On the Pareto Efficiency of Term Structure Targeting Policies

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Abstract

In a stochastic economy, monetary policies that rebalance a central bank’s portfolio of short and long term government debt (term structure targeting) can have real effects. When the number of assets equals the number of states of uncertainty, however, this paper shows that the equilibrium allocation is Pareto efficient. Further, all policies with the same short-term interest rate targets support the same equilibrium allocation.

Keywords complete markets – Pareto efficiency – asset span – term structure

JEL Classification D52, E43, E44, E52

1 Introduction

In a stochastic economy, monetary policies that rebalance a central bank’s portfolio of short and long term government debt (term structure targeting policies) can have real effects. This paper models a closed economy with heterogeneous households in which the asset structure consists of government debt (or bonds) of varying maturities and the number of assets is

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equal to the number of states of uncertainty (complete markets). In such a setting, this paper shows that term structure targeting policies always leads to a Pareto efficient allocation.

This result is not surprising, certainly in light of the conjectures from Eggertsson and Woodford (2003) and Cúrdia and Woodford (2008, 2010a, 2010b, 2011) that unconventional monetary policies, including term structure targeting, in settings with complete markets, do not change the real allocation.

In the present paper, I adapt the monetary model of Stokey and Lucas (1983) to allow households to trade bonds with varying maturities and to allow the government to choose term structure targeting policies. Within this class of monetary models, a cash-in-advance constraint, as in Clower (1967), determines the equilibrium value for money. The cash-in-advance friction is not vital to the main result of Pareto efficiency,¹ and any number of alternative monetary models would lead to the same result.

The cash-in-advance constraint does influence the equilibrium allocation, as the present paper does not include fiscal transfers (unlike Eggertsson and Woodford (2003) and Cúrdia and Woodford (2008, 2010a, 2010b, 2011)). With the cash-in-advance constraint and without fiscal transfers, changes in the short-term interest rates can have real effects as they can move the allocation along the Pareto frontier. In this and any other monetary model, the inclusion of fiscal transfers would lead to the predictions of Eggertsson and Woodford (2003) and Cúrdia and Woodford (2008, 2010a, 2010b, 2011) that unconventional monetary policies, in settings with complete markets, do not have real effects.

Abstracting away from the issue of fiscal transfers, the first-order theoretical question is which monetary policies support Pareto efficiency. If the payouts for the assets were exogenously specified, a sufficient condition for Pareto efficiency under complete markets is a full rank payout matrix in all periods (see Magill and Quinzii, 1996). Without a full rank payout matrix, the resulting equilibrium allocation would be generically Pareto inefficient (see Magill and Quinzii, 1996), where generically means that the result holds over an open and full measure subset of household endowments.

In the present monetary model, households face only implicit debt constraints (to rule out Ponzi schemes) and the aforementioned cash-in-advance constraints. The financial frictions are characterized by the payout matrices that the households face, specifically the ranks of the payout matrices. This paper abstracts away from other frictions common in financial markets, such as information frictions and lack of commitment frictions that lead to the emergence of collateralized borrowing.²

¹Lucas (1972) and Stokey and Lucas (1983) show that Pareto efficiency is achieved with effectively complete markets even in the presence of cash-in-advance constraints.
²See Geanakoplos and Zame (2014).
The payout matrix is endogenously determined as a function of the government policy choice, specifically the stochastic inflation rates and the state-contingent bond prices. With a full rank payout matrix, households can support any vector of contingent consumption choices with an appropriately chosen bond portfolio.

The government commits to policy choices consisting of its choice of the portfolio of debt positions (of varying maturities) and term structure of interest rates. A policy choice must be feasible, meaning that it satisfies government budget constraints and constraints that prevent the government from short-selling bonds. The government should be viewed as a joint monetary-fiscal authority. The fiscal authority issues government debt in terms of bonds of varying maturities. Monetary policy consists of buying and selling debt positions (of varying maturities) in order to change the term structure. The net debt positions of the government (for all maturities) cannot be negative. This means that the monetary authority cannot purchase more bonds than were initially issued by the fiscal authority. This nonnegativity constraint prevents the government from short-selling bonds.

There is a continuum of feasible policy choices. For all feasible policy choices, the resulting equilibrium allocation is Pareto efficient. Moreover, for all feasible policy choices supporting the same short-term interest rates, the resulting equilibrium allocation is identical.

In models with endogenous payout matrices, no known conditions exist to guarantee that such matrices have full rank. The result in this paper is theoretically important as it guarantees that the payout matrices will always have full rank in equilibrium. While it is important to know that, in settings with complete markets, the policy choice always supports Pareto efficiency, this finding has limited practical relevance. To understand the welfare implications of unconventional monetary policies, as adopted in the wake of the recent recession, the model would need to analyze all types of central bank purchases, not only the purchase and sale of bonds of varying maturities, but also the purchase of mortgage-backed securities and other real assets. The present paper, with complete markets, provides the baseline for the analysis in the companion paper Hoelle (2014), which considers settings with incomplete markets. With incomplete markets, the choice of a feasible policy has welfare implications.

One requirement for a feasible policy choice is that governments cannot short-sell bonds. In the present paper with complete markets, these constraints do not matter as they do not prevent a certain equilibrium allocation from being supported. In the companion paper Hoelle (2014), policy must be carefully designed to provide the correct risk-sharing incentives for households. For some economies, it may be that Pareto efficiency can only be supported if the government is able to short-sell bonds. In such economies, the constraints have real effects and prevent the economy from reaching the Pareto frontier.
This paper adopts a narrow definition of unconventional monetary policy in which, in addition to the short-term bonds typically bought and sold, the central bank buys and sells bonds with longer maturities. In reality, unconventional monetary policy can refer not only to the purchase and sale of bonds of varying maturities, but additionally to the purchase of mortgage-backed securities and other real assets. Empirical works by Krishnamurthy and Vissing-Jorgensen (2011), Gagnon et al. (2011), Lenza et al. (2010), Kapetanios et al. (2012), and Baumeister and Benati (2013) have analyzed the effects of central bank purchases, including long term treasury purchases, lending to financial institutions, liquidity to financial markets, and mortgage-backed securities, on the yield curve. In the present theoretical model, the central bank has complete information about the underlying parameters of the economy and understands perfectly the mapping from its purchases to the equilibrium term structure targets.

