Contents

I Foundations 1

1 Macroeconomic Accounting 3
   1.1 Product markets .............................................. 3
   1.2 Price indices ................................................ 13
   1.3 Labor markets ................................................. 17
   1.4 Money markets ............................................... 19
   1.5 Bond markets ................................................ 27
   1.6 Stock markets ................................................ 33

2 Mathematical Preliminaries 43
   2.1 Matrix Algebra ............................................... 43
   2.2 Basic Calculus ............................................... 52
   2.3 Optimization ................................................ 58
   2.4 Advanced Topics ............................................. 70

3 Microfoundations 87
   3.1 Utility functions ............................................. 87
   3.2 Equilibrium .................................................. 92
   3.3 Pareto efficiency ............................................. 96
   3.4 Equilibrium equations ...................................... 99
   3.5 Pareto efficiency equations ................................. 106
   3.6 Basic Welfare Theorems .................................... 109
   3.7 Edgeworth Box ............................................... 111
   3.8 Failures of the FBWT ....................................... 116
   3.9 Household demand properties .............................. 126
   3.10 Exercises ..................................................... 134
II  Growth Theory  

4 Neoclassical Growth Model  
  4.1 Introducing the model ................................. 141  
  4.2 Euler equation ........................................ 150  
  4.3 Dynamic programming ................................. 157  
  4.4 The AK model ......................................... 164  
  4.5 Equilibrium solution with taxes ...................... 173  
  4.6 Equilibrium solution with taxes and human capital 184  
  4.7 Exercises ............................................ 192  

5 Endogenous Growth Theory  
  5.1 Monopolistic competition ............................ 197  
  5.2 Technological change ................................ 205  
  5.3 Exercises ............................................ 216  

III  Classical Monetary Theory  

6 Cash-in-Advance Model  
  6.1 The basic model .................................... 223  
  6.2 Open market operations ............................. 233  
  6.3 Endogenous money demand ......................... 238  

7 Money in the Overlapping Generations Model  
  7.1 The model without money ............................ 249  
  7.2 The model with money ............................... 255  

IV  Real Business Cycle Theory  

8 Real Business Cycle Model  
  8.1 Stochastic growth model ............................ 267  
  8.2 Stochastic growth with endogenous labor .......... 274  
  8.3 Stochastic growth with taxes ....................... 280  
  8.4 Exercises ............................................ 286
## CONTENTS

### 9 Fiscal Policy
- 9.1 Tax revenue ......................................................... 291
- 9.2 Time-consistent fiscal policy ................................. 303
- 9.3 Debt in the overlapping generations model ................. 316
- 9.4 Exercises .......................................................... 325

### V New Keynesian Theory

#### 10 New Keynesian Monetary Theory
- 10.1 Introducing the model ........................................... 333
- 10.2 Household optimization ........................................ 339
- 10.3 Firm optimization ............................................... 342
- 10.4 Output gap ....................................................... 350
- 10.5 Effects of a monetary policy and technology shock ........ 354
- 10.6 Exercises .......................................................... 360

#### 11 New Keynesian Labor Market Theory
- 11.1 Extending the model ............................................. 365
- 11.2 Wage setting by the households .............................. 374
- 11.3 Price setting by the firms ...................................... 381
- 11.4 Output gap and wage gap ..................................... 383
- 11.5 Introducing unemployment .................................... 389
- 11.6 Solving the model ............................................... 392
- 11.7 Exercises .......................................................... 402

### VI Financial Economics

#### 12 Asset Pricing
- 12.1 General financial equilibrium ............................... 409
- 12.2 No arbitrage ...................................................... 415
- 12.3 Consumption asset pricing model (CAPM) ................. 421
- 12.4 Lucas asset pricing model .................................... 427
- 12.5 Bubbles .......................................................... 437
- 12.6 The Black-Scholes model ..................................... 447
12.7 Exercises ................................................................. 454

13 Leverage Cycle ......................................................... 459
  13.1 2-period model: No borrowing ................................. 459
  13.2 2-period model: With borrowing ............................... 464
  13.3 2-period model: Borrowing contracts ......................... 471
  13.4 3-period model: The leverage cycle ......................... 479
  13.5 Exercises ........................................................... 502

VII Search Theory ....................................................... 509

14 Monetary Search Theory ............................................. 511
  14.1 Pure exchange model ............................................. 511
  14.2 Bargaining model ................................................ 518

15 Labor Market Search Theory ...................................... 527
  15.1 Random labor search ............................................. 527
  15.2 Search with matching ........................................... 534
  15.3 Competitive search with matching ........................... 544
Preface

This manuscript provides the course material for a master’s level course on macroeconomics, with particular emphasis on theoretical and microfounded models of macroeconomic behavior. The course typically follows a master’s level course on microeconomics. However, the economics material in this text is entirely self-contained, including a chapter introducing all microfoundations needed to analyze the macroeconomic models. No prerequisite knowledge of any material in economics is required.

The material is slightly more advanced than a traditional course in intermediate macroeconomics (typically offered at the undergraduate level) as the use of microfounded models requires multivariate calculus. The material bridges the gap between intermediate macroeconomics and the core macroeconomic theory at the PhD level. The models covered are the same as those used by economists at the research frontier, but I focus on stylized settings and special economies that are easy to analyze. The treatment of these models and their applications is accessible to students with even the most primitive background in economics. The benefit of working with microfounded models is that the model predictions are more precise and less reliant on ad hoc assumptions that vary model to model. Using a consistent methodology, students are able to analyze the mechanisms by which macroeconomic policy affects the behavior of economic agents (households and firms) and market conditions.

While no prerequisite material in economics is required, the mathematical prerequisites include multivariate calculus and matrix algebra. Solving the macroeconomic models requires students to evaluate partial derivatives of multivariate functions and work with vectors and matrices. A comprehensive review of all mathematics needed in the text is contained in the mathematical preliminaries chapter.

The objective of this manuscript is to introduce students to the economic models used to evaluate macroeconomic policies in the real world, not least of which includes an evaluation of the policies enacted in response to the financial crisis of 2007-2008.
Organization  The manuscript is divided into seven parts.

Part I: Foundations (Chapters 1-3) covers chapters on Macroeconomic Accounting, Mathematical Preliminaries, and Microfoundations. The chapter on Macroeconomic Accounting focuses on 6 classes of macroeconomic variables: product market data, price indices, labor market data, money market data, bond market data, and stock market data. The chapter shows how the variables are measured in practice and provides an example of how the reported values are calculated from data. The chapter on Mathematical Preliminaries (no advanced mathematics is used in the first chapter) contains material on calculus and optimization required for all future chapters in the manuscript. The mathematical material is self-contained, but incredibly brief. It serves as an excellent refresher for students who have already completed the prerequisites of multivariate calculus and matrix algebra. The chapter on Microfoundations contains material on general equilibrium theory that forms the foundation for all modern macroeconomic models.

Part II: Growth Theory (Chapters 4-5) covers chapters on the complementary models of growth: Neoclassical Growth Theory and Endogenous Technological Change. The chapter on Neoclassical Growth Theory considers a growth model in which household savings and household labor are endogenously determined. As with all growth models, an infinite time horizon model is required. To solve such a model with endogenous savings and labor, the tools of dynamic programming are introduced. The chapter on Endogenous Technological Change provides an endogenous reason for economic growth. Growth can occur for one of two fundamental reasons: the inputs into the production process increase or a technology increase allows the same inputs to be used more productively. This chapter focuses on technological change, using the setting of monopolistic competition to drive firms’ profit incentives.

Part III: Classical Monetary Theory (Chapters 6-7) covers chapters on the two most common models of monetary economies: the Cash-in-Advance Model (Chapter 6) and the Overlapping Generations Model (Chapter 7).

Part IV: Real Business Cycle Theory (Chapters 8-9) includes chapters on Real Business Cycle Theory and Fiscal Policy. The chapter on Real Business Cycle Theory is the first chapter in this manuscript to consider stochastic dynamic economic models, where the stochastic nature is incorporated by using aggregate shocks to the economy. The real business cycle theory is founded on the principle that business cycle fluctuations, and therefore remedies to mitigate the effects of such fluctuations, have their origin in real shocks to the macroeconomy. Real shocks include fiscal policy shocks, supply side shocks, and demand side shocks. The chapter on Fiscal Policy looks more closely at how fiscal policy is determined by the
government. Topics to be covered include optimal taxation, Laffer curves, time-consistent policy, and debt dynamics.

Part V: New Keynesian Theory (Chapters 10-11) includes chapters on New Keynesian Monetary Theory and New Keynesian Labor Market Theory. Models in the New Keynesian tradition are microfounded and adhere to the central tenets of Keynesian theory, principally including that a monetary shock can have real effects in the economy. This provides an avenue for monetary policy, which is conducted according to a Taylor rule specification. The chapter on New Keynesian Monetary Theory focuses only on the effects of monetary policy on output and inflation. The chapter on New Keynesian Labor Market Theory extends the analysis to understand the effects of monetary policy on unemployment.

Part VI: Financial Economics (Chapters 12-13) includes chapters on Asset Pricing and Leverage Cycle. The chapter on Asset Pricing introduces the most important financial models. These financial models differ in their methods and the type of assets being studied, but all have one common feature: the notion of no arbitrage. No arbitrage is a condition used throughout all of economics and is the foundation for finance. Using this principle, we will study the leading asset pricing models, seeking to understand the relationship between macroeconomic policy, business cycle fluctuations, and financial shocks. The chapter on the Leverage Cycle focuses on one type of borrowing contracts in which borrowing is secured with collateral. The leverage ratio measures the collateral amount relative to the loan size, with higher leverage corresponding to easier lending conditions. This chapter considers a simple model showing how the interest rate and the leverage ratio can be endogenously determined. The key feature of the model is that investors have different beliefs about the likelihood of a financial market crash in the future. The model is able to capture the dynamic in which investors over-leverage in good times, and then de-leverage when a negative financial shock hits. This so-called leverage cycle is one of the most important explanations of the 2007-2008 financial crisis.

Part VII: Search Theory (Chapters 14-15) includes chapters analyzing the effects of bilateral, rather than market-based, trades. The chapter on Monetary Search Theory covers the basic models of monetary search theory. The theory of monetary search relies upon the fundamental observation that economic exchange takes place in decentralized meetings between pairs of agents. The key friction, and the reason that agents may choose to hold fiat money, is search and matching frictions that limit agents’ opportunities for economic exchange in the future. The chapter on Labor Market Search Theory applies the search and matching methodology to analyze the labor markets. The search and matching process is
between two groups of agents: workers and firms. The search and matching model is able to provide microfoundations that can explain cycles of unemployment.

**Text utilization** A semester-long course in macroeconomics (as taught for many years at Purdue University) would include Parts I and II, and selected chapters from Parts IV-VI. A year-long course in macroeconomics would be able to cover all topics. A stand-alone course in general equilibrium theory at the master's level can be developed using the material in Chapters 2 and 3. Students in an Economics PhD program without any prior exposure to macroeconomics will find the manuscript as a useful review that is easily accessible and also complements and previews the material to be covered during a PhD program.

Most chapters contain exercises that allow students to gain experience solving and applying the theoretical models.

**Mathematical notation** The following is a list of mathematical terminology and notation utilized throughout this manuscript:

- \( \mathbb{R} \) refers to the set of real numbers, \( \mathbb{R}^n \) to the set of \( n \)-dimensional vectors of real numbers, \( \mathbb{R}^n_{\geq} \) to the set of \( n \)-dimensional vectors of nonnegative real numbers, and \( \mathbb{R}^{n}_{++} \) to the set of \( n \)-dimensional vectors of strictly positive real numbers.
- \( \mathbb{N} \) refers to the set of natural numbers (including 0), \( \mathbb{N} = \{0, 1, 2, \ldots\} \).
- \( \in \) is the notation for "an element of." For example \( x \in \mathbb{R} \) means that \( x \) is an element of the set of real numbers.
- \( \notin \) is the notation for "not an element of."
- If \( x \in \{1, 2, 3\} \), then \( x \) is an element of the discrete set \( \{1, 2, 3\} \), meaning that it either has value 1, 2, or 3.
- If \( x \in \mathbb{R}^n \), then \( x \) is an \( n \)-dimensional vector which can be written as the column vector \( x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \). Unless otherwise specified, vectors are written as column vectors.
- If \( x \in \mathbb{R}^n \), the transpose \( x^T = (x_1, \ldots, x_n) \) is a row vector.
- If \( M \in \mathbb{R}^{n,m} \), then \( M \) is a matrix with \( n \) rows and \( m \) columns whose elements are all real numbers.
• The matrix $I_n \in \mathbb{R}^{n,n}$ is the $n$–dimensional identity matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \ldots & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

• $A \subseteq B$ means that the set $A$ is a subset of the set $B$ (equivalently written $B \supseteq A$). Specifically, for any element $a \in A$, it must be that $a \in B$.

• $A = B$ means that the set $A$ is equivalent to the set $B$, specifically $A \subseteq B$ and $B \subseteq A$.

• The set $A \setminus B$ is the set containing all elements that are in the set $A$, but not in the set $B$. Specifically, $A \setminus B$ is the set of all elements $a \in A$ such that $a \notin B$.

• The set $[a, b]$ is the closed interval between $a$ and $b$.

