}^k(s) - \mu^h \mathbf{e}^h(s) - \mu^{h'} \mathbf{e}^{h'}(s)\right)_{s\in\mathbf{S}} = 0. \tag{35}$$

The system (35) is equivalently expressed as

$$[\mathbf{e}]\left[\chi^h\right]_{h\in\mathbf{H}} = 0$$

for $\chi^h = 1 - \mu^h$, $\chi^{h'} = 1 - \mu^{h'}$, and $\chi^k = 1$ $\forall k \in \mathbf{H}\backslash\{h, h'\}$. Since $rank\,[\mathbf{e}] = H$, then $\left[\chi^h\right]_{h\in\mathbf{H}} = 0$. Since $\mathbf{H}\backslash\{h, h'\}$ is nonempty, the contradiction $\chi^k = 1$ and $\chi^k = 0$ completes the argument.

This implies that $rank\left(\left(\hat{\boldsymbol{\omega}}^h\right)_{h\in\mathbf{H}}\right) \geq 2$ and $rank\left(\mathbf{Q}_0^{J-1}\right) \geq 2$. Case 2 is not possible.[14]

A.5 Proof of Lemma 3

With stationary asset prices, the Euler equations (14) imply that the inflation rates $\pi(s^t, \sigma)$ only depend upon (s_t, σ). The sequence of inflation rates $\{\pi(s^t)\}$ must satisfy

$$\Phi = \left\langle \left(\Pi\left(s^t\right)\right)^{-1}\left(\mathbf{Q}_0^{J-1}\right)\right\rangle$$

for all date-events.

I claim that the asset span is constant. For the special case of $H = 2$, only $J = 1$ asset suffices for $\hat{\boldsymbol{\omega}}^h \in \Phi$ $\forall h \in \mathbf{H}$ and there exists a unique asset span Φ. When $H = J > 2$, $\exists!\Phi$ such that $\hat{\boldsymbol{\omega}}^h \in \Phi$ $\forall h \in \mathbf{H}$. If $H < J$, the asset span need not be constant. However, it is

[14] Additionally, this argument reveals that the Friedman rule (when $H > 2$) is never consistent with a Pareto efficient allocation.

innocuous (in terms of the real allocation) to restrict attention to policies with a constant asset span $\Phi = \left\langle (\Pi(s^t))^{-1} \left(\mathbf{Q}_0^{J-1}\right)\right\rangle$.

A constant asset span with stationary asset price vector dictates that $(\pi(s^t,\sigma))_{\sigma\in\mathbf{S}}$ are proportional for all date-events. Since $\hat{\boldsymbol{\omega}}^h \in \Phi\ \forall h \in \mathbf{H}$, then $\exists \left(\tilde{b}^h(s^t)\right)_{h\in\mathbf{H}}$ such that $\hat{\boldsymbol{\omega}}^h = (\Pi(s^t))^{-1}\left(\mathbf{Q}_0^{J-1}\right)\tilde{b}^h(s^t)\ \forall h \in \mathbf{H}$. If $(\pi(s^t,\sigma))_{\sigma\in\mathbf{S}} = \kappa(\pi(s^\tau,\sigma))_{\sigma\in\mathbf{S}}$ for some (t,τ), then $\tilde{b}^h(s^t) = \kappa\tilde{b}^h(s^\tau)\ \forall h \in \mathbf{H}$. Using the market clearing condition, $\tilde{B}(s^t) = \kappa\tilde{B}(s^\tau)$. The monetary authority constraint (12) for date-events (s^t,σ) and (s^τ,σ) yield:

$$\sum_{j\in\mathbf{J}} \mathbf{q}_j(\sigma)\,\tilde{B}_j\left(s^t,\sigma\right) = \sum_{j\in\mathbf{J}} \mathbf{q}_j(\sigma)\,\tilde{B}_j\left(s^\tau,\sigma\right).$$

This implies either that $\kappa = 1$ or $\tilde{B}(s^t) = 0$ for all date-events.