The general equilibrium framework of this paper connects it most closely to the works of Peiris and Polemarchakis (2013), Adão et al. (2014), and Magill and Quinzii (2014a,b). In standard settings with exogenous probabilities, both Peiris and Polemarchakis (2013) and Adão et al. (2014) analyze determinacy in a complete markets setting. In Adão et al. (2014), policy refers to the targets for the entire term structure and such targets uniquely determine the equilibrium stochastic inflation rates. In Peiris and Polemarchakis (2013), policy refers to the total value of purchases made by the central bank, but not the composition of such purchases. In such a setting, nominal indeterminacy persists. It is only by targeting the composition of purchases that Peiris and Polemarchakis (2013) arrive at a result equivalent to Adão et al. (2014). In both cases, such policies have no effect on welfare as they all support a Pareto efficient allocation.

The remainder of the paper is organized as follows. Section 2 introduces the model and defines an equilibrium. Section 3 introduces the requirements for a feasible policy choice. Section 4 shows that all feasible policy choices support a Pareto efficient allocation. Section 5 examines the role of the short-term interest rates in the determination of the equilibrium allocation along the Pareto frontier. Section 6 concludes and the proofs of the main results are contained in the Appendix.

3 Classic papers in the real business cycle tradition that focus on monetary policy include Sargent and Wallace (1975), Kydland and Prescott (1977), Lucas and Stokey (1983), Barro and Gordon (1983), Chari et al. (1991), and Calvo and Guidotti (1993). In terms of general equilibrium models with incomplete markets, Magill and Quinzii (1992, 1996) were the initial papers to analyze monetary policy.

4 In a model with incomplete markets and endogenous probabilities, Magill and Quinzii (2014a) focuses on how the targets on long-term bond yields suffice to ensure the uniqueness of the stochastic inflation rates. In related work using a similar setup, Magill and Quinzii (2014b) demonstrates how forward guidance, in which the short-term bond yield is fixed for a number of periods, replicates the outcome achieved when the entire term structure is targeted.
2 The Model

The model describes a closed economy with a single infinite-lived monetary-fiscal authority. Time is discrete and infinite with time periods \( t \in \{0, 1, \ldots\} \). The filtration of uncertainty follows a one-period Markov process with finite state space \( S = \{1, \ldots, S\} \). Specifically, the realized state of uncertainty in any period \( t \), denoted \( s_t \), is a random variable defined only in terms of the realized state in the previous period \( t - 1 \), denoted \( s_{t-1} \). This random process is characterized by a transition matrix \( \Gamma \in \mathbb{R}^{S \times S} \) whose elements are \( \Gamma(s, \sigma) \) for row \( s \) and column \( \sigma \).

The history of all realizations up to and including the current realization completely characterizes the date-event and is required to uniquely identify the markets, household decisions, and policy choices. Define the history of realizations up to and including the realization \( s_t \) in period \( t \) as \( s^t = (s_0, s_1, \ldots, s_t) \).

2.1 Households

The model contains a finite number of infinite-lived households \( h \in H = \{1, \ldots, H\} \). In each date-event, households trade and consume a single physical commodity. Households receive endowments \( \{e^h(s^t)\} \). I assume that the endowments are stationary, meaning that there exists a stationary endowment mapping \( e^h : S \rightarrow \mathbb{R}_{++} \) such that \( e^h(s^t) = e^h(s_t) \) for all date-events. Denote the aggregate endowment mapping \( E : S \rightarrow \mathbb{R}_{++} \) such that \( E(s) = \sum_{h \in H} e^h(s) \forall s \in S \). The model permits aggregate risk, i.e., \( E(s) \neq E(\sigma) \) for some \( s, \sigma \in S \).

The consumption by household \( h \) in date-event \( s^t \) is denoted \( c^h(s^t) \in \mathbb{R}_+ \).

The household preferences are represented by the following utility function:

\[
E_0 \sum_{t=0}^{\infty} \beta^t u^h(c^h(s^t)) . \tag{1}
\]

Assumption 1 The discount factor \( \beta \in (0, 1) \) and \( u^h \) is \( C^2 \), strictly increasing, strictly concave, and satisfies the Inada condition.

In each date-event \( s^t \), the money supply is \( M(s^t) > 0 \) and the nominal price level is \( p(s^t) > 0 \).

In each date-event, \( J \) nominal assets are traded. The assets are indexed by \( j \in J = \{1, \ldots, J\} \). Asset 1 is a 1-period nominally risk-free bond. The nominal payout of a 1-period
bond purchased in date-event $s^t$ equals 1 for all date-events $(s^t, \sigma)_{\sigma \in S}$ and 0 otherwise. The 1-period bond is referred to as the short-term bond.

The set of long-term bonds is the set of all $j$-period bonds for $j \in \{2, ..., J\}$. Each $j$-period bond is nominally risk-free, meaning that the nominal payouts of a $j$-period bond purchased in date-event $s^t$ equals 1 for all date-events $(s^t, \sigma_1, ..., \sigma_j)_{(\sigma_1, ..., \sigma_j) \in S_j}$ and 0 otherwise. The $j$-period nominally risk-free bond can be freely traded in all interim periods up to the maturation date.

**Assumption 2** Complete markets: $J = S$.

The nominal asset price for a $j$-period bond issued in date-event $s^t$ is denoted $q_j (s^t)$. Denote the bond price vector as $q (s^t) = (q_j (s^t))_{j \in J}$.

Each date-event is divided into two subperiods. In the initial subperiod, the money market and bond markets open. Denote $\hat{m}^h (s^t) \geq 0$ as the holding of money by household $h$ at the close of the money market in date-event $s^t$. Denote $b_j^h (s^t) \in \mathbb{R}$ as the holding of a $j$-period nominal bond by household $h$ at the close of the bond markets in date-event $s^t$. Each bond can either be held long or short by the household. Denote the entire portfolio as $b^h (s^t) = (b_j^h (s^t))_{j \in J} \in \mathbb{R}^J$.