• The set $(a, b)$ is the open interval between $a$ and $b$.

• $\sum_{t=0}^{T} x_t$ is the summation of the elements indexed from $t = 0$ to $t = T$, where we allow for the possibility that $T = \infty$.

• $\prod_{t=0}^{T} x_t$ is the product of the elements indexed from $t = 0$ to $t = T$.

• If $\{s_t\}_{t \in \mathbb{N}}$ is a sequence (all sequences contain an infinite number of terms), then $s = \lim_{t \to \infty} s_t$ is the limit point if and only if the distance $|s_t - s|$ becomes arbitrarily small as $t$ approaches $\infty$.

• $\forall t$ is the notation "for every $t$." For example, the tax rate $\tau(t) = 0.10 \forall t$ means that the tax rate is 10% in all time periods $t$ of the model.

• The derivative of a function $f : \mathbb{R} \to \mathbb{R}$ is denoted $Df(x)$.

• The partial derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$ with respect to the $n^{th}$ element is denoted $D_n f \left(x_1, \ldots, x_n\right)$.

• The integral $\int_{a}^{b} f(x)dx$ refers to the integral of the function $f(x)$ with respect to $x$ over the range from $a$ to $b$. 
Part I

Foundations
1

Macroeconomic Accounting

1.1 Product markets

1.1.1 Sneak peek

Summary

Product markets refers to the markets on which goods and services are traded. Data from these markets provides a measure of the aggregate output or production of an economy. The most common measure of product market output is gross domestic product, or GDP.

Product market data is a flow measure of value, meaning that it measures the number of units of currency (what economists call value) required to purchase the entire economy’s output over a 3-month period (the flow data for product markets is typically collected at a quarterly frequency, meaning 4 times per year).

Main takeaways

After completing this section, you will be able to answer the following questions:

- How is GDP measured?

- What are the two methods used to compute real GDP from nominal GDP?

- Given data in any two-good economy, what are the steps required to compute real GDP (for both methods)?
1.1.2 Background

Macroeconomic variables are of two types: stocks and flows. Economists view stocks as the number of physical units of machinery, or capital, used in the production process. Economists view flows as the amount of money that changes hands in a given period of time. In order to form a basis of comparison, both stocks and flows are measured in terms of the unit of account. If we are in the United States, the unit of account is the dollar. Other countries measure the stocks and flows in their own currency, though these values are often translated to US dollars (via the exchange rates) when international comparisons are required.

Nominal GDP

The most common measurement of economic activity is gross domestic product, or GDP. GDP is defined as the value of all final goods and services produced in an economy over a given time period. The period of time for GDP is one quarter of a year (3 months). There are two equivalent ways to think of GDP: the income approach and the expenditures approach.

GDP can be viewed as the total amount of money received by firms selling output over a given period of time. This is sometimes referred to as the "income approach" to measuring GDP. The total income of firms is used to pay labor costs, rental costs (on property, machines, and intellectual property, for a start), and interest payments. The total costs paid by all firms for labor must be equal to the total labor income collected by all households in the economy. In a similar fashion, the rental costs for all firms must equal the rental incomes for all households and the interest payments for all firms (plus the government) must equal the interest incomes for all households. After paying all costs, what is left over for the firms are profits. The national income can then be represented by the equation:

\[ NI = W + RI + Int + \pi. \]  

The variables are defined in the following table:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>NI</td>
<td>national income</td>
</tr>
<tr>
<td>W</td>
<td>labor incomes collected by households</td>
</tr>
<tr>
<td>RI</td>
<td>rental incomes collected by households</td>
</tr>
<tr>
<td>Int</td>
<td>interest earned by households</td>
</tr>
<tr>
<td>\pi</td>
<td>firm profits</td>
</tr>
</tbody>
</table>
1.1. PRODUCT MARKETS

There are still some pieces of GDP omitted by the national income measure. The relation is given by:

$$Y = NI + ST + \delta + NFFI.$$  

(1.3)

The variables are defined in the following table:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y$</td>
<td>GDP, or output</td>
</tr>
<tr>
<td>$ST$</td>
<td>sales tax</td>
</tr>
<tr>
<td>$\delta$</td>
<td>depreciation</td>
</tr>
<tr>
<td>$NFFI$</td>
<td>net foreign factor income</td>
</tr>
</tbody>
</table>

The final term $NFFI$ is a correction factor that accounts for the fact that foreign capital and labor may be used in US production and US capital and labor may be used in foreign production. $NFFI$ is the income of foreign factors of production in the US economy minus the income of US factors of production in foreign economies.

An equivalent approach measures GDP from the expenditure side and is referred to as the "expenditure approach." This approach is the more common definition of GDP and is useful to keep in mind when working through all the macroeconomic models in the text.

In this approach, GDP is equal to the total amount of money spent in the economy. Money is spent by all agents in the economy, where an agent can be a household, a firm, or the government. There is also a correction term to account for the net flows from exports, where the net flow is equal to the difference between the money earned by exporting from the US (to other countries) and the money spent to import goods to the US (from other countries). This is typically represented by the equation:

$$Y = C + I + G + Ex - Im.$$  

(1.5)

The variables are defined in the following table:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y$</td>
<td>GDP, or output</td>
</tr>
<tr>
<td>$C$</td>
<td>consumption by households</td>
</tr>
<tr>
<td>$I$</td>
<td>investment by firms</td>
</tr>
<tr>
<td>$G$</td>
<td>government purchases</td>
</tr>
<tr>
<td>$Ex$</td>
<td>exports (from US to foreign markets)</td>
</tr>
<tr>
<td>$Im$</td>
<td>imports (from foreign markets to US)</td>
</tr>
</tbody>
</table>
1. MACROECONOMIC ACCOUNTING

The reason that the two approaches to measuring GDP are equivalent is a result known as Walras’ Law, a famous relation in economics named after the 19th century French economist Leon Walras. The chapter on Microfoundations will discuss Walras’ Law in further detail. All that is required at this juncture is to know that Walras’ Law states that the total income of an economy must be equal to the total expenditures of an economy.

A few things are important to keep in mind about GDP:

- GDP only includes the sale of newly produced goods and services. The sale of used goods is not included.

- Firms may spend dollars to produce output that is not immediately sold. Such output would be added to inventory and can be sold at a later date. With inventory, the firm expenditures in the current period are larger than the household income by the amount of the output being inventoried. This is because the firm makes the investment to have these items produced, but it is only with the sale of these items that a household (a laborer) receives income from sales. For an appropriate balance, the value of any goods that are inventoried are included in the GDP measure. This of course requires that such goods can be sold at a later date. An inventory of a loaf of bread does not increase GDP, because bread is a perishable good.

- GDP only includes the value of all final goods and services. Intermediate goods, or those used as an input in another firm’s production process, are not included.

A different national product account is called gross national product, or GNP. This measure was in vogue prior to 1990, but has since been replaced as the fundamental measure of economic productivity by GDP. The difference between GDP and GNP is net factor payments from abroad. When factors of production (such as capital and labor) owned by the US are used in production in a foreign country, this output is included in US GNP, but not in US GDP. GDP measures all final goods and services produced in the US, while GNP measures all final goods and services produced using US factors of production (whether home or abroad). The net factor payments from abroad equal the payments received for US factors used in foreign production minus the payments made for US production using foreign factors.

The relation between GDP and GNP is given by:

\[
\text{GDP} = \text{GNP} - \text{net factor payments from abroad.}
\]
1.1. PRODUCT MARKETS

Real GDP

Thus far, we have introduced GDP. This measure is equivalently referred to as nominal GDP, as we are specifying the total number of dollars that exchange hands. The nominal value is the number of units of dollars, where the value of the dollar must be understood as the value of the dollar at that particular quarter of interest. For instance, if the Quarter 4, 2014 economy sells 7 cars at $20,000 per car, then the contribution to nominal GDP is equal to $140,000 in Quarter 4, 2014. The price of the car is $20,000 in Quarter 4, 2014. The value of the dollar in Quarter 4, 2014 is measured against how many units of other goods could be purchased with the same dollar at that time.

As we know, the price of cars, and most other goods, will increase over time. This is a phenomenon called inflation. GDP growth rates contain two terms: growth due to a change in price (inflation) and growth due to a change in quantity (real economic growth). We want to separate out the effects of inflation from the effects of real economic growth.

To do this, we will compute a new measurement variable called real GDP. In words, real GDP is a measure of the value of the final goods and services produced in an economy, while holding the aggregate price level fixed at the price level for a base year. Currently, the base year is 2009, meaning that all real GDP values are expressed in units of 2009 dollars. Given the purchasing power of the dollar in 2009, real GDP in Quarter 4, 2014 measures the number of 2009 dollars required to purchase the entire output of the Quarter 4, 2014 economy.

There are two methods for computing real GDP. The old method is referred to as the "fixed base year" method. This method was used up until 1996. Since that time, the method used is called the "chain weighting" method. Using either method (and I will introduce them both), we can define the GDP deflator as the ratio of the nominal GDP and the real GDP:

$$\text{GDP Deflator} = \frac{\text{Nominal GDP}}{\text{Real GDP}}. \quad (1.8)$$

The GDP deflator can be viewed as a measure of inflation. Nominal GDP values are measured in the current value of the dollar, so 2014 GDP values are measured in 2014 dollars. If the real GDP base year is 2009, then the ratio of nominal GDP to real GDP equals the ratio of the aggregate price level in 2014 to the price level in 2009. This ratio is typically bigger than 1, meaning that inflation has occurred from 2009 to 2014 (the price has increased).
**Fixed base year method** For the fixed base year method, the choice of the base year has a large impact on the real GDP values. Let’s suppose that the base year is 2009. The nominal GDP in the year 2014 is determined by summing up the value of all the goods sold in the economy. For any good, the value is equal to the number of units sold times the price. If we want to know the real GDP in 2014 (expressed in terms of 2009 dollars), the value is equal to the number of units sold times the price of the good in 2009. The important distinction is that we are valuing each good in terms of its 2009 prices.

For the year 2009 itself, the real GDP is identical to the nominal GDP as both are specified in 2009 dollars. In all other years, our real GDP measurements rely on the ad-hoc choice of 2009 for the price data.

The fixed base year method suffers from several shortcomings. Most notably, if the price changes between 2009 and 2014 differ across goods, the real GDP value tends to overstate the impact from goods with large price changes. Additionally, if the base year is changed, the real GDP values and the real GDP growth rates would also change. In essence, a change of the base year leads to new GDP growth rates in all time periods. "Rewriting history" is something that economists seek to avoid.

**Chain weighting method** Given these shortcomings of the fixed base year method, a new method was introduced in 1996. This method still specifies real GDP values in terms of a base year, but the choice of base year is largely irrelevant. If we want to change the base year, then we would only need to apply a constant scale factor to all real GDP values. The size of these real GDP values would change, but the relative differences would not. This means that the real GDP growth rates would not change as a result of a change in the base year.

The chain weighting method looks at changes over a large number of years as a chain of 1-year changes. Let’s suppose throughout that the base year for this exercise is 2009. If we have data for the years 2009, 2010, and 2011, then in order to determine the rate of change between 2009 and 2011, we first find the rate of change between 2009 and 2010 and then multiply this by the rate of change between 2010 and 2011.

The steps for the chain weighting method are given below:

1. Calculate the rate of change between 2009 and 2010.

   (a) Find the value of 2009 quantities in 2009 dollars (this is just the nominal GDP in 2009).
1.1. PRODUCT MARKETS

(b) Find the value of 2010 quantities in 2009 dollars.
(c) The relative growth in 2009 dollars is equal to \( \frac{\text{answer from b.}}{\text{answer from a.}} \).
(d) Find the value of 2009 quantities in 2010 dollars.
(e) Find the value of 2010 quantities in 2010 dollars (this is just the nominal GDP in 2010).
(f) The relative growth in 2010 dollars is equal to \( \frac{\text{answer from e.}}{\text{answer from d.}} \).
(g) The geometric average of the relative growth ratios is equal to

\[
\sqrt{(\text{answer from c.})(\text{answer from f.})}.
\] (1.9)

Call this term the relative growth ratio 2009-2010.

2. Real GDP and nominal GDP in 2009 are identical (when 2009 is the base year). Real GDP in 2010 is determined as:

\[
(\text{real GDP 2010}) = (\text{real GDP 2009}) \times (\text{relative growth ratio 2009-2010}).
\] (1.10)

3. Calculate the rate of change between 2010 and 2011 (following steps a.-g. as above for the years 2010 and 2011).

4. The real GDP in 2011 is determined as:

\[
\begin{align*}
(\text{real GDP 2011}) &= (\text{real GDP 2010}) \times (\text{relative growth ratio 2010-2011}) \\
(\text{real GDP 2011}) &= (\text{real GDP 2009}) \times (\text{relative growth ratio 2009-2010}) \times (\text{relative growth ratio 2010-2011})
\end{align*}
\] (1.11)

**Seasonal adjustments** The final thing I want to mention about real GDP is the seasonal adjustments. Economists are interested in real GDP growth rates on a quarterly basis. All measures of output, however, display a regular seasonal pattern. Specifically (due to the holiday season and the changing of the seasons), output peaks in the 4th quarter of the year and bottoms out during the 1st quarter of the year. In fact, nominal GDP can fall by about 8% from the 4th quarter to the 1st quarter (on average across the business cycle). Thus, we must account for these regular seasonal properties if we want to have meaningful growth rates on a quarterly basis. When these seasonal adjustments are made, the data series is said to be seasonally adjusted.
All quarterly data that you will find will be seasonally adjusted. Recent research suggests that the seasonal adjustments may be relying on methods that worked well for seasonal patterns in the 1990’s, but might be unable to capture the seasonal patterns in the 2010’s. Data collection agencies are aware of the problem and working to find the best methods for seasonal adjustments.