If $\kappa = 1$, $(\pi(s^t,\sigma))_{\sigma\in\mathbf{S}} = (\pi(s^\tau,\sigma))_{\sigma\in\mathbf{S}}$, $\tilde{b}^h(s^t) = \tilde{b}^h(s^\tau)$, and $\tilde{B}(s^t) = \tilde{B}(s^\tau)$ for all date-events. The inflation rates are stationary, meaning that there exists $(\boldsymbol{\pi}(s))_{s\in\mathbf{S}}$ such that $\pi(s^t,\sigma) = \boldsymbol{\pi}(\sigma)$ for all date-events.

If $\tilde{B}(s^t) = 0$ for all date-events, the monetary authority constraint (12) implies that $(\mathbf{q}_1(s))_{s\in\mathbf{S}} = \overrightarrow{1}$. If $\mathbf{q}_1(s) = 1\ \forall s$, then $\mathbf{Q}_0^{J-1}$ has rank 1, which violates the full rank condition. Additionally, Case 2 in the proof of Lemma 2 rules out $\mathbf{q}_1(s) = 1\ \forall s$. This contradiction implies that $\kappa = 1$ must hold and the inflation rates are stationary.

A.6 Proof of Claim 1

Recalling the expression for θ^h :

$$\theta^h = \frac{\left(I_S - \hat{\Gamma}\right)^{-1}_{(s_0)} \left(\omega^h(s_0)\,\mathbf{E}(s)\,(1-\mathbf{q}_1(s)) + W(s_0)\,\mathbf{q}_1(s)\,\mathbf{e}^h(s)\right)_{s\in\mathbf{S}}}{\left(I_S - \hat{\Gamma}\right)^{-1}_{(s_0)} \left(W(s_0)\,\mathbf{E}(s)\right)_{s\in\mathbf{S}}}.$$

The matrix $\left(I_S - \hat{\Gamma}\right)^{-1}_{(s_0)}$ contains only positive elements. Since $0 \le \mathbf{q}_1(s) \le 1\ \forall s \in \mathbf{S}$, then $\theta^h \ge 0$.

Select any s. Since $\mathbf{e}^h(s) \le \mathbf{E}(s)$, I claim that

$$\frac{\omega^h(s_0)\,\mathbf{E}(s)\,(1-\mathbf{q}_1(s)) + W(s_0)\,\mathbf{q}_1(s)\,\mathbf{e}^h(s)}{W(s_0)\,\mathbf{E}(s)} \le 1. \tag{36}$$

Algebraically, the previous inequality reduces to:

$$\mathbf{q}_1(s) \leq \left(\frac{1 - \frac{\omega^h(s_0)}{W(s_0)}}{\frac{\mathbf{e}^h(s)}{\mathbf{E}(s)} - \frac{\omega^h(s_0)}{W(s_0)}} \right).$$

Since $\mathbf{e}^h(s) \leq \mathbf{E}(s)$, then $\left(\frac{1 - \frac{\omega^h(s_0)}{W(s_0)}}{\frac{\mathbf{e}^h(s)}{\mathbf{E}(s)} - \frac{\omega^h(s_0)}{W(s_0)}} \right) \geq 1$ and the inequality $\mathbf{q}_1(s) \leq 1$ is always satisfied. The inequality (36) holds $\forall s \in \mathbf{S}$. Therefore, $\theta^h \leq 1$.

A.7 Proof of Lemma 4

It suffices to show that if $J < \frac{S(H-1)}{H}$, then generically over the subset of household endowments $\left(\mathbf{e}^h(s)\right)_{(h,s)\in H\times\mathbf{S}}$ and initial period wealth $\left(\omega^h(s_0)\right)_{h\in\mathbf{H}}$, the $H + SJ + SH$ equations (27), (23), (24), and (25) are linearly independent.