Denote $\omega^h (s^t) \in \mathbb{R}$ as the nominal wealth available for household $h$ at the beginning of date-event $s^t$. The initial period wealth $\omega^h (s_0)$ is a parameter of the model. The budget constraint at the close of the money market and bond markets in date-event $s^t$ is given by:

$$\hat{m}^h (s^t) + \sum_{j \in J} q_j (s^t) b_j^h (s^t) \leq \omega^h (s^t).$$

The budget constraint is specified in nominal terms.

In the second subperiod of each date-event, the commodity market opens. The purchase of the commodity is subject to the cash-in-advance constraint:

$$p (s^t) c^h (s^t) \leq \hat{m}^h (s^t).$$

At the same time that consumption is being purchased on the commodity market, the households receive income from selling their endowments. Denote $m^h (s^t)$ as the money holding of household $h$ by the close of the commodity market in date-event $s^t$:

$$m^h (s^t) = \hat{m}^h (s^t) + p (s^t) e^h (s_t) - p (s^t) c^h (s^t).$$
Given the money definition (4), the cash-in-advance constraint (3) can be rewritten as:

\[ m^h (s^t) \geq p (s^t) e^h (s_t). \] (5)

Entering date-events \((s^t, \sigma)_{\sigma \in \Sigma}\), the wealth available to household \(h\) is equal to the money holding plus the portfolio payout:

\[
\omega^h (s^t, \sigma) = m^h (s^t) + b^h_1 (s^t) + \sum_{j \in \mathcal{J}\setminus\{1\}} q_{j-1} (s^t, \sigma) b^h_j (s^t).
\]

For simplicity, I define \(q_0 (s^t) = 1\) for all date-events. The wealth can then be expressed as:

\[
\omega^h (s^t, \sigma) = m^h (s^t) + \sum_{j \in \mathcal{J}} q_{j-1} (s^t, \sigma) b^h_j (s^t). \] (6)

Households are permitted to short-sell the nominal bonds, so I require the following implicit debt constraint for the value of a household portfolio:

\[
\inf_{t,s^{t_t}} \left( \sum_{j \in \mathcal{J}} q_j (s^t) b^h_j (s^t) \right) > -\infty. \] (7)

The household optimization problem is:

\[
\max_{\{c^h(s^t), b^h(s^t), m^h(s^t)\}} E^0 \sum_{t=0}^{\infty} \beta^t u^h (c^h (s^t))
\]

subj. to

budget constraint (2) with (4) and (6) \(\forall t, s^t\).

cash-in-advance constraint (5) \(\forall t, s^t\)

debt constraint (7)

\(\forall t, s^t\).

### 2.2 Monetary-fiscal authority

The monetary-fiscal authority chooses the net debt positions \(B (s^t) = (B_j (s^t))_{j \in \mathcal{J}} \in \mathbb{R}_+^J\) in each date-event \(s^t\). Since \(B_j (s^t)\) is the net debt position for the monetary-fiscal authority, it must be non-negative, meaning that the monetary authority cannot purchase more than the amount of debt initially issued by the fiscal authority. The no short-sale constraint is given by:

\[ B_j (s^t) \geq 0 \text{ for all } j. \] (9)

\(5\) The fiscal authority issues debt, while the monetary authority buys or sells the debt issued by the fiscal authority. It is assumed that the two authorities are acting in perfect concert.
The monetary-fiscal authority issues the money supply $M(s^t) \geq 0$ in the date-event $s^t$. In the initial period $s_0$, the monetary-fiscal authority has the nominal obligation $W(s_0) > 0$. The model does not contain any taxes or fiscal transfers.

In the initial period, the budget constraint for the monetary-fiscal authority is given by:

$$W(s_0) = M(s_0) + \sum_{j \in J} q_j(s_0) B_j(s_0).$$

In all future periods, the budget constraint for the monetary-fiscal authority is given by:

$$M(s^{t-1}) + \sum_{j \in J} q_{j-1}(s^t) B_j(s^{t-1}) = M(s^t) + \sum_{j \in J} q_j(s^t) B_j(s^t) \ \forall t, s^t. \quad (10)$$

This specification of the monetary-fiscal authority budget constraint is equivalent to a constraint requiring that the current period debt obligations must be equal to the discounted expected seigniorage revenues, together with a solvency constraint.

With the budget constraints as written, the following implicit debt constraint is imposed (in lieu of a solvency constraint):

$$\sup_{t, s^t} \left( \sum_{j \in J} q_j(s^t) B_j(s^t) \right) < \infty. \quad (11)$$

In equilibrium, the implicit debt constraint (11) is redundant given the implicit debt constraints (7) for all households and market clearing.

### 2.3 Sequential competitive equilibrium

**Definition 1** A sequential competitive equilibrium (SCE) is the sequence of household variables $\{c^h(s^t), b^h(s^t), m^h(s^t)\}_{h \in H}$, the sequence of monetary-fiscal authority variables $\{B(s^t), M(s^t)\}$, and the sequence of price variables $\{p(s^t), q(s^t)\}$ such that:

1. Given $\{p(s^t), q(s^t)\}$ and $\omega^h(s_0)$, each household chooses $\{c^h(s^t), b^h(s^t), m^h(s^t)\}$ to solve the household problem (8).

2. Given $W(s_0)$, the monetary-fiscal authority variables $\{q(s^t), B(s^t), M(s^t)\}$ satisfy (9) and (10) in all date-events and satisfy (11).

3. Markets clear:

   $$(a) \sum_{h \in H} c^h(s^t) = \sum_{h \in H} e^h(s_t) \text{ for every } t, s^t.$$
Existence of a SCE is guaranteed using standard techniques.

The Friedman rule in date-event $s^t$ is such that $q_1(s^t) = 1$. Under the Friedman rule, money and the short-term bond are perfect substitutes. Market clearing for both implies that the sum of the two is pinned down for all households and the monetary-fiscal authority, but not the composition. The cash-in-advance constraints (5) need not bind under the Friedman rule. Outside the Friedman rule, $q_1(s^t) < 1$ and the cash-in-advance constraints (5) bind.