### 1.1.3 Measurement example

The following example will calculate the real GDP values using both the fixed base year method and the chain weighting method. Data is provided below for our idealized economy with two goods: apples and bananas.

<table>
<thead>
<tr>
<th></th>
<th>Quantities</th>
<th>Prices</th>
</tr>
</thead>
<tbody>
<tr>
<td>Apples</td>
<td>3</td>
<td>$4</td>
</tr>
<tr>
<td>Bananas</td>
<td>2</td>
<td>$3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Quantities</th>
<th>Prices</th>
</tr>
</thead>
<tbody>
<tr>
<td>Apples</td>
<td>4</td>
<td>$4.50</td>
</tr>
<tr>
<td>Bananas</td>
<td>2.5</td>
<td>$3.25</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Quantities</th>
<th>Prices</th>
</tr>
</thead>
<tbody>
<tr>
<td>Apples</td>
<td>3</td>
<td>$6</td>
</tr>
<tr>
<td>Bananas</td>
<td>3</td>
<td>$3.75</td>
</tr>
</tbody>
</table>

**Fixed base year method**

The fixed base year will be 2009.

#### 2009

The nominal GDP in 2009 is equal to

\[
2009 \text{ nominal GDP} = 3(\$4) + 2(\$3) = \$18. \tag{1.12}
\]

The real GDP (in terms of 2009 dollars) is equal to $18.
1.1. PRODUCT MARKETS

2010 The nominal GDP in 2010 is equal to

\[
2010 \text{ nominal GDP} = 4(\$4.50) + 2.5(\$3.25) = \$26.125. \tag{1.13}
\]

The real GDP is equal to the 2010 quantities in terms of the 2009 prices:

\[
2010 \text{ real GDP} = 4(\$4) + 2.5(\$3) = \$23.5. \tag{1.14}
\]

The GDP deflator is equal to:

\[
\text{GDP deflator} = \frac{\$26.125}{\$23.5} = 1.11. \tag{1.15}
\]

This implies 11% price growth between 2009 and 2010.

2011 The nominal GDP in 2011 is equal to

\[
2011 \text{ nominal GDP} = 3(\$6) + 3(\$3.75) = \$29.25. \tag{1.16}
\]

The real GDP is equal to the 2011 quantities in terms of the 2009 prices:

\[
2011 \text{ real GDP} = 3(\$4) + 3(\$3) = \$21. \tag{1.17}
\]

The GDP deflator is equal to:

\[
\text{GDP deflator} = \frac{\$29.25}{\$21} = 1.39. \tag{1.18}
\]

This implies 39% price growth between 2009 and 2011.

Chain weighting method

2009 The nominal GDP in 2009 is equal to

\[
2009 \text{ nominal GDP} = 3(\$4) + 2(\$3) = \$18. \tag{1.19}
\]

The real GDP (in terms of 2009 dollars) is equal to $18.
The value of 2009 quantities in terms of 2009 dollars is equal to $18. The value of 2010 quantities in terms of 2009 dollars is equal to $23.5 (from above). The relative growth ratio in 2009 dollars is equal to:

\[
\text{relative growth ratio 2009 dollars} = \frac{23.5}{18} = 1.3056. \tag{1.20}
\]

The value of 2009 quantities in terms of 2010 dollars is equal to

\[
3(4.50) + 2(3.25) = 20. \tag{1.21}
\]

The value of 2010 quantities in terms of 2010 dollars is equal to $26.125. The relative growth ratio in 2010 dollars is equal to:

\[
\text{relative growth ratio 2010 dollars} = \frac{26.125}{20} = 1.30625. \tag{1.22}
\]

The geometric average of the relative growth ratios is equal to:

\[
\text{relative growth ratio 2009-2010} = \sqrt{(1.3056)(1.30625)} = 1.3059. \tag{1.23}
\]

This means that the real GDP in 2010 is equal to:

\[
\text{real GDP 2010} = 18(1.3059) = 23.51. \tag{1.24}
\]

Compared to the real GDP determined using the fixed base year method ($23.50), these values are very close. The GDP deflator can also be computed and we will see that its value is very similar to what we found for the fixed base year method.

Let’s see what happens as we move further away from the base year.

The value of 2010 quantities in terms of 2010 dollars is equal to $26.125. The value of 2011 quantities in terms of 2010 dollars is equal to

\[
3(4.50) + 3(3.25) = 23.25. \tag{1.25}
\]

The relative growth ratio in 2010 dollars is equal to:

\[
\text{relative growth ratio 2010 dollars} = \frac{23.25}{26.125} = 0.89. \tag{1.26}
\]
1.2. **PRICE INDICES**

The value of 2010 quantities in terms of 2011 dollars is equal to

\[ 4(\$6) + 2.5(\$3.75) = \$33.375. \quad (1.27) \]

The value of 2011 quantities in terms of 2011 dollars is equal to $29.25. The relative growth ratio in 2011 dollars is equal to:

\[
\text{relative growth ratio 2011 dollars} = \frac{\$29.25}{\$33.375} = 0.8764. \quad (1.28)
\]

The geometric average of the relative growth ratio is equal to:

\[
\text{relative growth ratio 2010-2011} = \sqrt{(0.89)(0.8764)} = 0.8832. \quad (1.29)
\]

This means that the real GDP in 2011 is equal to:

\[ \text{real GDP 2011} = \$18 (1.3059)(0.8832) = \$20.76. \quad (1.30) \]

Compared to the real GDP determined using the fixed base year method ($21), we see that the chain weighted value for real GDP in 2011 is quite a bit different than its fixed base year counterpart. The reasons are two-fold: (i) we are further away from the base year and that creates a bias in the fixed base year method and (ii) the production of apples in 2011 slowed (only 3 apples compared to 4 in 2010), but the price increased by a large amount (from $4.50 to $6).

1.2 **Price indices**

1.2.1 **Sneak peek**

**Summary**

It is not possible to discuss the computation of real GDP without discussing the aggregate price level in the economy. There are numerous ways to measure the aggregate price level in the economy, and by extension the rate of change of the price level. Economists refer to the rate of change of the price level as the inflation rate.

There are many ways to measure the inflation rate. Between any two points in time, the price of a particular good will likely change. The price changes across all goods will never
be identical. When calculating an inflation rate for the economy, we want a single measure that tells us how the prices have changed over time. One of the possible measures could be the change in price for Fuji apples, while a different possible measure could be the change in price for 4-door Nissan sedans. In order to get a picture of the change across all goods in the economy, we must construct a bundle of goods that we wish to consider and then assign weights to each good in the bundle. The inflation rate can then tell us how much the price changed for this particular bundle of goods.

The aggregate price level is measured by a single number that is referred to as a price index. With numerous price indices, it is important to understand how they differ and what macroeconomic factors have a relatively stronger effect on each. One price index was previously introduced, namely the GDP deflator. This section will focus on the most popular price index in the popular press, the consumer price index (CPI).

Main takeaways

After completing this section, you will be able to answer the following questions:

- What are the leading price indices?
- How is CPI determined?
- Given data in a 2-good economy, what are the steps required to compute CPI?

1.2.2 Background

The GDP deflator, introduced in the previous section, measures the average ratio of the prices for all goods and services used in the computation of GDP. This is certainly a measure of the inflation rate. Algebraically, the GDP deflator is equal to 1 plus the inflation rate. The GDP deflator can be subdivided into price changes caused by the components of GDP (expenditure definition). One common price index is PCE, personal consumption expenditure, which is a version of the GDP deflator applied only to the household consumption component of GDP.

Any price index provides a single number for the aggregate price level by constructing a bundle of goods. The bundle of goods used in the GDP deflator changes over time. The bundle in 2009 is the final goods and services produced in 2009, while the bundle in 2010 is the final goods and services produced in 2010. These bundles include many goods and services that are produced and purchased by firms, and might not be a good measure of the price changes that are relevant for a typical consumer.
1.2. **PRICE INDICES**

**Consumer price index**

For that reason, a second measure of inflation is derived from the consumer price index, or CPI for short. The CPI data is collected by the Bureau of Labor Statistics. This government agency sends its employees out into major metropolitan areas around the country to find the current prices for a representative basket of goods and services. These include specific items such as food, clothing, housing, and fuel. The base period for CPI is 1983. The CPI is equal to the cost of the basket of goods and services today divided by the cost of the basket in 1983, multiplied by 100:

\[
\text{CPI} = \frac{\text{basket cost today}}{\text{basket cost in 1983}} \times 100.
\]

By definition, the CPI equals 100 in 1983.

The inflation rate between two consecutive years (current year and last year) is defined by:

\[
\text{Inflation rate} = \left( \frac{\text{CPI (current year)}}{\text{CPI (last year)}} - 1 \right) \times 100.
\]

**Other price indices**

Another price index (one favored by the Federal Reserve) is the personal consumption expenditures, or PCE. This measures the consumer prices from the household consumption component of GDP. The key difference between the CPI and the PCE is that the CPI is a measure of the prices being charged in the stores, while the PCE is a measure of the actual prices that consumers are paying. Consider a scenario in which consumers purchase equal amounts of ketchup and mustard in 2014. Suppose that the price of ketchup doubles in 2015, but the price of mustard stays the same. Logically, consumers will entirely substitute from ketchup purchases to mustard purchases. The PCE has not changed, because the total amount spent has remained the same. However, the CPI has increased. Ketchup was one of the goods in the basket of goods for 2014, so therefore an increase in the price of ketchup in 2015 (even if no one buys it) will cause the CPI to increase.

As food and energy are typically the most volatile components of consumption expenditures, another price index often used is the PCE excluding food and energy. Energy includes petrol for automobiles and electricity or natural gas required to heat or cool a home.

If the policy question is more interested in the effects of inflation on firms, we can use the producer price index, or PPI. The exact same collection approach is used with PPI as
is used with CPI, except now the prices that are gathered are the prices for inputs into the production processes of firms.

Finally, as we will see in models with monetary policy, the inflation expectations of households and firms play a key role in determining the actual inflation that arises in the current period. While it is typically more accurate to backward engineer an individual’s inflation expectations based upon their investment decisions on the bond markets, another possible method to gather this expectation data is to conduct a survey. The University of Michigan conducts a survey of the inflation expectations of households and this can be seen as another price index.

1.2.3 Measurement example

The following example will calculate the CPI values. The data to be considered is identical to Section 1:

<table>
<thead>
<tr>
<th>Year</th>
<th>Quantities</th>
<th>Prices</th>
</tr>
</thead>
<tbody>
<tr>
<td>2009</td>
<td>Apples: 3</td>
<td>$4</td>
</tr>
<tr>
<td></td>
<td>Bananas: 2</td>
<td>$3</td>
</tr>
<tr>
<td>2010</td>
<td>Apples: 4</td>
<td>$4.50</td>
</tr>
<tr>
<td></td>
<td>Bananas: 2.5</td>
<td>$3.25</td>
</tr>
<tr>
<td>2011</td>
<td>Apples: 3</td>
<td>$6</td>
</tr>
<tr>
<td></td>
<td>Bananas: 3</td>
<td>$3.75</td>
</tr>
</tbody>
</table>

Let 2009 be the base year for CPI. This means that the CPI value in 2009 is equal to 100.

2010

The cost of purchasing the 2009 quantities at 2009 prices is given by:

\[ 3(\$4) + 2(\$3) = \$18. \] (1.33)
1.3. LABOR MARKETS

The cost of purchasing the 2009 quantities at 2010 prices is given by:

$$3(\$4.50) + 2(\$3.25) = \$20.$$  \hspace{1cm} (1.34)

This implies that the CPI value in 2010 is given by:

$$\text{CPI (2010)} = \frac{\$20}{\$18} \times 100 = 111.11.$$  \hspace{1cm} (1.35)

The inflation rate between 2009 and 2010 (using the CPI data) is equal to:

$$\text{Inflation rate} = \left( \frac{111.11}{100} - 1 \right) \times 100 = 11.11\%.$$  \hspace{1cm} (1.36)

2011

The cost of purchasing the 2009 quantities at 2009 prices is given by $18 (from above). The cost of purchasing the 2009 quantities at 2011 prices is given by:

$$3(\$6) + 2(\$3.75) = \$25.5.$$  \hspace{1cm} (1.37)

This implies that the CPI value in 2011 is given by:

$$\text{CPI (2011)} = \frac{\$25.5}{\$18} \times 100 = 141.67.$$  \hspace{1cm} (1.38)

The inflation rate between 2010 and 2011 (using the CPI data) is equal to:

$$\text{Inflation rate} = \left( \frac{141.67}{111.11} - 1 \right) \times 100 = 27.5\%.$$  \hspace{1cm} (1.39)

1.3 Labor markets

1.3.1 Sneak peek

Summary

The next market that we analyze is labor markets. Rather than units of currency or units of physical goods produced, the labor market is typically measured in numbers of people working or numbers of total hours worked. In order to standardize the measures of the labor
market, these are typically expressed as rates. For instance, the unemployment rate is a fraction that relates the number of unemployed people to the total size of the labor force. A second popular measure is the labor force participation rate, which is a fraction relating the size of the labor force to the total working age population.