The variables $\xi \in \mathbb{R}^{H+SJ+S+HJ}$ are

$$\xi = \left((p(s_0)), \left(\theta^h\right)_{h\in\mathbf{H}}, \left(\mathbf{q}_j(s)\right)_{(j,s)\in\mathbf{J}\times\mathbf{S}}, \left(\boldsymbol{\pi}(s)\right)_{s\in\mathbf{S}}, \left(\mathbf{b}_j^h\right)_{(h,j)\in\mathbf{H}\times\mathbf{J}} \right)$$

and the parameters $\theta \in \mathbb{R}^{H(S+1)}$ are

$$\theta = \left(\left(\mathbf{e}^h(s)\right)_{(h,s)\in\mathbf{H}\times\mathbf{S}}, \left(\omega^h(s_0)\right)_{h\in\mathbf{H}} \right).$$

Define the system of equations as

$$\Phi : \mathbb{R}^{H+SJ+S+HJ+H(S+1)} \longrightarrow \mathbb{R}^{H+SJ+SH}$$

where $\Phi(\xi, \theta) = 0$ iff the following equations are satisfied: (i) initial period discounted present value equations (27), (ii) Euler equations (23) and (24), and (iii) budget constraints (25).

Define the projection $\phi : \mathbb{R}^{H+SJ+S+HJ+H(S+1)} \longrightarrow \mathbb{R}^{H(S+1)}$ as the mapping $(\xi, \theta) \mapsto \theta$ such that $\Phi(\xi, \theta) = 0$. The mapping ϕ is proper iff for any compact subset of the range Y', the inverse image $\phi^{-1}(Y')$ is also compact. The payout matrix has full rank since the only variables considered are those with linearly independent payouts. This implies that the projection ϕ is proper.

To complete the argument, it suffices to prove that $D_{\xi,\theta}\Phi(\xi, \theta)$ has full row rank $H + SJ + SH$ and apply the parametric transversality theorem.

Consider the columns for derivatives with respect to the price variables $p(s_0)$ and the parameters $\left(\omega^h(s_0)\right)_{h\in\mathbf{H}}$. The submatrix in $D_{\xi,\theta}\Phi(\xi,\theta)$ for the initial period discounted present value equations (27) and the derivatives with respect to $p(s_0)$ and $\left(\omega^h(s_0)\right)_{h\in\mathbf{H}}$ has full row rank. Moreover, the variables $p(s_0)$ and the parameters $\left(\omega^h(s_0)\right)_{h\in\mathbf{H}}$ do not appear in any other equations. The initial period discounted present value equations are therefore linearly independent of all other equations.

Consider the budget constraints (25):

$$\left(\theta^h\mathbf{E}(s) - \mathbf{q}_1(s)\,\mathbf{e}^h(s) + \sum_{j\in\mathbf{J}}\mathbf{b}_j^h\left(\mathbf{q}_j(s) - \frac{\mathbf{q}_{j-1}(s)}{\boldsymbol{\pi}(s)}\right)\right)_{h,s\in\mathbf{H}\times\mathbf{S}} = 0.$$

Recall that $\mathbf{Q}_0^{J-1} = \begin{bmatrix} 1 & \mathbf{q}_1(1) & .. & \mathbf{q}_{J-1}(1) \\ : & : & : & : \\ 1 & \mathbf{q}_1(S) & .. & \mathbf{q}_{J-1}(S) \end{bmatrix}$. Define the stationary inflation rate matrix $\boldsymbol{\Pi} = \begin{bmatrix} \boldsymbol{\pi}(1) & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & \boldsymbol{\pi}(S) \end{bmatrix}$. From the Euler equation, the expression $\left(\mathbf{q}_j(s) - \frac{\mathbf{q}_{j-1}(s)}{\boldsymbol{\pi}(s)}\right)_{s\in\mathbf{S}}$ is given by $\left(\hat{\Gamma} - I_S\right)\boldsymbol{\Pi}^{-1}\mathbf{Q}_0^{J-1}$. The updated household budget constraints are given by:

$$\left(\theta^h\mathbf{E}(s) - \mathbf{q}_1(s)\,\mathbf{e}^h(s) + \left(\hat{\Gamma} - I_S\right)\boldsymbol{\Pi}^{-1}\mathbf{Q}_0^{J-1}\mathbf{b}^h\right)_{h,s\in\mathbf{H}\times\mathbf{S}} = 0,$$

where $\mathbf{b}^h$ is a $J-$dimensional column vector. To this system of equations, I add the total endowment expressions

$$\left(\mathbf{E}(s) - \sum_{h\in\mathbf{H}}\mathbf{e}^h(s)\right)_{s\in\mathbf{S}} = 0$$

to account for the fact that individual endowments and total endowment are related.