It is innocuous (i.e., without real effects) under the Friedman rule to set the household money holdings such that the cash-in-advance constraints (5) bind. If the cash-in-advance constraints bind, adding the cash-in-advance constraints over all households, together with the market clearing condition for the money market, yields the Quantity Theory of Money:

$$M(s^t) = p(s^t) \sum_{h \in H} e^h(s_t) = p(s^t) E(s_t).$$

Corollary 1 proves that the Friedman rule applied in all date-events is not a feasible policy choice. In particular, the Friedman rule violates the no short-sale constraints (9).

## 3 Feasible Policy Choices

Define the real debt positions for the monetary-fiscal authority and the real bond positions for the households as:

$$\hat{B}_j(s^t) = \frac{B_j(s^t)}{p(s^t)} \quad \forall j, t, s^t.$$  

$$\hat{b}^h_j(s^t) = \frac{b^h_j(s^t)}{p(s^t)} \quad \forall h, j, t, s^t.$$  

The portfolios are denoted $\hat{B}(s^t) = \left(\hat{B}_j(s^t)\right)_{j \in J}$ and $\hat{b}^h(s^t) = \left(\hat{b}^h_j(s^t)\right)_{j \in J}$, respectively. Market clearing in terms of nominal bond positions occurs if and only if market clearing in the real bond positions occurs.
3.1 Monetary-fiscal authority constraints

The monetary-fiscal authority constraints (10) in real terms, after using the Quantity Theory of Money (12), are given by:

\[
\frac{p(s_{t-1})}{p(s_t)} \left( E(s_t) + \sum_{j \in J} a_{j-1}(s_t) \hat{B}_j(s_{t-1}) \right) = E(s_t) + \sum_{j \in J} a_j(s_t) \hat{B}_j(s_t). \tag{13}
\]

3.2 Household constraints and optimization

The household budget constraint in real terms is given by:

\[
c^h(s_t) + \sum_{j \in J} a_j(s_t) \hat{B}_j(s_t) \leq \frac{p(s_{t-1})}{p(s_t)} \left( c^h(s_{t-1}) + \sum_{j \in J} a_{j-1}(s_t) \hat{B}_j(s_{t-1}) \right). \tag{14}
\]

The Euler equations for bond \( j \) are given by:

\[
a_j(s_t) = \beta \sum_{\sigma \in \mathcal{S}} \Gamma(s_t, \sigma) \frac{Du_h(c^h(s_t, \sigma))}{Du_h(c^h(s_t))} \frac{p(s_t)}{p(s_t, \sigma)} a_{j-1}(s_t, \sigma). \tag{15}
\]

These first order conditions must hold for all households.

3.3 Feasibility conditions

A monetary-fiscal authority policy choice is the sequence \( \{q(s_t), B(s_t), M(s_t)\} \). Recall that the definition of equilibrium does not require optimality on the part of the monetary-fiscal authority and no objective function for the monetary-fiscal authority is specified. A feasible policy choice is the sequence \( \{q(s_t), B(s_t), M(s_t)\} \) that is consistent with equilibrium. Necessary conditions for a feasible policy choice include:

1. Nonnegative asset prices

   The asset prices must be such that \( q_j(s_t) \geq 0 \) for all \( j \) and for all date-events.

2. Short-term bond returns dominate money returns (the zero lower bound constraint)

   The asset prices must be such that \( q_1(s_t) \leq 1 \) for all date-events. If not, then the return on money dominates the return on the short-term bonds, meaning that all households sell 1-period bonds, which violates the market clearing condition for this asset.

3. No arbitrage
For each date-event $s^t$, there exists a state-price vector $\alpha (s^t) \in \mathbb{R}^S_{++}$ such that

$$\left( q (s^t) \right)^T = \left( \alpha (s^t) \right)^T \begin{bmatrix} 1 & q_1 (s^t, 1) & \ldots & q_{J-1} (s^t, 1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & q_1 (s^t, S) & \ldots & q_{J-1} (s^t, S) \end{bmatrix}.$$ (16)

The no-arbitrage conditions (16) imply that the following inequalities must be satisfied for all $j > 1$:

$$q_j (s^t) \left[ \min_{\sigma} q_{j-1} (s^t, \sigma) \right] \leq q_j (s^t) \leq q_1 (s^t) \left[ \max_{\sigma} q_{j-1} (s^t, \sigma) \right].$$

4. The monetary-fiscal authority constraints (13) are satisfied in all date-events.
5. The no short-sale constraints (9) are satisfied in all date-events.

These 5 conditions are only necessary, and not sufficient, as the sequence $\{q (s^t), B (s^t), M (s^t)\}$ must also be consistent with equilibrium, in particular with household optimization as characterized by the first order conditions (15). Appendix A.2 provides two examples of policy choices that satisfy the 5 conditions above and yet are not consistent with equilibrium. This is the main point of the paper, to which I now turn in the following section, with Theorem 1 providing the main result.

4 Pareto Efficiency

For any date-event $s^t$, define the real payout matrix $R (s^t) \in \mathbb{R}^{S,J}$ as:

$$R (s^t) = \begin{bmatrix} \frac{p(s^t)}{p(s^t, 1)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{p(s^t)}{p(s^t, S)} \end{bmatrix} \begin{bmatrix} 1 & q_1 (s^t, 1) & \ldots & q_{J-1} (s^t, 1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & q_1 (s^t, S) & \ldots & q_{J-1} (s^t, S) \end{bmatrix}.$$ 

Under complete markets ($J = S$), $R (s^t)$ is a square matrix. By definition, the rank of the matrix $R (s^t)$ is equal to the rank of the matrix

$$\begin{bmatrix} 1 & q_1 (s^t, 1) & \ldots & q_{J-1} (s^t, 1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & q_1 (s^t, S) & \ldots & q_{J-1} (s^t, S) \end{bmatrix}.$$ 

For an equilibrium supporting Pareto efficiency, household consumption is stationary, meaning that there exist stationary mappings $c^h : S \rightarrow \mathbb{R}$ such that $c^h (s^t) = c^h (s_i)$ for all households and all date-events. Further, the ratio of marginal utilities are equal for all
households:

\[
\frac{D u^h \left(c^h(s', \sigma)\right)}{D u^h \left(c^h(s')\right)} = \frac{D u^1 \left(c^1(\sigma)\right)}{D u^1 \left(c^1(s)\right)} \quad \forall \sigma \in S \text{ and } \forall h \in H. \tag{17}
\]