**Main takeaways**

After completing this section, you will be able to answer the following questions:

- What are the common measures used to gauge the health of the labor markets?
- How is the unemployment rate determined?
- How is the labor force participation rate determined?

### 1.3.2 Background

If economic growth is priority 1 for the Federal Reserve, then low unemployment is priority 1(b). The unemployment rate is computed from a survey of about 60,000 households. There are three categories into which any adult (age 16 and older) can fall: (i) employed, (ii) unemployed, and (iii) not in the labor force. "Unemployed" refers to anyone that is out of work, but actively searching for employment (search effort made within the past 4 weeks). "Not in the labor force" refers to students, retired persons, and anyone that is not actively searching for employment.

The unemployment rate is calculated as follows:

\[
\text{Unemployment Rate} = \frac{\text{Number of Unemployed}}{\text{Number of Employed} + \text{Unemployed}} \times 100. \tag{1.40}
\]

Another statistic is the labor force participation rate, which is computed as:

\[
\text{Labor Force Participation Rate} = \frac{\text{Number of Employed} + \text{Unemployed}}{\text{Total Number of Adults}} \times 100. \tag{1.41}
\]

This data is collected and compiled by the US Department of Labor on a monthly basis.

If we look at the data on the unemployment rate beginning in 1948, the lowest quarterly average in the US is equal to 2.57%. Economists recognize that some unemployment is inevitable as workers switch between jobs. Job switching is beneficial for the economy as it matches workers and firms that are most productive together. Economists believe that the
natural rate of unemployment is equal to roughly 4%. This can be viewed as an ideal economic situation. While there are still a number of unemployed persons when the unemployment rate equals 4%, this is desirable for the economy as a whole as it allows firms to grow their business and expand by hiring new workers.

### 1.3.3 Measurement example

The following data was available from the US Department of Labor in September 2016 (numbers in thousands):

<table>
<thead>
<tr>
<th>Category</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Civilian noninstitutional population</td>
<td>254,091</td>
</tr>
<tr>
<td>Civilian labor force</td>
<td>159,907</td>
</tr>
<tr>
<td>Employed</td>
<td>151,968</td>
</tr>
<tr>
<td>Unemployed</td>
<td>7,939</td>
</tr>
<tr>
<td>Not in labor force</td>
<td>94,184</td>
</tr>
</tbody>
</table>

From the table, the unemployment rate for September 2016 (using our formula above) would be:

\[
\text{Unemployment Rate} = \frac{\text{Number of Unemployed}}{\text{Number of Employed} + \text{Unemployed}} \times 100
\]

\[
= \frac{7,939}{151,968 + 7,939} \times 100 = 5.0\%.
\]

The labor force participation rate would be:

\[
\text{Labor Force Participation Rate} = \frac{\text{Number of Employed} + \text{Unemployed}}{\text{Total Number of Adults}} \times 100
\]

\[
= \frac{151,968 + 7,939}{254,091} \times 100 = 62.9\%.
\]

### 1.4 Money markets

#### 1.4.1 Sneak peek

**Summary**

After product markets and labor markets, we are now ready to analyze money markets. Money has many purposes in the economy, including as a store of value and a medium
of exchange. For both of these reasons, agents in the economy may seek to exchange one currency for another. These markets are referred to as foreign exchange markets (or FX markets).

Each market, including a foreign exchange market, has a price. In a foreign exchange market, the exchange rate between the two currencies serves as the price. The exchange rate adjusts so that currency demand equals currency supply. The exchange rate is usually specified in terms of a base currency. The exchange rate between a 2nd currency and the base currency is the number of units of the 2nd currency that must be traded for every one unit of the base currency.

**Notation**

The variables to be introduced in this section are given in the following table:

\[
\begin{align*}
M & \quad \text{money supply} \\
v & \quad \text{velocity of money} \\
P & \quad \text{price level} \\
Y & \quad \text{real GDP (or output)} \\
\xi & \quad \text{exchange rate}
\end{align*}
\]

**Main takeaways**

After completing this section, you will be able to answer the following questions:

- What are the leading measures of money supply?
- What does the Quantity Theory of Money predict about the relation between the money supply and the price level?
- What is an exchange rate and what does it mean if a currency appreciates (or depreciates)?
- What does the theory of Purchasing Power Parity (and the Law of One Price) predict?
- What does the term "exchange rate pass-through" refer to?
- Given data in a 2-good economy, what are the steps required to analyze the magnitudes of exchange rate fluctuations?
1.4. MONEY MARKETS

1.4.2 Background

Money

Money has a long and complicated history, and in many ways the history of money is the history of macroeconomics as a whole. Before discussing international money markets (foreign exchange markets), we begin with a look at monetary data in a closed economy.

The Federal Reserve uses three main definitions of money. The first is the M1 money stock. M1 (for short) is the most liquid measure of the money supply and includes all physical money, such as coins and currency, as well as demand deposits and checking accounts. This totals all money that is readily available to be spent in the economy.

As the nature of money has changed and its role as a medium of exchange has changed, so too has the definition of money. The second definition of money is an extension of the first and is called the M2 money stock. M2 (for short) is a less liquid measure of the money supply and includes M1 plus savings deposits and money market mutual funds. M2 includes a less liquid set of financial assets that are typically held by households.

A more recent measure used to confront the realities of the 2007-2008 financial crisis is the St. Louis Adjusted Monetary Base. The monetary base is a little bit different than the aforementioned monetary stocks as it includes the role of the intra-banking sector (particularly the deposits by commercial banks at the Federal Reserve). The St. Louis Adjusted Monetary Base includes currency plus deposits held by member banks at Federal Reserve Banks. It is, in essence, a measure of the total liabilities of the monetary system. The use of the adjective ‘adjusted’ refers to the fact that adjustments are made whenever the legally mandated reserve requirements are changed. The monetary base seeks to measure the deposits held by the Federal Reserve system above and beyond what member banks are required to hold to satisfy reserve requirements.

An important property of money is its velocity, which is defined as the rate at which money circulates in the economy. The Quantity Theory of Money postulates that a country’s money supply and its price level are directly related. Since it is not possible to directly measure the velocity of money, as it is infeasible to track every unit of money in the economic system, velocity is determined to satisfy the following economic identity (which is commonly referred to, albeit with a slight abuse of the notion of a Theory, as the Quantity Theory of Money):

\[ Mv = PY. \]

Here, \( M \) is the money supply, as measured according to either the M1 or M2 notion of the
money stock, \( v \) is the velocity, \( P \) is the price level, and \( Y \) is the real GDP (or output) of the economy. The term \( PY \) is the nominal GDP as it is the value of the goods and services sold in the economy.

**Exchange rates**

We are now ready to begin our analysis of exchange rates in foreign exchange markets. Consider two countries, country A and country B, each with its own currency. Let country A be the base country. The exchange rate, typically denoted as \( \xi \), is defined as the number of units of currency B for every one unit of currency A.

The exchange rate is bilateral, meaning that one unit of currency B is equal to \( \frac{1}{\xi} \) units of currency A. The exchange rate is the price at which money is traded in foreign exchange markets.

To understand the dynamics involving international markets, it is essential to keep track of the dynamics of exchange rates. Beginning in period \( t \), the exchange rate in the following period can either go up \( (\xi(t + 1) > \xi(t)) \), it can go down \( (\xi(t + 1) < \xi(t)) \), or it can stay the same \( (\xi(t + 1) = \xi(t)) \).

If \( \xi(t + 1) > \xi(t) \), the country A currency has appreciated (relative to currency B) and the country B currency has depreciated (relative to currency A). If the country for comparison is understood, then the phrase ‘relative to currency such and such’ is omitted. But it is important to bear in mind that any movements in the exchange rates always involve a relation between two currencies. The country A currency has become stronger and the country B currency has become weaker. Why is this? One unit of currency A can be traded for more units of currency B in period \( t + 1 \) compared to period \( t \). On the opposite side, one unit of currency B can be traded for fewer units of currency A in period \( t + 1 \) compared to period \( t \).

In such a scenario, what types of agents in the economy benefit? The importers in country A and the exporters in country B benefit. Consider the importers in country A. This includes any households and firms that purchase goods from country B. This could even include residents of country A taking a vacation in country B. The goods purchased from country B are priced in country B currency. Suppose that the market is such that the country B good sells for 100 units of currency B. To find the optimal demand by the residents of country A, we have to translate this into currency A. Since \( \xi(t + 1) > \xi(t) \), then \( \frac{100}{\xi(t+1)} < \frac{100}{\xi(t)} \). This means that the price in terms of currency A is strictly lower in period \( t + 1 \) compared to period \( t \). This is advantageous for country A consumers.
In the same manner, since the country B good is effectively cheaper in country A after
the depreciation of currency B, it is easier for exporting firms in country B to sell more units
of the good. This means that exporters in country B benefit.

The two types of agents that are harmed by an appreciation of currency A (equivalently
a depreciation of currency B) are the exporters in country A and the importers in country
B. The logic is the same as above, but working in the opposite direction.

Purchasing power parity

A theory about the relation between exchange rate movements and prices in the two countries
is called either purchasing power parity (PPP) or the law of one price (LOP). PPP refers
to the aggregate price level in each of the countries, where the aggregate price level may
refer to any of the price indices previously introduced. LOP, on the other hand, refers to
the price of a particular good. So PPP presents a theory for the entire economy, while LOP
refers to one particular good. The predictions of both theories are true when the foreign
exchange markets do not have any transaction frictions and the countries do not have any
pricing frictions. In practice, neither theory will hold with 100% accuracy, but economists
are generally interested in whether or not the theories can provide a useful approximation
for the relation between the exchange rates and the prices in the two countries.

We will introduce the details for PPP, where LOP is identical but only refers to the price
for a particular good. The theory of PPP states that the exchange rate is equal to the ratio
of the price levels in the two countries. Consider the exchange rate between countries A and
B. If the aggregate price level in country A is $p(A)$ and the aggregate price level in country
B is $p(B)$, then PPP predicts that that

$$\xi = \frac{p(B)}{p(A)}.$$

The theory makes perfect sense if you can envision a frictionless economy. Recall that the
exchange rate is defined as the number of units of currency B for every one unit of currency
A. An equilibrium in this frictionless setting requires that the real prices for commodities
must be the same in both countries. If the exchange rate $\xi = 2$, then consumers receive twice
as many units of currency B for every unit of currency A. They can have twice as many units
of currency B, but if the price level is twice as high in country B, then the consumer is able
to purchase the same commodities in country B as in country A. This is the rationale behind
PPP: equal purchasing power. The nominal income is higher in country B (by a factor of
but the real income is the same in both countries.

A concept called the real exchange rate looks at the ratio of the nominal exchange rate relative to the ratio of the price levels in the two countries. The real exchange rate measures how much the real price for commodities differs across the two countries. The theory of PPP states that the real exchange rate equals 1.

**Exchange rate pass-through**

The concept of exchange rate pass-through measures how close or far the actual price-setting mechanism is to the theory of LOP. If currency A appreciates, then the effective price of exports from country B to country A decreases. The change in effective price is the surplus created by the change in the exchange rate. This surplus can be split between the exporting firm and the importing consumers. If the surplus goes entirely to the consumers, then the firm does not change the price it charges in its own currency, meaning that all benefits of the exchange rate change are reaped by the consumers. If the surplus goes entirely to the firm, then the firm will increase the price it charges so that country A consumers are paying the same price as before the exchange rate change. In reality, the surplus is typically shared between the two parties.

Exchange rate pass-through measures the ratio of the observed price change (in the import country) to the exchange rate change. If pass-through equals 1, then LOP holds and the entire surplus is collected by the consumers. If pass-through is strictly less than 1, then LOP fails. If pass-through equals 0, then the entire surplus is collected by the exporting firm.

Pass-through works on both sides of the bilateral transaction. An appreciation of currency A means that the country A importers and country B exporters have a positive surplus to split. The question is how much of the effective price drop will the exporting firms allow the consumers to receive (how to split a surplus)? When currency A appreciates, country A exporters and country B importers have a deficit that must be borne. The question in this direction is how much of the effective price increase will the exporting firms require the consumers to bear (how to share a deficit)?

**1.4.3 Measurement example**

Consider a 2-country economy in which country A produces automobiles and country B produces jeans. The exchange rate is defined as the number of units of currency B for every
one unit of currency A.

Suppose the price of automobiles is equal to 1,000 units of currency A and the price of jeans is equal to 10 units of currency B. When the exchange rate is equal to $\xi(t) = 1.5$, what price do country B consumers pay for automobiles (in currency B) and what price do country A consumers pay for jeans (in currency A)?