I claim that the submatrix in $D_{\xi,\theta}\Phi(\xi,\theta)$ for the budget constraints (25) and the derivatives with respect to $(\boldsymbol{\pi}(s))_{s\in\mathbf{S}}$ and $\left(\mathbf{e}^h(s)\right)_{(h,s)\in\mathbf{H}\times\mathbf{S}}$ has full row rank. A sufficient condition for full row rank is that for any

$$\alpha^T = \left(\left(\Delta a_h^T\right)_{h\in\mathbf{H}}, \left(\Delta E_s\right)_{s\in\mathbf{S}}\right) \in \mathbb{R}^{SH+S},$$

the product

$$\alpha^T D_{\boldsymbol{\pi},e}\Phi'(\xi,\theta) = 0 \tag{37}$$

implies $\alpha^T = 0$, where $D_{\boldsymbol{\pi},e}\Phi'(\xi,\theta)$ is the submatrix in $D_{\xi,\theta}\Phi(\xi,\theta)$ for the budget constraints (25) and the derivatives with respect to $(\boldsymbol{\pi}(s))_{s\in\mathbf{S}}$ and $\left(\mathbf{e}^h(s)\right)_{(h,s)\in\mathbf{H}\times\mathbf{S}}$.

Consider the columns for the derivatives with respect to $\left(\mathbf{e}^h(s)\right)_{(h,s)\in\mathbf{H}\times\mathbf{S}}$. The equations $\alpha^T D_{\boldsymbol{\pi},e}\Phi'(\xi,\theta)=0$ for these columns are:

$$\Delta a_h^T \begin{bmatrix} -\mathbf{q}_1(1) & 0 & 0 \\ 0 & .. & 0 \\ 0 & 0 & -\mathbf{q}_1(S) \end{bmatrix} - (\Delta E_1, ..., \Delta E_S) = 0. \tag{38}$$

Since this holds $\forall h$, then $\Delta a_h^T = \Delta a^T\ \forall h$.

Consider the columns for the derivatives with respect to $\left(\frac{1}{\boldsymbol{\pi}(s)}\right)$. The equations $\alpha^T D_{\boldsymbol{\pi},e}\Phi'(\xi,\theta) = 0$ for these columns are:

$$\sum_{h\in\mathbf{H}} \Delta a_h(s)\left(\hat{\Gamma}-I_S\right)_{(s)} \mathbf{Q}_0^{J-1}\mathbf{b}^h = 0, \tag{39}$$

where $\left(\hat{\Gamma}-I_S\right)_{(s)}$ refers to row s of the matrix $\hat{\Gamma}-I_S$. Since $\Delta a_h^T = \Delta a^T\ \forall h$, then we obtain $\Delta a(s)\left(\hat{\Gamma}-I_S\right)_{(s)} \mathbf{Q}_0^{J-1}\sum_{h\in\mathbf{H}}\mathbf{b}^h = 0$.

I claim $\sum_{h\in\mathbf{H}}\mathbf{b}^h \neq 0$. If, in order to obtain a contradiction, $\sum_{h\in\mathbf{H}}\mathbf{b}^h = 0$, then $\mathbf{B}=0$. This violates the inequality $\hat{B}_1(s^t)\geq 0$ as $\mathbf{B}_1 = \tilde{B}_1(s^t) = \mathbf{E}(s_t) + \hat{B}_1(s^t)$ and $\mathbf{E}(s_t)>0$.