The Euler equations can be written only in terms of the marginal utilities for household 1:

\[
q_j \left(s^t\right) = \beta \sum_{\sigma \in S} \Gamma \left(s_t, \sigma\right) \frac{D u^1 \left(c^1(\sigma)\right)}{D u^1 \left(c^1(s)\right)} \frac{p(s^t)}{p(s, \sigma)} q_{j-1} \left(s^t, \sigma\right). \tag{18}
\]

**Theorem 1** Under Assumptions 1-2, for any feasible policy choice, the SCE allocation is Pareto efficient.

The proof of Theorem 1 is contained in Appendix A.1. I now sketch the argument. Assume that the equilibrium allocation is Pareto efficient. Under this assumption, there are two cases to consider:

1. The policy choice is such that the payout matrices \(R \left(s^t\right)\) have full rank in all date-events. Household optimization results in an allocation satisfying Pareto efficiency.

2. The policy rule is such that a payout matrix \(R \left(s^t\right)\) has less than full rank in some date-event. Appendix A.2 contains 2 examples of policy rules (together with specific economies) under which the payout matrix \(R \left(s^t\right)\) has rank 1 in all date-events (under the assumption that the equilibrium allocation is Pareto efficient). The first example considers policies of interest rate targeting in economies without aggregate risk. The second example considers policies of nominal GDP targeting in economies with Cobb-Douglas preferences.

If the equilibrium allocation is Pareto efficient (which occurs over a closed and measure zero subset of household endowments), then the conclusion of the theorem holds.

If, however, the equilibrium allocation is not Pareto efficient, then the household marginal utilities are no longer proportional. If the household marginal utilities are no longer proportional, then the asset prices are determined using the Euler equations:

\[
q_j \left(s^t, \sigma\right) = \beta \sum_{\phi \in S} \Gamma \left(\sigma, \phi\right) \frac{D u^h \left(c^h(s^t, \sigma, \phi)\right)}{D u^h \left(c^h(s^t, \sigma)\right)} \frac{p(s^t, \sigma)}{p(s^t, \sigma, \phi)} q_{j-1} \left(s^t, \sigma, \phi\right). \]

The same policy choice, together with the different marginal utility ratios \(\left(\frac{D u^h \left(c^h(s^t, \sigma, \phi)\right)}{D u^h \left(c^h(s^t, \sigma)\right)} \right)_{h \in H}\), results in a different payout matrix \(R \left(s^t\right)\). The updated payout matrix \(R \left(s^t\right)\) has full rank, contradicting that we are in Case 2. The missing details are contained in Appendix A.1.
This contradiction argument using the logic of Hart (1975) proves that it is impossible to have both (i) a rank-deficient payout matrix $R(s^t)$ in some date-event and (ii) the payout matrices and household optimization result in a Pareto inefficient allocation. The typical outcome is that the payout matrix has full rank in all date-events, though the possibility exists that Pareto efficiency can be supported by a rank-deficient payout matrix.\footnote{Intuition suggests that the latter outcome only occurs over a closed and measure zero subset of policy choices, but that distinction is irrelevant for the main result that Pareto efficiency is always obtained.}

5 Effects of Policy on Allocation

With the cash-in-advance constraint (5), the equilibrium allocation may differ across policies. If, however, two different policy choices target the same sequence of short-term interest rates, then both policy choices support the same equilibrium allocation.

**Theorem 2** Under Assumptions 1-2, all policy choices with the same sequence of short-term bond prices $\{q_1(s^t)\}$ support the same equilibrium allocation.

**Proof.** See Appendix A.3. ■

The following corollary proves that the Friedman rule, in which $q_1(s^t) = 1$ in all date-events, is not a feasible policy choice (it is not consistent with equilibrium).

**Corollary 1** The Friedman rule violates the no short-sale constraints (9) in all date-events.

**Proof.** See Appendix A.4. ■

6 Conclusions

Under complete markets, any and all policies of term structure targeting support a Pareto efficient allocation. Policies that target the short-term bond yields suffice to uniquely determine the equilibrium allocation under complete markets. Additional policies that target the long-term bond yields do not have real effects.

In settings with incomplete markets, however, targets for the long-term bond yields can have real effects and the choice of term structure targeting policy matters (see Hoelle, 2014). Future work on term structure targeting policies should focus exclusively on settings with incomplete markets.
References


A Appendix

A.1 Proof of Theorem 1

A.1.1 Part 1

Assume that a SCE allocation is Pareto efficient. This implies that consumption is stationary. There are two cases to consider for the monetary-fiscal authority variables:

1. Case 1

The monetary-fiscal authority variables are such that in all date-events $s^t$, there exists $\sigma \in S$ such that $\left( \frac{M(s^t, \sigma, \phi)}{M(s^t, \sigma)} \right)_{\phi \in S}$ is not proportional to $\left( \frac{Du^1(c^t(\phi))}{Du^1(c^t(\sigma))} \frac{E(\phi)}{E(\sigma)} \right)_{\phi \in S}$.

2. Case 2

The monetary-fiscal authority variables are such that in some date-event $s^t$, $\left( \frac{M(s^t, \sigma, \phi)}{M(s^t, \sigma)} \right)_{\phi \in S} \propto \left( \frac{Du^1(c^t(\phi))}{Du^1(c^t(\sigma))} \frac{E(\phi)}{E(\sigma)} \right)_{\phi \in S}$ for all $\sigma \in S$.

Case 1  Consider the matrix

$$ Q(s^t) = \begin{bmatrix} 1 & q_1(s^t, 1) & \ldots & q_{J-1}(s^t, 1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & q_1(s^t, S) & \ldots & q_{J-1}(s^t, S) \end{bmatrix}. $$
Recall that the rank of $R(s^t)$ is equal to the rank of $Q(s^t)$. The first column is $\mathbf{1}$, which is a $S \times 1$ vector of ones. The second column is $(q_1(s^t, \sigma))_{\sigma \in S}$.