Recall that

$$\xi(t) = \frac{\# \text{ units of currency B}}{1 \text{ unit of currency A}}.$$  

The key to these equations is simply to get the units correct. First, we want to translate 1,000 units of currency A into units of currency B:

$$\text{automobile price (currency B)} = (1,000 \text{ current A}) \times \frac{\# \text{ units of currency B}}{1 \text{ unit of currency A}} = 1000 \times 1.5 = 1500.$$ 

Next, we want to translate 10 units of currency B into units of currency A:

$$\text{jeans price (currency A)} = (10 \text{ current B}) \div \frac{\# \text{ units of currency B}}{1 \text{ unit of currency A}} = \frac{10}{1.5} = 6.67.$$ 

Now we want to consider two exchange rate movements: (i) the exchange rate increases by 10% from period $t$ to period $t+1$ and (ii) the exchange rate decreases by 10% from period $t$ to period $t+1$. Case (i) refers to $\xi(t+1) = 1.65$ and case (ii) refers to $\xi(t+1) = \frac{1.5}{1.1} = 1.36$, but it is usually easier to work with the % changes. In case (i), currency A has appreciated and currency B has depreciated (both by 10%), while in case (ii), currency A has depreciated and currency B has appreciated (both by 10%).

In case (i), suppose the exchange rate pass-through is equal to 100%. What is the new automobile price in currency B and what is the new jeans price in currency A? Currency A appreciation means that country A importers (of jeans) are better off and country B importers (of automobiles) are worse off.

With 100% exchange rate pass-through, LOP holds and the entire exchange rate movement is captured in the new prices. This means that the automobile price in currency B will
increase by 10% and the jeans price in currency A will decrease by 10%:

\[
\text{automobile price (currency B)} = 1.10 \times (1500) = 1650.
\]

\[
\text{jeans price (currency A)} = \frac{6.67}{1.10} = 6.06.
\]

It is easy to verify that these prices can be determined by using the equation for the new exchange rate:

\[
\text{automobile price (currency B)} = 1000 \times 1.65 = 1650.
\]

\[
\text{jeans price (currency A)} = \frac{10}{1.65} = 6.06.
\]

Now consider case (ii) and assume that the exchange rate pass-through is only 60%.
Currency A depreciation means that country A exporters (of automobiles) are better off and country B exporters (of jeans) are worse off.

Since the pass-through is not complete, then we cannot determine the price based upon the new exchange rate of \(\xi(t + 1) = 1.36\). This is because automobile producers in country A are raising the price they charge in currency A (to extract some of the surplus from the exchange rate change) and jeans producers in country B are lowering the price they charge in currency B (to absorb some of the deficit). The first thing to recognize is that a currency A depreciation means that the automobile price in currency B must decrease and the jeans price in currency A must increase. The amount that each of these prices change is equal to 60% of the total exchange rate change of 10%:

\[
\text{automobile price (currency B)} = (60\%) \left( \frac{1500}{1.10} \right) + (40\%) \times (1500) = 1418.2.
\]

\[
\text{jeans price (currency A)} = (60\%) \times (1.10)(6.67) + (40\%)(6.67) = 7.067.
\]

For the firms in country A, they were originally charging a price of 1000 units of currency A and are now charging \(\frac{1418.2}{1.36} = 1040\) units of currency A. This represents a 4% increase in the price that they receive (in their home currency). If the firms wished to extract the entire surplus, then they would charge \(\frac{1500}{1.36} = 1100\) units of currency A. Since the country B consumers are only paying 1040 units of currency A, they have a 6% decrease in the price that they would otherwise have to pay (relative to the original price of 1000 units of currency A).
In a similar fashion, for the firms in country B, they were originally charging 10 units of currency B and are now charging 7.067(1.36) = 9.636 units of currency B. If the firms were to bear the entire burden of the exchange rate movement, then would charge 6.67(1.36) = 9.091 units of currency B. The country A consumers are paying 9.636 units of currency B, which is 6% higher than the price of 9.091 units of currency B if the firm had born the entire cost of the exchange rate change. The firms are losing 10 – 9.636 = 0.3636, which equals 4% of the price of 9.091 units of currency B if the firm had born the entire cost of the exchange rate change.

In both cases, the total exchange rate change equals 10%, with 60% of that total change going to consumers and 40% of that total change going to firms. This means that consumers experience a 6% change and firms experience a 4% change.

1.5 Bond markets

1.5.1 Sneak peek

Summary

One role of money is as a store of value. This is evident from the fact that investors use the foreign exchange markets as an investment opportunity, so money plays the role of an asset. Continuing with the theme of financial assets, this section introduces bonds. Governments in their dual roles of establishing peace through the creation and enforcement of laws (especially property rights) and providing public goods must collect revenue. This is typically done through taxation. Most federal governments are not required to balance their budget. If government spending exceeds revenue, the government has a deficit. The government must fund the deficit by borrowing. The government borrows by issuing (selling) government bonds (treasury bills or T-bills). A bond is an IOU from the government to the bond holder promising a fixed repayment at a later date. The total amount of all government borrowing at any point in time is referred to as the government debt. To finance the debt, the government must redeem the promises made to all bond holders, both the initial amount borrowed plus the interest payment.

The government issues many forms of bonds. A bond’s maturity is the number of years until it can be redeemed for the full promised repayment value. In practice, the maturities for treasury bills vary from very short (1 month) to very long (30 years).
28

1. MACROECONOMIC ACCOUNTING

Notation

The variables to be introduced in this section are given in the following table:

- \( n \) nominal interest rate
- \( r \) real interest rate
- \( \pi \) inflation rate

Main takeaways

After completing this section, you will be able to answer the following questions:

- What is the relation between the return, the payout, and the price (of any asset)?
- What does the Fisher equation imply about the relation between the nominal and the real interest rate?
- What does the uncovered interest rate parity condition imply about the relation between nominal interest rates across countries?
- Given data in a sample economy, what are the steps required to compute the price of long-term bonds?

1.5.2 Background

Price, payout, return

Governments, in order to raise revenue for government spending, will borrow by issuing government bonds. In the US, government bonds are called US Treasury bonds. The bonds are nominal assets, meaning that their payouts in the future are specified in terms of US dollars. Most bonds are nominally risk-free, meaning that the nominal payouts in the future are identical regardless of the state of the economy. Inflation-indexed bonds (less common) offer risk-free real payouts, meaning that the nominal payouts adjust in response to the inflation rate to leave the real payouts unchanged. Bonds differ in terms of the amount of time before they mature.

The face value of the bond is the risk-free payout that is collected in the period of maturation. Suppose that a bond matures in 2020. In each year from now until 2020, the bond has a coupon payout. The bond can be traded in any year prior to 2020. The bond
has value in these interim periods, because whomever owns the bond in 2020 is entitled to collect the face value.

The price of a bond is the amount of dollars that must be given up for 1 unit of the bond. The payout of a bond is the amount of dollars that are paid out in the following period for each 1 unit of the bond. The return of a bond is defined as the payout divided by the price:

\[
\text{Return} = \frac{\text{Payout}}{\text{Price}}. \tag{1.44}
\]

In the jargon of bonds, the term 'yield' is used to refer to the interest earned on a bond, and is defined by:

\[1 + \text{Yield} = \text{Return}. \tag{1.45}\]

Also in the jargon of bonds, a basis point is 1/100 of 1%, meaning that if the yield increases by 0.5%, then it has increased by 50 basis points.

Such equations are straightforward for a bond that matures in 1 year. For bonds with longer maturities, let's address the equations in further detail. Formally, the yield on a bond is equal to the interest rate that its holder would receive every year from now until the bond matures. Consider a bond with $100 face value and a coupon payout of $1 every year. The face value means that the bond can be redeemed for $100 when it matures. The coupon payout is collected by the bond holder in every year until the bond matures. If the price of a 2-year bond is $95, then the yield must satisfy the following equation:

\[95 = \frac{1}{1 + \text{Yield}} + \frac{101}{(1 + \text{Yield})^2}. \tag{1.46}\]

Notice that the coupon payout of 1 is made every year (both the interim year before maturation and the year of maturation). In this example, Yield = 3.6%.

For bonds with maturities longer than 1 year, the simplest way to think of the payout is in terms of a new bond. Consider a 2-year bond traded in 2016. The payout of the bond in 2017 is equal to the price of a 1-year bond in 2017. The price of a 1-year bond is given by

\[\text{Price} = \frac{\text{Payout}}{\text{Return}} = \frac{101}{1 + \text{Yield}} = 97.5, \tag{1.47}\]

since the 1-year bond has a payout of $101 (face value plus coupon) and the yield was already found to be 3.6%.

The 2-year bond payout is equal to the sum of the coupon ($1) and the price of a 1-year
bond ($97.5):\)
\[
Payout = \$1 + \$97.5. \tag{1.48}
\]
This means that we have an expression for the 2-year bond return
\[
Return = \frac{Payout}{Price} = \frac{\$98.50}{\$95} = 1.036. \tag{1.49}
\]
The equation
\[
1 + \text{Yield} = \text{Return} \tag{1.50}
\]
implies that the yield is equal to 3.6%. This is the same value that we found by solving the following equation directly:
\[
\$95 = \frac{\$1}{1 + \text{Yield}} + \frac{\$101}{(1 + \text{Yield})^2}. \tag{1.51}
\]
I have just shown that there are two equivalent ways to determine a bond yield when the maturity is longer than 1 year. My preference is for the second method:
\[
\text{Return} = \frac{\text{Payout}}{\text{Price}} \quad 1 + \text{Yield} = \text{Return}. \tag{1.52}
\]

**Yield curve**

As mentioned, bonds are issued with a variety of maturation dates, typically ranging from 3 months to 30 years in the future. The relationship among the interest rates (or yields) for bonds of varying maturities is called the term structure of interest rates. The yield curve is a plot of the bond yields (on the y-axis) as a function of the time to maturity (on the x-axis). A "typical" yield curve is strictly increasing and strictly concave, but importantly, the yield curve will have a different shape right before a recession takes place.

The yield of a bond is identical to the nominal interest rate for that bond. This is because a bond typically specifies payouts in terms of units of currency, and is not indexed for inflation. A less common type of bond is an inflation-indexed bond in which the payout is adjusted for inflation. In essence, the payout is specified in units of a commodity equivalent. An inflation-indexed bond is also called a real bond. The yield of a real bond is the real interest rate.
1.5. BOND MARKETS

Fisher equation

The Fisher equation provides a relation between the nominal interest rate and the real interest rate. If $\pi$ is the inflation rate, $n$ is the nominal interest rate, and $r$ is the real interest rate, the Fisher equation states that:

$$1 + n = (1 + r)(1 + \pi).$$

Multiplying the right-hand side, and since the product $r\pi \approx 0$, the Fisher equation is more commonly presented as the following approximation:

$$n \approx r + \pi.$$

Uncovered interest rate parity

Finally, the previous section on money introduced the notion of exchange rates. Uncovered interest rate parity (UIP) is a theoretical condition that provides a relation between the nominal interest rates in two countries in a setting with a single short-term bond available for trade. Consider time periods $t$ and $t + 1$. Denote $n_A(t)$ as the nominal interest rate in country A (for a bond purchased in $t$ and paying out in $t + 1$) and $n_B(t)$ as the nominal interest rate in country B. If the exchange rate in period $t$ is $\xi(t)$ (i.e., the number of units of currency B that can be traded for every one unit of currency A), the uncovered interest rate parity condition is given by:

$$1 + n_B(t) = \frac{E_t(\xi(t + 1))}{\xi(t)}(1 + n_A(t)).$$

Consider the effects if the expected exchange rate is higher than the current exchange rate ($\frac{E_t(\xi(t + 1))}{\xi(t)} > 1$). In such a case, the currency for country A has appreciated (stronger currency for country A). UIP predicts that $n_A(t) < n_B(t)$. The nominal interest rate in country A is lower than the nominal interest rate of country B. Consider that the same investment in the country B bond markets (with payouts in units of currency B) now yields fewer units of currency A after the exchange rate change. This means that the nominal interest rate of country B must increase to compensate for the exchange rate change. Remember that this theory is only valid in settings without any frictions on the foreign exchange markets.

Under an exchange rate peg, the expected exchange rate must be equal to the current exchange rate. In such a scenario, the nominal interest rates of both countries must be
1.5.3 Measurement example

Consider a 5-year bond with face value equal to $100 and a coupon payout of $1 every year. Suppose that the price of the 5-year bond is equal to $89.

First, use the following equation to calculate the yield for the 5-year bond:

$$89 = \frac{1}{1 + \text{Yield}} + \frac{1}{(1 + \text{Yield})^2} + \frac{1}{(1 + \text{Yield})^3} + \frac{1}{(1 + \text{Yield})^4} + \frac{101}{(1 + \text{Yield})^5}. \quad (1.53)$$

The easiest way to solve this equation for the yield is to use the Goal Seek function on Excel. The answer is that the yield is equal to 3.4%.

Once we know the value for the yield, then we can find the value for the price of a 1-year bond, a 2-year bond, a 3-year bond, and a 4-year bond.

The price of a 1-year bond is given by

$$\text{Price} = \frac{\text{Payout}}{\text{Return}} = \frac{101}{1 + \text{Yield}} = 97.6, \quad (1.54)$$

since the 1-year bond has a payout of $101 (face value plus coupon) and the yield was already found to be 3.4%.