I claim $\left(\hat{\Gamma}-I_S\right)_{(s)} \mathbf{Q}_0^{J-1}\sum_{h\in\mathbf{H}}\mathbf{b}^h \neq 0$. If $\left(\hat{\Gamma}-I_S\right)_{(s)} \mathbf{Q}_0^{J-1}\sum_{h\in\mathbf{H}}\mathbf{b}^h = 0$, then

$$\left(\hat{\Gamma}-I_S\right)_{(s)} \mathbf{\Pi}^{-1}\mathbf{Q}_0^{J-1}\sum_{h\in\mathbf{H}}\mathbf{b}^h = 0.$$

The budget constraints require that $\sum_{h\in\mathbf{H}}\left(\theta^h\mathbf{E}(s) - \mathbf{q}_1(s)\,\mathbf{e}^h(s)\right) = \mathbf{E}(s)\left(1-\mathbf{q}_1(s)\right) = 0$. If $\mathbf{q}_1(s) = 1\ \forall s$, then $\mathbf{Q}_0^{J-1}$ has rank 1, which violates the full rank condition. Additionally, Case 2 in the proof of Lemma 2 rules out $\mathbf{q}_1(s)=1\ \forall s$.

For the special case of $\mathbf{q}_1(s)=1$, replace one of the household budget constraints (suppose, without loss of generality, the budget constraint for household H) in state s with the Euler equation for $\mathbf{q}_1(s)$:

$$1 = \beta\sum_{\sigma\in\mathbf{S}}\Gamma(s,\sigma)\left(\frac{\mathbf{E}(\sigma)}{\mathbf{E}(s)}\right)^{-\rho}\frac{1}{\boldsymbol{\pi}(\sigma)}.$$

Consider the columns for the derivatives with respect to $\left(\frac{1}{\boldsymbol{\pi}(s)}\right)$. The equations $\alpha^T D_{\boldsymbol{\pi},e}\Phi'(\xi,\theta) =$

0 for these columns are:

$$\sum_{h\in\mathbf{H}\setminus\{H\}} \Delta a_h(s)\left(\hat{\Gamma}-I_S\right)_{(s)}\mathbf{Q}_0^{J-1}\mathbf{b}^h+\Delta a_H(s)\beta\Gamma(s,s)=0. \tag{40}$$

Recall that $\Delta a_h(s)=\Delta a(s)\ \forall h$. From the budget constraint with $\mathbf{q}_1(s)=1$,

$$\boldsymbol{\pi}(s)\left(\theta^H\mathbf{E}(s)-\mathbf{e}^H(s)\right)=-\left(\hat{\Gamma}-I_S\right)\mathbf{Q}_0^{J-1}\mathbf{b}^H.$$

The sum $\left(\hat{\Gamma}-I_S\right)_{(s)}\mathbf{Q}_0^{J-1}\sum_{h\in\mathbf{H}\setminus\{H\}}\mathbf{b}^h=-\left(\hat{\Gamma}-I_S\right)\mathbf{Q}_0^{J-1}\mathbf{b}^H$. Equation (40) is updated to:

$$\Delta a(s)\left(-\boldsymbol{\pi}(s)\left(\theta^H\mathbf{E}(s)-\mathbf{e}^H(s)\right)\right)+\Delta a(s)\beta\Gamma(s,s)=0. \tag{41}$$

Without loss of generality, the household H is chosen such that $\theta^H\mathbf{E}(s)-\mathbf{e}^H(s)<0$. From Assumption 1, such a household exists. Since

$$\Delta a(s)\left\{-\boldsymbol{\pi}(s)\left(\theta^H\mathbf{E}(s)-\mathbf{e}^H(s)\right)+\beta\Gamma(s,s)\right\}=0$$

and $\left\{-\boldsymbol{\pi}(s)\left(\theta^H\mathbf{E}(s)-\mathbf{e}^H(s)\right)+\beta\Gamma(s,s)\right\}>0$, then $\Delta a(s)=0$.