From the Euler equation (18),

$$q_1(s^t, \sigma) = \beta \sum_{\phi \in S} \Gamma(\sigma, \phi) \frac{Du^1(c^1(\phi))}{Du^1(c^1(\sigma))} \frac{p(s^t, \sigma)}{p(s^t, \sigma, \phi)}.$$ 

From the Quantity Theory of Money (12),

$$\frac{p(s^t, \sigma)}{p(s^t, \sigma, \phi)} = \frac{M(s^t, \sigma)}{M(s^t, \sigma, \phi)} \frac{E(\phi)}{E(\sigma)}.$$ 

This implies that the Euler equation is:

$$q_1(s^t, \sigma) = \beta \sum_{\phi \in S} \Gamma(\sigma, \phi) \frac{Du^1(c^1(\phi))}{Du^1(c^1(\sigma))} \frac{E(\phi)}{E(\sigma)} \frac{M(s^t, \sigma)}{M(s^t, \sigma, \phi)} q_1(s^t, \sigma, \phi).$$

(19)

The second column $(q_1(s^t, \sigma))_{\sigma \in S}$ is proportional to the first column $\mathbf{1}$ iff

$$\left(\frac{Du^1(c^1(\phi))}{Du^1(c^1(\sigma))} \frac{E(\phi)}{E(\sigma)} \frac{M(s^t, \sigma)}{M(s^t, \sigma, \phi)}\right)_{\sigma \in S}$$

for all $\sigma \in S$. Given the Case 1 assumption, columns 1 and 2 are linearly independent.

The third column is $(q_2(s^t, \sigma))_{\sigma \in S}$. From the Euler equation (18) and the Quantity Theory of Money (12):

$$q_2(s^t, \sigma) = \beta \sum_{\phi \in S} \Gamma(\sigma, \phi) \frac{Du^1(c^1(\phi))}{Du^1(c^1(\sigma))} \frac{E(\phi)}{E(\sigma)} \frac{M(s^t, \sigma)}{M(s^t, \sigma, \phi)} q_1(s^t, \sigma, \phi).$$

Using the same analysis as above, the columns $(q_1(s^t, \sigma, \phi))_{\phi \in S}$ and $\mathbf{1}$ are not proportional. Thus, the first 3 columns of $Q(s^t)$ are linearly independent.

Proceeding by induction, the matrix $Q(s^t)$ has full rank $S = J$.

Repeating for all date-events, the payout matrix $R(s^t)$ has full rank in all date-events. With a full rank payout matrix in all date-events, the equilibrium allocation is Pareto efficient.

**Case 2** Suppose that

$$\left(\frac{M(s^t, \sigma, \phi)}{M(s^t, \sigma)}\right)_{\sigma \in S}$$

is proportional to

$$\left(\frac{Du^1(c^1(\phi))}{Du^1(c^1(\sigma))} \frac{E(\phi)}{E(\sigma)}\right)_{\sigma \in S}$$

for all $\sigma \in S$.

Then, using the steps from Case 1, columns 1 and 2 in $Q(s^t)$ are linearly dependent.

If the equilibrium allocation satisfies Pareto efficiency, Theorem 1 holds. If the equilibrium allocation does not satisfy Pareto efficiency, proceed to Part 2.
A.1.2 Part 2

The equilibrium allocation does not satisfy Pareto efficiency. This implies that there exists some date-event \( s^t \) in which the next period household marginal utilities are not proportional, specifically that \((Du^h (c^h (s^t, \sigma)))_{\sigma \in S}\) is not proportional to \((Du^{h'} (c^{h'} (s^t, \sigma)))_{\sigma \in S}\) for some \( h, h' \in H \).

The ordered vector \( s^t \subset s^\tau \) for \( t \leq \tau \) iff \( s^\tau = (s^t, \sigma_1, ..., \sigma_{\tau-t}) \) for some \( (\sigma_1, ..., \sigma_{\tau-t}) \in S^{\tau-t} \). In words, the date-event \( s^t \) precedes the date-event \( s^\tau \) (by \( \tau - t \) periods).

**Lemma 1** Household optimality implies that if \((Du^h (c^h (s^t, \sigma)))_{\sigma \in S}\) is not proportional to \((Du^{h'} (c^{h'} (s^t, \sigma)))_{\sigma \in S}\) for some households \( h, h' \in H \), then \( \forall \tau \geq t \) such that \( s^t \subset s^\tau \), \((Du^h (c^h (s^\tau, \sigma)))_{\sigma \in S}\) is not proportional to \((Du^{h'} (c^{h'} (s^\tau, \sigma)))_{\sigma \in S}\).

Choose time period \( t \) such that there exists households \( h, h' \in H \) such that \((Du^h (c^h (s^t, \sigma)))_{\sigma \in S}\) is not proportional to \((Du^{h'} (c^{h'} (s^t, \sigma)))_{\sigma \in S} \forall \tau \geq t \). Such a time period \( t \) exists from Lemma 1. Consider the Euler equation (15) for the SCE allocation (after using the Quantity Theory of Money (12)):

\[
q_1 (s^t, \sigma) = \beta \sum_{\phi \in S} \Gamma (\sigma, \phi) \frac{Du^h (c^h (s^t, \sigma, \phi))}{Du^h (c^h (s^t, \sigma))} \frac{E(\phi)}{E(\sigma)} M (s^t, \sigma, \phi). \tag{20}
\]

Suppose, in order to obtain a contradiction, that \((M(s^t, \sigma, \phi))_{\sigma \in S} \propto (Du^h (c^h (s^t, \sigma, \phi)) E(\phi))_{\phi \in S}\) for all \( \sigma \in S \) and for all households \( h \in H \). Together with the commodity market clearing conditions, this implies that for all states \( \sigma \in S \), the ratio of marginal utilities \(\frac{Du^h (c^h (s^t, \sigma, \phi))}{Du^h (c^h (s^t, \sigma))}\) must be identical for all households \( h \in H \). This is a contradiction to the fact that \((Du^h (c^h (s^\tau, \sigma)))_{\sigma \in S}\) is not proportional to \((Du^{h'} (c^{h'} (s^\tau, \sigma)))_{\sigma \in S} \forall \tau \geq t \).