The 2-year bond payout is equal to the sum of the coupon ($1) and the price of a 1-year bond ($97.6):

$$\text{Payout} = 1 + 97.6. \quad (1.55)$$

This means that price of a 2-year bond is given by

$$\text{Price} = \frac{\text{Payout}}{\text{Return}} = \frac{98.6}{1 + \text{Yield}} = 95.3. \quad (1.56)$$

The 3-year bond payout is equal to the sum of the coupon ($1) and the price of a 2-year bond ($95.3):

$$\text{Payout} = 1 + 95.3. \quad (1.57)$$
This means that price of a 3-year bond is given by

\[
\text{Price} = \frac{\text{Payout}}{\text{Return}} = \frac{\$96.3}{1 + \text{Yield}} = \$93.1. \quad (1.58)
\]

The 4-year bond payout is equal to the sum of the coupon ($1) and the price of a 3-year bond ($93.1):

\[
\text{Payout} = \$1 + \$93.1. \quad (1.59)
\]

This means that price of a 4-year bond is given by

\[
\text{Price} = \frac{\text{Payout}}{\text{Return}} = \frac{\$94.1}{1 + \text{Yield}} = \$91.0. \quad (1.60)
\]

The 5-year bond payout is equal to the sum of the coupon ($1) and the price of a 4-year bond ($91.0):

\[
\text{Payout} = \$1 + \$91.0. \quad (1.61)
\]

As a verification, we can continue and find the price of a 5-year bond. The price is equal to

\[
\text{Price} = \frac{\text{Payout}}{\text{Return}} = \frac{\$92.0}{1 + \text{Yield}} = \$89.0. \quad (1.62)
\]

In this particular example, the yield is held fixed. The yield for a bond of any maturity is equal to 3.4%. The yield curve would be a flat line equal to 3.4% for all maturities.

1.6 Stock markets

1.6.1 Sneak peek

Summary

The final market that we will analyze will be the stock market. Compared to bonds, stocks are more complicated financial assets. First, bond payouts are typically risk-free payouts, whereas the price and dividends of stocks vary in response to the shocks hitting the economy at the time of payment. Second, bonds are finite-lived assets as they have a finite number of periods until maturation. Stocks, by contrast, are infinite-lived assets that investors can use to receive payouts in the infinite future.

As with prices in commodity markets, there are a great number of prices in the stock
markets. To obtain a measure that represents the health of a particular stock market, a stock market index is created. In the United States, the two major stock markets (exchanges) are the New York Stock Exchange (NYSE) and NASDAQ. The two major stock market indices are the Dow Jones Industrial Average (Dow, Dow Jones, or DJIA) and the Standard & Poor’s 500 (S&P 500). Both are determined based upon stock prices from both NYSE and NASDAQ.

**Main takeaways**

After completing this section, you will be able to answer the following questions:

- What is an option and how is it related to a stock?
- What are the two types of stock market indices (one used for the Dow Jones Industrial Average and one used for the Standard & Poor’s 500) and how are each computed?
- What firms are included in each of the aforementioned stock market indices?
- If a stock market index changes its composition, how does the stock market index formula change?

### 1.6.2 Background

A stock represents ownership in a firm. A stock will periodically pay out dividends. The dividends are the profits of the firm. Consequently, stocks are real assets, in that their payouts are determined by the amount of final goods and services that the firm is able to produce. When the firm sells these final goods and services, and after subtracting off costs, the firm earns profits that can be distributed as dividends to the stock holders. A stock is a risky asset, because its payout can change depending upon the fortunes of the firm.

A stock is a long-lived asset. That means that in any time period, the payout of a stock is equal to the sum of the following two components: (i) the value of the stock (the stock can be traded on the stock market) and (ii) the dividends that the stock pays out. Expressing the stock payouts in this manner, the stock return can be defined exactly as with bonds:

\[
\text{Return} = \frac{\text{Payout} \cdot (\text{Stock Price} + \text{Dividend})}{\text{Price}}.
\]  

(1.63)
Options

As a function of stock prices, new assets can be created. These new assets have payouts that are derived from the stock prices and are called derivative assets. A futures contract is a derivative asset such that a stock is traded for a certain price at a certain point in time in the future. The most common type of futures contracts are options. A put option is the right, but not the obligation, to sell a stock at a fixed strike price at a future date. A call option is the right, but not the obligation, to buy a stock at a fixed strike price at a future date. These contracts are called options, because the investors have the option at the specified future date to either exercise the option (sell or buy the stock at the fixed strike price) or not.

Consider the following scenario:

<table>
<thead>
<tr>
<th>Current price</th>
<th>Strike price</th>
<th>Next year price if market is...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$350</td>
<td>$400</td>
<td>$420 $395 $360</td>
</tr>
</tbody>
</table>

Suppose an investor holds a put option. The put option is the right, but not the obligation, to sell the asset at the strike price of $400. The put option is exercised when the actual price is lower than the strike price. Consider that the investor could exercise the put option (and receive $400) and then turn around and buy the stock back at the actual price on the market. Thus, the instantaneous payouts for the investor from holding a put option are:

<table>
<thead>
<tr>
<th>Put option payouts if market is...</th>
</tr>
</thead>
<tbody>
<tr>
<td>...strong ...average ...sluggish</td>
</tr>
<tr>
<td>$0 $5 $40</td>
</tr>
</tbody>
</table>

The price of the put option in the current period equals the expected value of an asset that pays out $5 if the market is average and $40 if the market is sluggish.

Suppose an investor has a call option. The call option is the right, but not the obligation, to buy the asset at the strike price of $400 (the strike prices for a call and put option are typically different, but for simplicity I set them equal). The call option is exercised when the actual price is higher than the strike price. Thus, the instantaneous payouts for the investor
Call option payouts if market is...

...strong ...average ...sluggish 

$20 $0 $0

(1.66)

The price of the call option in the current period equals the expected value of an asset that pays out $20 if the market is strong next period.

I will now discuss the two major stock market indices and how they are computed.

**Dow Jones Industrial Average**

The Dow Jones Industrial Average is a stock index that includes the 30 largest publicly traded US companies. That list changes from time to time, but currently includes the following companies:

<table>
<thead>
<tr>
<th>3M</th>
<th>ExxonMobil</th>
<th>McDonald’s</th>
<th>United Technology</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amex</td>
<td>GE</td>
<td>Merck</td>
<td>Verizon</td>
</tr>
<tr>
<td>Apple</td>
<td>Goldman Sachs</td>
<td>Microsoft</td>
<td>Visa</td>
</tr>
<tr>
<td>Boeing</td>
<td>Home Depot</td>
<td>Nike</td>
<td>Wal-Mart</td>
</tr>
<tr>
<td>Caterpillar</td>
<td>Intel</td>
<td>Pfizer</td>
<td>Walt Disney</td>
</tr>
<tr>
<td>Chevron</td>
<td>IBM</td>
<td>P&amp;G</td>
<td></td>
</tr>
<tr>
<td>Cisco</td>
<td>Johnson &amp; Johnson</td>
<td>Travelers</td>
<td></td>
</tr>
<tr>
<td>Coke</td>
<td>JPMorgan Chase</td>
<td>UnitedHealth</td>
<td></td>
</tr>
<tr>
<td>DuPont</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The Dow Jones average is an adjusted sum of the stock prices of these 30 companies. Specifically, the Dow Jones average is calculated as follows:

\[
\text{Dow Jones} = \frac{\sum_{i=1}^{30} p_i}{\text{Dow Divisor}}. 
\]

(1.68)

Here, \( p_i \) is the stock price for company \( i \in \{1, \ldots, 30\} \). The original Dow Jones average had a Dow Divisor equal to the number of companies. However, as the composition of the Dow Jones changes and stock splits are made, steps have to be taken to ensure that the Dow Jones average remains continuous. On the day that a change to the Dow Jones is made
1.6. STOCK MARKETS

(either the composition changes or one of the 30 companies issues a stock split), the new Dow Divisor is defined such that:

\[
\frac{\sum_{i=1}^{30} p_i(\text{old})}{\text{Dow Divisor (old)}} = \frac{\sum_{i=1}^{30} p_i(\text{new})}{\text{Dow Divisor (new)}}.
\]

(1.69)

Currently, the Dow Divisor is roughly 0.16, meaning that the Dow Jones average is equal to roughly 6 times the summed prices of all 30 companies.

As we can see from the formula, the Dow Jones average is a price-weighted average. A $1 increase in the stock price for a lower-priced stock has the same effect as a $1 increase in the stock price for a higher-priced stock. In terms of percent changes, if two stocks experience the same percent change in their price, the higher-priced stock would have a greater influence on the Dow Jones average.

Though the Dow Jones average purports to include the 30 largest publicly traded US companies, the actual composition of the index is selected by a committee. One of the requirements for inclusion in the index is that the company has a stock price that is somewhere in the middle of the stock prices of current companies in the index. This helps to prevent the bias from higher-priced stocks.

**Standard and Poor’s 500 (S&P 500)**

The S&P 500 (Standard and Poor’s) perhaps gives a better picture of the health of the financial sector as it looks at the 500 largest US companies. Additionally, the S&P 500 is a market-value weighted average and does not afford higher-priced stocks greater influence on the index. As with the Dow Jones, the composition of the S&P 500 is selected by committee, without strict adherence to the 500 largest US companies provision.

The market capitalization is defined as the product of the share price times the number of shares outstanding. Rather than looking at all outstanding shares, the index only focuses on those shares that are floating (or traded) in the market. This does not include shares held by company officers or the government. The S&P 500 is a float-weighted market-capitalization weighted average, meaning that the index is calculated as follows:

\[
\text{S&P 500} = \frac{\sum_{i=1}^{500} p_i Q_i}{\text{S&P 500 Divisor}}.
\]

(1.70)
As before, \( p_i \) is the stock price for company \( i \in \{1, \ldots, 500\} \). Here, \( Q_i \) is the number of freely floating shares of that company. As before, the S&P 500 Divisor must be adjusted every time a change to the composition of the index is made.

### 1.6.3 Measurement example

Consider the following table containing stock price data for 3 companies:

<table>
<thead>
<tr>
<th></th>
<th>May 11, 2016</th>
<th>May 12, 2016</th>
<th>% Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>3M</td>
<td>$40</td>
<td>$42</td>
<td>5%</td>
</tr>
<tr>
<td>Amex</td>
<td>$50</td>
<td>$52.50</td>
<td>5%</td>
</tr>
<tr>
<td>AT&amp;T</td>
<td>$60</td>
<td>$55</td>
<td>-8.3%</td>
</tr>
</tbody>
</table>

#### Original Dow composition

Suppose that Dow Jones Industrial Average includes 3M and AT&T on May 11, 2016 and the average is 500. What is the value of the Dow Divisor on May 11, 2016?

The formula is given by:

\[
500 = \frac{40 + 60}{\text{Dow Divisor}}. \tag{1.72}
\]

This means that the Dow Divisor is equal to 0.2.

For this composition of the Dow Jones average, what impact does a 1% increase in the 3M stock price have? What impact does a 1% increase in the AT&T stock price have?

If 3M increases 1%, then the Dow Jones average is given by:

\[
\text{Dow Jones} = \frac{40.4 + 60}{0.2} = 502. \tag{1.73}
\]

This is a 0.4% increase in the Dow Jones average.

If AT&T increases 1%, then the Dow Jones average is given by:

\[
\text{Dow Jones} = \frac{40 + 60.6}{0.2} = 503. \tag{1.74}
\]

This is a 0.6% increase in the Dow Jones average.
New Dow composition

Now suppose that the Dow Jones changes its composition on May 12, 2016 to include 3M and Amex (AT&T has been replaced by Amex). Suppose that continuity requires the Dow Jones average with 3M and AT&T on May 12 must be equal to the Dow Jones average with 3M and Amex on May 12. What must the new value for the Dow Divisor be equal to?

The May 12 value for the Dow Jones average under the old composition is given by:

\[
\text{Dow Jones} = \frac{42 + 55}{0.2} = 485. \tag{1.75}
\]

The formula for the new composition is given by:

\[
485 = \frac{42 + 52.50}{\text{Dow Divisor}}. \tag{1.76}
\]

This means that the new Dow Divisor is equal to 0.195.

For this composition of the Dow Jones average, what impact does a 1% increase in the 3M stock price have? What impact does a 1% increase in the Amex stock price have?

If 3M increases 1%, then the Dow Jones average is given by:

\[
\text{Dow Jones} = \frac{42.42 + 52.50}{0.195} = 487.16. \tag{1.77}
\]

This is a 0.44% increase in the Dow Jones average.

If Amex increases 1%, then the Dow Jones average is given by:

\[
\text{Dow Jones} = \frac{42 + 53.025}{0.195} = 487.69. \tag{1.78}
\]

This is a 0.56% increase in the Dow Jones average.
1. MACROECONOMIC ACCOUNTING
Bibliography


2

Mathematical Preliminaries

2.1 Matrix Algebra

2.1.1 Vectors

A vector is a finite collection of elements. The vector \( x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \) is a column vector with two elements. The column vector has 2 rows and 1 column. The vector \( p = (p_1, p_2) \) is a row vector with two elements. The row vector has 1 row and 2 columns.