For the special case with $\mathbf{q}_1(s)=1$ and $\mathbf{q}_1(s')=1$, replace the (s,H) budget constraint with $1=\beta\sum_{\sigma\in\mathbf{S}}\Gamma(s,\sigma)\left(\frac{\mathbf{E}(\sigma)}{\mathbf{E}(s)}\right)^{-\rho}\frac{1}{\boldsymbol{\pi}(\sigma)}$ and the (s',H') budget constraint with $1=\beta\sum_{\sigma\in\mathbf{S}}\Gamma(s',\sigma)\left(\frac{\mathbf{E}(\sigma)}{\mathbf{E}(s')}\right)^{-\rho}\frac{1}{\boldsymbol{\pi}(\sigma)}$. Without loss of generality, household H is chosen such that $\theta^H\mathbf{E}(s)-\mathbf{e}^H(s)<0$ and household H' is chosen such that $\theta^{H'}\mathbf{E}(s')-\mathbf{e}^{H'}(s')>0$. It may be the case that $H=H'$. Consider the columns for the derivatives with respect to $\left(\frac{1}{\boldsymbol{\pi}(s)}\right)$ and $\left(\frac{1}{\boldsymbol{\pi}(s')}\right)$. Using (41), the equations $\alpha^T D_{\boldsymbol{\pi},e}\Phi'(\xi,\theta)=0$ for these columns are:

$$\begin{aligned}\Delta a(s)\left\{-\boldsymbol{\pi}(s)\left(\theta^H\mathbf{E}(s)-\mathbf{e}^H(s)\right)+\beta\Gamma(s,s)\right\}+\Delta a(s')\beta\Gamma(s,s') &= 0.\\ \Delta a(s')\left\{-\boldsymbol{\pi}(s')\left(\theta^{H'}\mathbf{E}(s')-\mathbf{e}^{H'}(s')\right)+\beta\Gamma(s',s')\right\}+\Delta a(s)\beta\Gamma(s',s) &= 0.\end{aligned}$$

If either $\Gamma(s,s')=0$ or $\Gamma(s',s)=0$, then $\Delta a(s)=\Delta a(s')=0$ as before. If $\Gamma(s,s')>0$ and $\Gamma(s',s)>0$, the first equation implies $\Delta a(s)\Delta a(s')\leq 0$ and the second equation implies $\Delta a(s)\Delta a(s')\geq 0$. This is only satisfied if either $\Delta a(s)=0$ or $\Delta a(s')=0$. Once one of them equals 0, then $\Delta a(s)=\Delta a(s')=0$ as before.

The same approach works for any number of states s such that $\mathbf{q}_1(s)=1$, provided that $(\mathbf{q}_1(s))_{s\in\mathbf{S}}\neq 1$.

Therefore $\Delta a(s) = 0$ and the same argument holds $\forall s$, implying $\Delta a_h^T = 0\ \forall h \in \mathbf{H}$. From (38), $\Delta E_s = 0\ \forall s \in \mathbf{S}$. This implies $\alpha^T = 0$, so $D_{\boldsymbol{\pi},e}\Phi'(\xi,\theta)$ has full row rank.

The submatrix in $D_{\xi,\theta}\Phi(\xi,\theta)$ for the Euler equations (23) and (24) and the derivatives with respect to $(\mathbf{q}_j(s))_{(j,s)\in\mathbf{J}\times\mathbf{S}}$ has full row rank. Given the above findings, the Euler equations are linearly independent of all other equations.

This completes the argument.