Therefore, there exists \( \sigma \in S \) such that \((M(s^t, \sigma, \phi))_{\phi \in S} \propto (Du^h (c^h (s^t, \sigma, \phi)) E(\phi))_{\phi \in S}\).

As in Case 1, columns 1 and 2 in \( Q (s^t) \) are linearly independent. By induction, citing Lemma 1 again, all columns of \( Q (s^t) \) are linearly independent. With a full rank payout matrix, the equilibrium allocation is Pareto efficient, contradicting the assumption of Part 2. Therefore, Part 2 is not possible.

**Proof of Lemma 1** Suppose, in order to obtain a contradiction and without loss of generality, that there exists households \( h, h' \in H \) such that \((Du^h (c^h (s^t, \sigma)))_{\sigma \in S}\) is not proportional to \((Du^{h'} (c^{h'} (s^t, \sigma)))_{\sigma \in S}\) and yet \((Du^h (c^h (s^t, \sigma, \phi)))_{\phi \in S} \propto (Du^{h'} (c^{h'} (s^t, \sigma, \phi)))_{\phi \in S}\) for all \( \sigma \in S \).
Households always have access to a risk-free bond. Define $\kappa_\sigma$ such that:

$$Du^h (c^h (s^t, \sigma, \phi)) = \kappa_\sigma Du^{h'} (c^{h'} (s^t, \sigma, \phi)) \quad \forall \phi \in S.$$ 

If $Du^h (c^h (s^t, \sigma)) > \kappa_\sigma Du^{h'} (c^{h'} (s^t, \sigma))$, then decreasing $b^h (s^t, \sigma)$ and increasing $b^{h'} (s^t, \sigma)$ leads to strictly higher utility for both (given that utility is strictly concave). If $Du^h (c^h (s^t, \sigma)) < \kappa_\sigma Du^{h'} (c^{h'} (s^t, \sigma))$, then the opposite prescription leads to strictly higher utility for both. If $Du^h (c^h (s^t, \sigma)) = \kappa_\sigma Du^{h'} (c^{h'} (s^t, \sigma))$ for all $\sigma \in S$, then we arrive at a contradiction to the initial supposition that $(Du^h (c^h (s^t, \sigma)))_{\sigma \in S}$ is not proportional to $(Du^{h'} (c^{h'} (s^t, \sigma)))_{\sigma \in S}$.

### A.2 Examples of Rank Deficient Payout Matrices

The following examples are constructed under the assumption that the equilibrium allocation is Pareto efficient.

#### A.2.1 No aggregate risk and inflation rate targeting

Consider an economy with identical and homothetic preferences and without aggregate risk (i.e., $E(s) = E(\sigma) \forall s, \sigma \in S$). Identical and homothetic preferences implies that household consumption is a constant fraction of total endowment in all periods.

Suppose that the monetary authority variables satisfy inflation rate targeting. Inflation rate targeting implies that there exists $\psi > 0$ such that $\frac{p(s^t)}{p(s^t, \sigma)} = \psi \forall (s^t, \sigma)$.

For these economies and under inflation rate targeting, the household Euler equations

(18) are given by:

$$q_j (s^t, \sigma) = \beta \psi \sum_{\phi \in S} \Gamma (\sigma, \phi) q_{j-1} (s^t, \sigma, \phi).$$

The equations (21) imply that $q_1 (s^t, \sigma) = \beta \psi$ and that $q_2 (s^t, \sigma) = \beta \psi \sum_{\phi \in S} \Gamma (\sigma, \phi) q_1 (s^t, \sigma, \phi)$.

By induction, $q_1 (s^t, \sigma, \phi) = \beta \psi \forall (s^t, \sigma, \phi) \in S$, meaning that $q_2 (s^t, \sigma) = (\beta \psi)^2$. Continuing by induction:

$$(q_1 (s^t, \sigma), ..., q_J (s^t, \sigma)) = \left(\beta \psi, (\beta \psi)^2, ..., (\beta \psi)^J\right).$$

In matrix form,

$$\begin{bmatrix}
1 & q_1(s^t, 1) & ... & q_{J-1}(s^t, 1) \\
: & : & : & : \\
1 & q_1(s^t, S) & ... & q_{J-1}(s^t, S)
\end{bmatrix}
= \begin{bmatrix}
1 & \beta \psi & ... & (\beta \psi)^{J-1} \\
: & : & : & : \\
1 & \beta \psi & ... & (\beta \psi)^{J-1}
\end{bmatrix}$$

---

7The inflation rate is constant and equal to $\frac{1}{\psi} - 1$. 19
has rank 1, meaning that $R(s^t)$ has rank 1.

### A.2.2 Cobb-Douglas preferences and nominal GDP targeting

Consider an economy with identical Cobb-Douglas preferences (i.e., $u^h(c) = \ln(c)$ for all $h$).

Suppose that the monetary authority variables satisfy nominal GDP (or money supply) targeting. Nominal GDP targeting implies that there exists $\mu > 0$ such that $\frac{M(s^t)}{M(s^t, \sigma)} = \mu \forall (s^t, \sigma).$\(^8\)

Using the Quantity Theory of Money (12), the inverse inflation rates are given by:

$$\frac{p(s^t)}{p(s^t, \sigma)} = \mu \frac{E(\sigma)}{E(s^t)} \forall \sigma \in S.$$ 

For these economies and under nominal GDP targeting, the household Euler equations (18) are given by:

$$q_j(s^t, \sigma) = \beta \mu \sum_{\phi \in S} \Gamma(\sigma, \phi) q_{j-1}(s^t, \sigma, \phi). \quad (22)$$

As above, induction implies both that

$$(q_1(s^t, \sigma), ..., q_J(s^t, \sigma)) = \left( \beta \mu, (\beta \mu)^2, ..., (\beta \mu)^J \right)$$

and

$$\begin{bmatrix} 1 & q_1(s^t, 1) & \ldots & q_{J-1}(s^t, 1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & q_1(s^t, S) & \ldots & q_{J-1}(s^t, S) \end{bmatrix} = \begin{bmatrix} 1 & \beta \mu & \ldots & (\beta \mu)^{J-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \beta \mu & \ldots & (\beta \mu)^{J-1} \end{bmatrix}.$$ 

The payout matrix $R(s^t)$ has rank 1.