Vector addition

Suppose that there are two column vectors \( x^1 = \begin{pmatrix} x^1_1 \\ x^1_2 \end{pmatrix} \) and \( x^2 = \begin{pmatrix} x^2_1 \\ x^2_2 \end{pmatrix} \), where \( x^1 \) and \( x^2 \) can be viewed as the consumption bundles of consumers 1 and 2, respectively. To add vectors, we simply add up each of the elements:

\[
x^1 + x^2 = \begin{pmatrix} x^1_1 \\ x^1_2 \end{pmatrix} + \begin{pmatrix} x^2_1 \\ x^2_2 \end{pmatrix} = \begin{pmatrix} x^1_1 + x^2_1 \\ x^1_2 + x^2_2 \end{pmatrix}.
\] (2.1)

Example 1: Total Consumption Vector

Consider an economy with two commodities: apples and bananas. Suppose consumer 1 chooses a consumption bundle consisting of \( x^1_1 = 3 \) apples and \( x^1_2 = 4 \) bananas, while consumer 2 chooses a consumption bundle consisting of \( x^2_1 = 6 \) apples and \( x^2_2 = 2 \) bananas. The consumption bundle for consumer 1 is \( x^1 = \begin{pmatrix} x^1_1 \\ x^1_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \) and the consumption bundle for consumer 2 is
\[ x^2 = \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}. \]

What is the total consumption for these two consumers? Notice, of course, that we cannot add apples and bananas together as they are different commodities, so we must have the total consumption of apples and the total consumption of bananas as separate values. The vector representation is perfect for this task. The total consumption vector is given by:

\[ x^1 + x^2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \end{pmatrix}. \] (2.2)

This means that the total consumption of apples is 9 and the total consumption of bananas is 6.

**Vector multiplication and dot product**

If we want to consider a scalar multiple of the household \( h = 1 \) consumption, we simply multiply each element by the scalar:

\[ \kappa x^1 = \kappa \begin{pmatrix} x_1^1 \\ x_2^1 \end{pmatrix} = \begin{pmatrix} \kappa x_1^1 \\ \kappa x_2^1 \end{pmatrix}. \] (2.3)

Finally, as a precursor to matrix multiplication, we can look at the dot product of two vectors. The two vectors must be compatible, meaning that the first vector in the dot product is a row vector with \( n \) elements and the second vector is a column vector with \( n \) elements (where \( n \) is any finite number).

Consider the price vector \( p = (p_1, p_2) \). In economics, price vectors are typically represented as row vectors and commodity vectors are typically represented as column vectors. Consider the commodity vector \( x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \). The dot product \( p \circ x \) is the same as matrix product \( px \). The first vector \( p \) is a row vector with 2 elements and the second vector \( x \) is a column vector with 2 elements. The two vectors are compatible for multiplication as \( p \) is a row vector with the same number of elements as the column vector \( x \). The dot product is obtained by adding together the product of the first elements in the vectors and the product of the second elements in the vectors:

\[ p \circ x = \begin{pmatrix} p_1 & p_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = p_1 x_1 + p_2 x_2. \] (2.4)
Example 2: Total Expenditure  Consider an economy with two commodities: apples and bananas. The price of an apple is \( p_1 = \$5 \) and the price of a banana is \( p_2 = \$2 \). What is the total expenditure of a consumer with consumption bundle consisting of \( x_1 = 3 \) apples and \( x_2 = 4 \) bananas?

The consumption bundle is represented as a column vector \( x = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \). The price vector is represented as a row vector \( p = (5, 2) \). The dot product \( p \cdot x \) determines the total expenditure of the consumer:

\[
p \cdot x = \begin{pmatrix} 5 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 5 \cdot 3 + 2 \cdot 4 = \$23.
\]

2.1.2 Matrices

A matrix is a vector of vectors. For instance, if the two column vectors \( x^1 = \begin{pmatrix} x^1_1 \\ x^1_2 \end{pmatrix} \) and \( x^2 = \begin{pmatrix} x^2_1 \\ x^2_2 \end{pmatrix} \) are put together we have a matrix \( X = (x^1 \ x^2) = \begin{pmatrix} x^1_1 & x^2_1 \\ x^1_2 & x^2_2 \end{pmatrix} \). The dimension of a matrix is equal to the number of rows and the number of columns in a matrix. The matrix \( X \) is a \( 2 \times 2 \) matrix. The matrix \( M = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \) is a \( 3 \times 2 \) matrix. The set of all \( 3 \times 2 \) matrices is \( \mathbb{R}^{3,2} \) and we indicate that the matrix \( M \) is an element of this set by writing \( M \in \mathbb{R}^{3,2} \).

Matrix addition and multiplication

Matrix addition occurs for two matrices of the same dimension. Matrix addition entails the adding together of each element in the two matrices. For instance, if \( N = \begin{pmatrix} g & h \\ i & j \\ k & l \end{pmatrix} \), then the matrix sum \( M + N = \begin{pmatrix} a + g & b + h \\ c + i & d + j \\ e + k & f + l \end{pmatrix} \).

A matrix can also be multiplied by a scalar. In this case, all elements of the matrix are
multiplied by that scalar. For instance, \( kM = k \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \\ ke & kf \end{pmatrix} \).

For matrix multiplication of \( A \) and \( B \), we require that the number of columns of \( A \) is equal to the number of rows of \( B \). Let \( A \) be an \( I \times J \) matrix with elements \([A]_{ij} = a_{ij}\). Let \( B \) be a \( J \times K \) matrix with elements \([B]_{jk} = b_{jk}\). The matrix product \( AB \) is an \( I \times K \) matrix with elements

\[
[AB]_{ik} = \sum_{j=1}^{J} a_{ij}b_{jk}.
\] (2.6)

**Example 3: Total Expenditures with Multiple States**  
Consider an economy with two commodities: apples and bananas. There are two possible states of the world. In the first state of the world, the fruit prices are high, while in the second state of the world, the fruit prices are low. In the high state, the price of an apple is \( p_1(H) = $5 \) and the price of a banana is \( p_2(H) = $2 \). In the low state, the price of an apple is \( p_1(L) = $2 \) and the price of a banana is \( p_2(L) = $1 \). In both states, two consumers make the same choices for the consumption bundles. Suppose consumer 1 chooses a consumption bundle consisting of \( x_1^1 = 3 \) apples and \( x_2^1 = 4 \) bananas, while consumer 2 chooses a consumption bundle consisting of \( x_1^2 = 6 \) apples and \( x_2^2 = 2 \) bananas. What is the total expenditure for each consumer in each state of the world?

To answer this question, we can write the prices in a matrix:

\[
P = \begin{pmatrix} p_1(H) & p_2(H) \\ p_1(L) & p_2(L) \end{pmatrix}.
\] (2.7)

The commodity matrix was previously given by \( X = \begin{pmatrix} x^1 & x^2 \end{pmatrix} = \begin{pmatrix} x_1^1 & x_2^1 \\ x_1^2 & x_2^2 \end{pmatrix} \). Since there are 2 consumers and 2 states of the world, there are 4 total expenditures that we must calculate: one for each consumer in each state of the world. These are found by taking the matrix
2.1. MATRIX ALGEBRA

The product

\[ PX = \begin{pmatrix} p_1(H) & p_2(H) \\ p_1(L) & p_2(L) \end{pmatrix} \begin{pmatrix} x_1^1 & x_2^1 \\ x_1^2 & x_2^2 \end{pmatrix} \]

\[ = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 6 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 5 \cdot 3 + 2 \cdot 4 & 5 \cdot 6 + 2 \cdot 2 \\ 2 \cdot 3 + 1 \cdot 4 & 2 \cdot 6 + 1 \cdot 2 \end{pmatrix} \]

\[ = \begin{pmatrix} 23 & 54 \\ 10 & 14 \end{pmatrix}. \]  

After performing the multiplication, we only have to line up the results. The amount $23$ is the expenditure by consumer 1 in the high state. The amount $34$ is the expenditure by consumer 2 in the high state. The amount $10$ is the expenditure by consumer 1 in the low state. The amount $14$ is the expenditure by consumer 2 in the low state.

Matrix rank and matrix inversion

Just because the matrix product \( AB \) is well-defined, this does not necessarily imply that the matrix product \( BA \) is well-defined. The matrix product \( AB \) is well-defined when the number of columns of \( A \) equals the number of rows of \( B \). However, this does not imply that the number of columns of \( B \) equals the number of rows of \( A \). In fact, \( AB \) and \( BA \) are only both well-defined when \( A \) and \( B \) are square matrices of the same size.

A matrix with the same number of rows and columns is referred to as a square matrix. A square matrix whose only nonzero elements are along the diagonal is referred to as a diagonal matrix. A special example of a diagonal matrix is the identity matrix in which each of the diagonal elements is equal to 1. For instance, the \( 2 \times 2 \) identity matrix is given by:

\[ I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]

\[ (2.9) \]

Given any matrix \( M = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \), the transpose is a \( 2 \times 3 \) matrix obtained by flipping the rows and column. The transpose is given by:

\[ M^T = \begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix}. \]

\[ (2.10) \]
If a square matrix $M$ is such that $M = M^T$, the matrix $M$ is said to be symmetric.

The row rank of a matrix is equal to the number of linearly independent rows. The column rank of a matrix is equal to the number of linearly independent columns. One nice property of matrices is that the row rank is always equal to the column rank, which we simply call the rank of a matrix. The rank of a matrix is always less than or equal to both the number of rows and the number of columns in the matrix.

A matrix of dimension $I \times J$ has full row rank if the rank of the matrix is equal to $I$. Since the rank must be less than or equal to the number of columns, this requires that $I \leq J$. The matrix $P = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ has full row rank if the equation

\[(v_1, v_2) \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}\]

(2.11)

implies that both $v_1 = 0$ and $v_2 = 0$. In the matrix product

\[(v_1, v_2) \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix},\]

(2.12)

we have the product of a $1 \times 2$ matrix (equivalently called a row vector) and a $2 \times 3$ matrix, resulting in a $1 \times 3$ matrix. The product is equal to:

\[
\begin{pmatrix} v_1 + v_2d, & v_1b + v_2e, & v_1c + v_2f \end{pmatrix}.
\]

(2.13)

The matrix $P$ has full rank if the equations

\[
\begin{pmatrix} v_1 + v_2d, & v_1b + v_2e, & v_1c + v_2f \end{pmatrix} = \begin{pmatrix} 0, & 0, & 0 \end{pmatrix}
\]

(2.14)

are only satisfied when $(v_1, v_2) = (0, 0)$.

If a square matrix has full rank, it is said to be invertible. Consider the matrix $S = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. The inverse matrix $S^{-1}$ is such that both matrix products $SS^{-1} = I_2$ and $S^{-1}S = I_2$, where $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the 2-dimensional identity matrix.
It is easy to see for this simple matrix that
\[
S^{-1} = \begin{pmatrix}
\frac{1}{a} & 0 \\
0 & \frac{1}{b}
\end{pmatrix}.
\] (2.15)

In general, the inverse of the 2 × 2 matrix \( T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is given by the following formula:
\[
T^{-1} = \frac{1}{\text{det}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},
\] (2.16)
where the scalar that we multiply by is \( \frac{1}{\text{det}} \) and the determinant \( \text{det} \) is defined by:
\[
\text{det} = ad - bc.
\] (2.17)

An easy way to verify that the square matrix \( T \) has full rank is to verify that \( \text{det} \neq 0 \). Without full rank, \( \text{det} = 0 \), the ratio \( \frac{1}{\text{det}} \) is undefined, and the matrix is not invertible.

When we get to multivariate calculus shortly, we will see that the first derivative is a matrix called the Jacobian matrix and the second derivative is a matrix called the Hessian matrix.

**Example 4: Supporting Prices** Suppose consumer 1 has income \( m^1 = \$10 \) and consumer 2 has income \( m^2 = \$18 \). Suppose the consumption bundle for consumer 1 is \( x^1 = \begin{pmatrix} x^1_1 \\ x^1_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \) and the consumption bundle for consumer 2 is \( x^2 = \begin{pmatrix} x^2_1 \\ x^2_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \).

There is only one possible set of prices \((p_1, p_2)\) such that both consumer expenditures exactly equal their available income. To find the prices \((p_1, p_2)\), we set up a system of equations and use matrix inversion.

The expenditure constraints (2 total, 1 for each consumer) are represented by the following system of equations:
\[
(p_1, p_2) \begin{pmatrix} x^1_1 & x^2_1 \\ x^1_2 & x^2_2 \end{pmatrix} = (m^1, m^2).
\] (2.18)

Plugging in the known values:
\[
(p_1, p_2) \begin{pmatrix} 3 & 6 \\ 4 & 2 \end{pmatrix} = (10, 18).
\] (2.19)
To solve for \((p_1, p_2)\), we perform matrix inversion:

\[
(p_1, p_2) \begin{pmatrix} 3 & 6 \\ 4 & 2 \end{pmatrix}^{-1} = (10, 18) \begin{pmatrix} 3 & 6 \\ 4 & 2 \end{pmatrix}^{-1}.
\]

(2.20)

By definition, \(
\begin{pmatrix} 3 & 6 \\ 4 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\), so the left-hand side of the equation simplifies to \((p_1, p_2)\).

Using the matrix inverse formula,

\[
\begin{pmatrix} 3 & 6 \\ 4 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{1}{9} & \frac{1}{3} \\ \frac{2}{9} & -\frac{1}{6} \end{pmatrix}.
\]

(2.21)

The matrix product

\[
(10, 18) \begin{pmatrix} -\frac{1}{9} & \frac{1}{3} \\ \frac{2}{9} & -\frac{1}{6} \end{pmatrix} = \begin{pmatrix} \frac{26}{9} & \frac{1}{3} \end{pmatrix}.
\]

(2.22)

Thus, for prices \((p_1, p_2) = \left(\frac{26}{9}, \frac{1}{3}\right)\), both consumers have expenditures exactly equal to their incomes.