A.8 Proof of Theorem 4

Pareto efficiency requires that for all state variables $(s,\hat{\omega})$, the corresponding consumption fractions $\left(\theta^h\right)_{h\in\mathbf{H}}$ are constant. Consider the recursive derivation of the policy correspondence. Take as given the policy correspondence $\mathbf{V}^n : \mathbf{S}\times\mathbb{R}^{H-1} \rightrightarrows \Delta_b \times \Delta_q$. For a given vector of state variables $(s,\hat{\omega})$, the following algorithm determines the image $\mathbf{V}^{n+1}(s,\hat{\omega})$ for a dynamic policy rule consistent with Pareto efficiency. Guess the Pareto efficient consumption fractions $\left(\theta^h\right)_{h\in\mathbf{H}}$. For each $\sigma\in\mathbf{S}$, the state variables $\hat{\omega}'(\sigma)$ are determined such that: (i) $\left(\hat{b}'(\sigma), q'(\sigma)\right) \in \mathbf{V}^n(\sigma,\hat{\omega}'(\sigma))$ and (ii) $\theta^h\mathbf{E}(\sigma) = \hat{\omega}^{h\prime}(\sigma) - \sum_{j\in\mathbf{J}} q_j'(\sigma) b_j^{h\prime}(\sigma)$ for $h\in\mathbf{H}\backslash\{H\}$.[15] Recall that $\hat{\omega}'(\sigma) = \left(\hat{\omega}^{h'}(\sigma)\right)_h \in \mathbf{H}\backslash\{H\}$ does not include the real wealth for household $h=H$. For household $h=H$, $\hat{\omega}^{H\prime}(\sigma) = \theta^H\mathbf{E}(\sigma) + \sum_{j\in\mathbf{J}} q_j'(\sigma) b_j^{H\prime}(\sigma)$. Notice that the equilibrium values $\left(\hat{\omega}'(\sigma), \hat{b}'(\sigma), q'(\sigma)\right)_{\sigma\in\mathbf{S}}$ only depend on $\left(\theta^h\right)_{h\in\mathbf{H}}$ and not $(s,\hat{\omega})$ (at this point in the iteration).

The equilibrium variables $\left(\hat{b}^h\right)_{h\in\mathbf{H}}$ are determined from the simplified version of the wealth equations (15):

$$\hat{\omega}'^h(\sigma) = \frac{\mathbf{e}^h(s) + \sum_{j\in\mathbf{J}} q_{j-1}'(\sigma)\hat{b}_j^h}{\mathbf{E}(s) + \sum_{j\in\mathbf{J}} q_{j-1}'(\sigma)\sum_{h\in\mathbf{H}}\hat{b}_j^h}\left(\mathbf{E}(\sigma) + \sum_{j\in\mathbf{J}} q_j'(\sigma)\sum_{h\in\mathbf{H}}\hat{b}_j^{h\prime}(\sigma)\right) \quad \forall (h,\sigma)\in\mathbf{H}\backslash\{H\}\times\mathbf{S}.$$

The wealth equation for household $h=H$ is redundant. The number of unknowns equals HJ and the number of equations equals $S(H-1)$. Using the same arguments as in the proof of Lemma 4, if $J < \frac{S(H-1)}{H}$, then over a generic subset of household endowments, there does not exist a solution to the wealth equations. Using contraposition, then $J \geq \frac{S(H-1)}{H}$ is a generic necessary condition for a solution.

Given $\left(\hat{b}^h\right)_{h\in\mathbf{H}}$, the inflation rates $(\pi'(\sigma))_{\sigma\in\mathbf{S}}$ are uniquely determined from the monetary

[15] The policy rule is recursively updated. At this stage in the iteration, $q'(\sigma) = \mathbf{T}^n\left(\sigma,\hat{\omega}'(\sigma)\right)$. As the policy correspondence converges, so too does the policy rule.

authority constraints (16):

$$\frac{1}{\pi'(\sigma)}\left(\mathbf{E}(s)+\sum_{j\in\mathbf{J}}q'_{j-1}(\sigma)\sum_{h\in\mathbf{H}}\hat{b}^h_j\right)=\mathbf{E}(\sigma)+\sum_{j\in\mathbf{J}}q'_j(\sigma)\sum_{h\in\mathbf{H}}\hat{b}^{h\prime}_j(\sigma)\quad\forall\sigma\in\mathbf{S}.$$