### A.3 Proof of Theorem 2

Define the variables $\left( \tilde{b}^h(s^t), \tilde{B}(s^t) \right)$ such that

$$\begin{align*}
\tilde{b}_1^h(s^t) &= e^h(s^t) + \tilde{b}_1^h(s^t). \\
\tilde{b}_j^h(s^t) &= \tilde{b}_j^h(s^t) \text{ for } j > 1. \\
\tilde{B}_1(s^t) &= E(s^t) + \tilde{B}_1(s^t). \\
\tilde{B}_j(s^t) &= \tilde{B}_j(s^t) \text{ for } j > 1. 
\end{align*}$$

\(^8\)The monetary growth rate is constant and equal to $\frac{1}{\mu} - 1.$
The household budget constraints under Pareto efficiency are given by:

\[ c^h(s_t) - q_1(s^t) e^h(s_t) + \sum_{j \in J} q_j(s^t) \tilde{b}^h_j(s^t) = \frac{p(s^{t-1})}{p(s^t)} \left( \sum_{j \in J} q_{j-1}(s^t) \tilde{b}^h_j(s^{t-1}) \right). \] (23)

The exact same constraint can be written for date-event \((s^t, \sigma)\):

\[ c^h(\sigma) - q_1(s^t, \sigma) e^h(\sigma) + \sum_{j \in J} q_j(s^t, \sigma) \tilde{b}^h_j(s^t, \sigma) = \frac{p(s^t)}{p(s^t, \sigma)} \left( \sum_{j \in J} q_{j-1}(s^t, \sigma) \tilde{b}^h_j(s^t) \right). \] (24)

Multiply both sides of (24) by \(\beta \frac{Du^1(c^1(\sigma))}{Du^1(c^1(s))}\) and take the conditional expectation:

\[ E_t \left[ \beta \frac{Du^1(c^1(\sigma))}{Du^1(c^1(s))} \left( c^h(\sigma) - q_1(s^t, \sigma) e^h(\sigma) + \sum_{j \in J} q_j(s^t, \sigma) \tilde{b}^h_j(s^t, \sigma) \right) \right] = \sum_{j \in J} \tilde{b}^h_j(s^t) \left\{ \beta \sum_\sigma \Gamma(s_t, \sigma) \frac{Du^1(c^1(\sigma))}{Du^1(c^1(s))} \frac{p(s^t)}{p(s^t, \sigma)} q_{j-1}(s^t, \sigma) \right\}. \] (25)

The Euler equations (18) imply that:

\[ \beta \sum_\sigma \Gamma(s_t, \sigma) \frac{Du^1(c^1(\sigma))}{Du^1(c^1(s))} \frac{p(s^t)}{p(s^t, \sigma)} q_{j-1}(s^t, \sigma) = q_j(s^t) \ \forall j \in J. \]

This means that (25) is given by:

\[ E_t \left[ \beta \frac{Du^1(c^1(\sigma))}{Du^1(c^1(s))} \left( c^h(\sigma) - q_1(s^t, \sigma) e^h(\sigma) + \sum_{j \in J} q_j(s^t, \sigma) \tilde{b}^h_j(s^t, \sigma) \right) \right] = \sum_{j \in J} q_j(s^t) \tilde{b}^h_j(s^t). \] (26)

The real household wealth vectors are defined as:

\[ \frac{\omega^h(s^t)}{p(s^t)} = \frac{p(s^{t-1})}{p(s^t)} \left( \sum_{j \in J} q_{j-1}(s^t) \tilde{b}^h_j(s^{t-1}) \right). \] (27)

Inserting the new expression (26) back into the date-event \(s^t\) budget constraint (23) and iterating forward yields:

\[ \frac{\omega^h(s^t)}{p(s^t)} = \sum_{k=0}^\infty \beta^k E_t \left[ \frac{Du^1(c^1(s_{t+k}))}{Du^1(c^1(s_t))} \left( c^h(s_{t+k}) - q_1(s_{t+k}) e^h(s_{t+k}) \right) \right], \] (28)

after citing the transversality condition.
Sum the budget constraints over all households and apply the market clearing condition to yield:

\[
\frac{W(s^t)}{p(s^t)} = \sum_{h \in H} \frac{\omega^h(s^t)}{p(s^t)} = \sum_{k=0}^\infty \beta^k E_t \left[ \frac{Du^1(c^1(s_{t+k}))}{Du^1(c^1(s_t))} E(s_{t+k}) \left(1 - q_1(s^{t+k})\right) \right].
\] (29)

The equilibrium equations (28) and (29) must hold in all date-events \(s^t\), including the initial period. In the initial period, the initial nominal wealth \(\omega^h(s_0)\) \(h \in H\) and \(W(s_0)\) are parameters of the model. If the sequence \(\{q_1(s^t)\}\) is held fixed, equation (29) implies a unique value for \(p(s_0)\). The set of Pareto efficient allocations has dimension \(H - 1\). If the sequence \(\{q_1(s^t)\}\) is held fixed, unique stationary mappings \(c^h : S \rightarrow \mathbb{R}\) exist satisfying the equilibrium equation (28) for \(H - 1\) households. Commodity market clearing yields the mapping for the omitted household.

A.4 Proof of Corollary 1

Using market clearing and (27):

\[
\frac{W(s^t)}{p(s^t)} = \frac{p(s^{t-1})}{p(s^t)} \left( \sum_{j \in J} q_{j-1}(s^t) \tilde{B}_j(s^{t-1}) \right).
\]

The no short-sale constraints (9) require that:

\[
\frac{W(s^t)}{p(s^t)} \geq \frac{p(s^{t-1})}{p(s^t)} E(s_{t-1}) > 0.
\]

Under the Friedman rule, requiring that \(q_1(s^{t+k}) = 1\) for all future date-events, (29) implies that \(\frac{W(s^t)}{p(s^t)} = 0\). This contradiction finishes the claim.