Given $(\pi'(\sigma))_{\sigma\in\mathbf{S}}$, the asset prices q are determined from the Euler equations under Pareto efficiency:

$$q_j=\beta\sum_{\sigma\in\mathbf{S}}\Gamma(s,\sigma)\left(\frac{\mathbf{E}(\sigma)}{\mathbf{E}(s)}\right)^{-\rho}\frac{q'_{j-1}(\sigma)}{\pi'(\sigma)}\quad\forall j\in\mathbf{J}.$$

Notice that the equilibrium values $\left(\left(\hat{b}^h\right)_{h\in\mathbf{H}},(\pi'(\sigma))_{\sigma\in\mathbf{S}},q\right)$ only depend on $\left(\theta^h\right)_{h\in\mathbf{H}}$ and not s (at this point in the iteration).

To complete the determination of the image $\mathbf{V}^{n+1}(s,\hat{\omega})$, the initial guess $\left(\theta^h\right)_{h\in\mathbf{H}}$ must be updated. The consumption fractions are updated such that:

$$\theta^h\mathbf{E}(s)=\hat{\omega}^h-\sum_{j\in\mathbf{J}}q_j\hat{b}^h_j\quad\text{for } h\in\mathbf{H}\backslash\{H\}. \tag{42}$$

Iteration continues until convergence to $\left(\theta^h\right)_{h\in\mathbf{H}}$ for each element $(s,\hat{\omega})$ in the domain of $\mathbf{V}^{n+1}:\mathbf{S}\times\mathbb{R}^{H-1}\rightrightarrows\Delta_b\times\Delta_q$.

Pareto efficiency requires that $\left(\theta^h\right)_{h\in\mathbf{H}}$ is identical for state variables $(s,\hat{\omega}_A)$ and $(s,\hat{\omega}_B)$. Using the algorithm above, identical consumption fractions $\left(\theta^h\right)_{h\in\mathbf{H}}$ imply identical values for $\left(\hat{\omega}'(\sigma),\hat{b}'(\sigma),q'(\sigma)\right)_{\sigma\in\mathbf{S}}$ and $\left(\left(\hat{b}^h\right)_{h\in\mathbf{H}},(\pi'(\sigma))_{\sigma\in\mathbf{S}},q\right)$. Returning to the iterative step, (42) is satisfied iff $\hat{\omega}_A=\hat{\omega}_B$. Pareto efficiency implies that the the real wealth vectors $\hat{\omega}^h(s^t)$ are stationary, meaning that there exists $\left(\hat{\boldsymbol{\omega}}^h(s)\right)_{(h,s)\in\mathbf{H}\times\mathbf{S}}$ such that $\hat{\omega}^h(s^t)=\hat{\boldsymbol{\omega}}^h(s_t)$ for all h and all date-events.

With stationary real wealth vectors, the recursive representation beginning in date-event s^t is given by:

$$\hat{\boldsymbol{\omega}}^h=\left(\theta^h\mathbf{E}(\sigma)-q_1\left(s^t,\sigma\right)\mathbf{e}^h(\sigma)\right)_{\sigma\in\mathbf{S}}+\hat{\Gamma}\hat{\boldsymbol{\omega}}^h.$$

Matrix algebra requires that $q_1(s^t,\sigma)$ are stationary, namely that there exists $(\mathbf{q}_1(s))_{s\in\mathbf{S}}$ such that $q_1(s^t,\sigma)=\mathbf{q}_1(\sigma)$ for all date-events. Lemma 2 can now be applied.

From Lemma 2, if $H>2$, then $rank\left(\left(\hat{\boldsymbol{\omega}}^h\right)_{h\in\mathbf{H}}\right)=H$. The necessary condition for $\hat{\boldsymbol{\omega}}^h\in\left\langle(\Pi(s^t))^{-1}\left(\mathbf{Q}_0^{J-1}\right)\right\rangle\ \forall h\in\mathbf{H}$ is $J\geq H$ or $(H,J)=(2,1)$.

In conclusion, Condition SC is a generic necessary condition for Pareto efficiency under dynamic policy rules.