**Example 5: Supporting Portfolio** Consider an economy with two financial assets and two states of the world. The first financial asset is a risk-free bond, meaning that it has a payout equal to 1 in both states of the world. Denote this payout as \(r^1 = \begin{pmatrix} r^1_1 \\ r^1_2 \end{pmatrix} \). The second financial asset is a stock, with payout equal to 2 in the first state of the world (the good state) and payout equal to \(\frac{1}{2}\) in the second state of the world (the bad state). Denote this payout as \(r^2 = \begin{pmatrix} r^2_1 \\ r^2_2 \end{pmatrix} \). The two payouts can be gathered into a payout matrix \(R = (r^1, r^2) = \begin{pmatrix} r^1_1 & r^2_1 \\ r^1_2 & r^2_2 \end{pmatrix} \).

An investor chooses a portfolio \(z = \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \) consisting of both assets. What portfolio will the investor choose if it wants a total portfolio payout of \(y_1 = 1\) in the good state and \(y_2 = 2\) in the bad state? The total portfolio payouts (2 total, 1 for each state) are represented by
the following system of equations:

\[
\begin{pmatrix}
  r_1^1 & r_1^2 \\
  r_2^1 & r_2^2
\end{pmatrix}
\begin{pmatrix}
z_1^1 \\
z_2^1
\end{pmatrix}
=
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix}.
\]

(2.23)

Plugging in the known values:

\[
\begin{pmatrix}
  1 & 2 \\
  1 & \frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
z_1^1 \\
z_2^1
\end{pmatrix}
=
\begin{pmatrix}
  1 \\
  2
\end{pmatrix}.
\]

(2.24)

To solve for \( \begin{pmatrix} z_1^1 \\ z_2^1 \end{pmatrix} \), we perform matrix inversion:

\[
\begin{pmatrix}
  1 & 2 \\
  1 & \frac{1}{2}
\end{pmatrix}^{-1}
\begin{pmatrix}
  1 & 2 \\
  1 & \frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
z_1^1 \\
z_2^1
\end{pmatrix}
=
\begin{pmatrix}
  1 & 2 \\
  1 & \frac{1}{2}
\end{pmatrix}^{-1}
\begin{pmatrix}
  1 \\
  2
\end{pmatrix}.
\]

(2.25)

By definition, \( \begin{pmatrix}
  1 & 2 \\
  1 & \frac{1}{2}
\end{pmatrix}^{-1}
\begin{pmatrix}
  1 & 2 \\
  1 & \frac{1}{2}
\end{pmatrix} = \begin{pmatrix}
  1 & 0 \\
  0 & 1
\end{pmatrix} \), so the left-hand side of the equation simplifies to \( \begin{pmatrix} z_1^1 \\ z_2^1 \end{pmatrix} \).

Using the matrix inverse formula,

\[
\begin{pmatrix}
  1 & 2 \\
  1 & \frac{1}{2}
\end{pmatrix}^{-1} = \begin{pmatrix}
  \frac{-1}{3} & \frac{4}{3} \\
  \frac{2}{3} & -\frac{2}{3}
\end{pmatrix}.
\]

(2.26)

The matrix product

\[
\begin{pmatrix}
  \frac{-1}{3} & \frac{4}{3} \\
  \frac{2}{3} & -\frac{2}{3}
\end{pmatrix}
\begin{pmatrix}
  1 \\
  2
\end{pmatrix}
= \begin{pmatrix}
  \frac{7}{3} \\
  -\frac{2}{3}
\end{pmatrix}.
\]

(2.27)

Thus, the portfolio \( \begin{pmatrix} z_1^1 \\ z_2^1 \end{pmatrix} = \begin{pmatrix} \frac{7}{3} \\ -\frac{2}{3} \end{pmatrix} \) allows the investor to achieve the desired payouts of \( \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \). The portfolio specifies that the investor buys \( \frac{7}{3} \) units of asset 1 and sells \( \frac{2}{3} \) units of asset 2 (i.e., increases asset 1 holdings by \( \frac{7}{3} \) units and decreases asset 2 by \( \frac{2}{3} \) units).

If the investor initially owns at least \( \frac{2}{3} \) units of asset 2, then it is easy to sell the required
2. MATHEMATICAL PRELIMINARIES

Let's suppose that we are asked to take a second derivative. What is this? A second derivative is the derivative of a derivative, so we are taking two derivatives. For instance, I can rewrite all of the rules above when moving from the first derivative to the second derivative:

1. If $Df(x) = k$ for a constant $k$, then $D^2f(x) = 0$.

   For example, $f(x) = kx$ has a derivative $Df(x) = k$ and a second derivative $D^2f(x) = 0$.

2. If $Df(x) = kx$ for a constant $k$, then $D^2f(x) = k$.

   For example, $f(x) = kx^2$ has a derivative $Df(x) = kx$ and a second derivative $D^2f(x) = k$. 

3. If $Df(x) = kx^p$ for constants $k$ and $p$, then $D^2f(x) = kpx^{p-1}$.

For example, $f(x) = \frac{k}{p+1}x^{p+1}$ has a derivative $Df(x) = kx^p$ and a second derivative $D^2f(x) = kpx^{p-1}$.

4. If $Df(x) = k\ln(x)$ for constant $k$, then $D^2f(x) = \frac{k}{x}$.

5. If $Df(x) = ke^{px}$ for constants $k$ and $p$, then $D^2f(x) = kpe^{px}$.

For example, $f(x) = \frac{k}{p}e^{px}$ has a derivative $Df(x) = ke^{px}$ and a second derivative $D^2f(x) = kpe^{px}$.

We could keep going with third derivatives and so forth, but in economics we typically only need the first and second derivatives.

These 5 formulas capture most of the derivatives that we need to take in economics. Sometimes, a function is written as the sum, product, quotient, or composition of other functions. In that case, we can apply 4 rules of differentiation in order to evaluate the derivatives of these compound functions. This is addressed in the following subsection.

**Example 6: Marginal Utility** Let’s consider the first and second derivatives of a standard utility function in economics: $U(x) = \ln(x)$. The first derivative is given by $DU(x) = \frac{1}{x}$. The second derivative is given as follows:

$$DU(x) = \frac{1}{x} = x^{-1}$$

$$D^2U(x) = -1 \cdot x^{-2} = -\frac{1}{x^2}.$$  \hspace{1cm} (2.28)

The first derivative $DU(x)$ is referred to as marginal utility.

### 2.2.2 Rules of differentiation

We will address the following 4 rules of differentiation:

1. **Summation Rule**: If $f(x) = g(x) + h(x)$, what is the formula for $Df(x)$?

2. **Chain Rule**: If $f(x) = h(g(x))$, which is equivalently written as the composition $f(x) = (h \circ g)(x)$, what is the formula for $Df(x)$?

3. **Product Rule**: If $f(x) = g(x)h(x)$, what is the formula for $Df(x)$?
4. **Quotient Rule**: If \( f(x) = \frac{g(x)}{h(x)} \) for \( h(x) \neq 0 \), what is the formula for \( Df(x) \)?

With the 5 derivative formulas in the previous subsection, combined with the 4 rules of differentiation previewed above, we will be able to evaluate the derivatives of nearly all functions that we come across in economics.

1. **Summation Rule**: If \( f(x) = g(x) + h(x) \), then

\[
Df(x) = Dg(x) + Dh(x).
\] (2.29)

2. **Chain Rule**: If \( f(x) = h(g(x)) \), then

\[
Df(x) = Dh(g(x)) \cdot Dg(x).
\] (2.30)

3. **Product Rule**: If \( f(x) = g(x)h(x) \), then

\[
Df(x) = Dg(x)h(x) + g(x)Dh(x).
\] (2.31)

4. **Quotient Rule**: If \( f(x) = \frac{g(x)}{h(x)} \), then

\[
Df(x) = \frac{h(x)Dg(x) - g(x)Dh(x)}{[h(x)]^2}.
\] (2.32)

The following is a useful application of the chain rule: the **Inverse Function Theorem** (in one dimension). Consider the function such that \( y = f(x) \). If the function \( f \) is invertible, then there exists an inverse function \( g = f^{-1} \) such that \( x = g(y) \). By composition, \( y = f(g(y)) \) must hold. Taking the derivative of both sides and applying the chain rule

\[
1 = Df(g(y)) \cdot Dg(y).
\] (2.33)

Since \( f \) is invertible, then it must be that \( Df(g(y)) \neq 0 \), so the derivative of the inverse function \( g \) is evaluated as:

\[
Dg(y) = \frac{1}{Df(g(y))}.
\] (2.34)

In words, the derivative of the inverse is equal to the (multiplicative) inverse of the (original) derivative.

We will receive practice with each of these differentiation rules in short order.
2.2. BASIC CALCULUS

2.2.3 Multivariate calculus

We will now consider multivariate functions \( f : \mathbb{R}^n \to \mathbb{R} \), where \( n > 1 \). For instance, consider the function \( f(K,L) \) that depends upon the 2 variables \((K,L)\). To evaluate the partial derivative of \( f \) with respect to \( K \), we will take the derivative while treating \( L \) as a constant.

The notation for the partial derivative of \( f \) with respect to \( K \) can be any of the following: \( D_1 f(K,L) \), \( D_K f(K,L) \), or \( \frac{\partial f(K,L)}{\partial K} \). This manuscript will use the first notation (here, \( K \) is the 1st term in the vector \((K,L)\)).

The derivative vector has dimension \( 1 \times n \) (i.e., a row vector of length \( n \)) and contains the partial derivatives with respect to each of the \( n \) variables. This derivative vector is denoted \( Df(K,L) \). For instance,

\[
Df(K,L) = (D_1 f(K,L), D_2 f(K,L)).
\] (2.35)

We can also take second derivatives of a multivariate function. When we do this, we recognize that this is equivalent to taking first derivatives of the multi-valued function 
\[
\begin{pmatrix}
D_1 f(K,L) \\
D_2 f(K,L)
\end{pmatrix}
\]. Since there are \( n \) first derivative functions (\( n \) rows) and we take derivatives of these functions with respect to all variables (\( n \) columns), the second derivative will be a \( n \times n \) matrix. This matrix is called the Hessian matrix. For instance, the Hessian matrix associated with \( f(K,L) \) is given by:

\[
D^2 f(K,L) = \begin{bmatrix}
D^2_{11} f(K,L) & D^2_{12} f(K,L) \\
D^2_{21} f(K,L) & D^2_{22} f(K,L)
\end{bmatrix}.
\] (2.36)

A useful property of second derivatives is that \( D^2_{12} f(K,L) = D^2_{21} f(K,L) \). The matrix \( D^2 f(K,L) \) is a square matrix and it is symmetric. Recall that a matrix is symmetric if \( D^2 f(K,L) = [D^2 f(K,L)]^T \), where

\[
[D^2 f(K,L)]^T = \begin{bmatrix}
D^2_{11} f(K,L) & D^2_{21} f(K,L) \\
D^2_{12} f(K,L) & D^2_{22} f(K,L)
\end{bmatrix}.
\] (2.37)

The Hessian matrix is negative semi-definite if for any \( a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \), the quadratic form

\[
a^T D^2 f(K,L) a \leq 0.
\] (2.38)
The Hessian matrix is negative definite if for any nonzero \( a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \), the quadratic form

\[ a^T D^2 f(K, L) a < 0. \] (2.39)

Another possibility with multivariate functions is that we have a vector of functions, which is also called a multi-valued function. Suppose that \( g : \mathbb{R}^n \to \mathbb{R}^m \) for any values of \( n > 1 \) and \( m > 1 \). The function \( g \) is a system of equations \( g = \begin{pmatrix} g_1 \\ \vdots \\ g_m \end{pmatrix} \), where \( g_1 : \mathbb{R}^n \to \mathbb{R} \), \( g_2 : \mathbb{R}^n \to \mathbb{R} \), and so forth all the way until \( g_m : \mathbb{R}^n \to \mathbb{R} \). We can view this system of equations as a vector of functions.

If we want to take the first derivative of the function \( g \), we will obtain a Jacobian matrix. The Jacobian matrix has dimension \( m \times n \). The rows in this matrix correspond to the different functions \( g_1 \) up to \( g_m \), while the columns correspond to the derivatives with respect to each of the \( n \) variables. The Jacobian matrix is given by:

\[
Dg = \begin{bmatrix}
D_1 g_1 & \ldots & D_n g_1 \\
\vdots & \ddots & \vdots \\
D_1 g_m & \ldots & D_n g_m
\end{bmatrix}.
\] (2.40)

The Jacobian matrix is sometimes simply called the derivative matrix.

**Example 7: Marginal Products of Capital and Labor**  Let’s consider the first derivatives of a standard production function in economics: \( f(K, L) = K^\alpha L^{1-\alpha} \), where \( 0 < \alpha < 1 \). A production function of this form is called a Cobb-Douglas production function. The variable \( K \) is commonly referred to as capital and the variable \( L \) is commonly referred to as labor. The partial derivatives are given by:

\[
D_1 f(K, L) = \alpha K^{\alpha-1} L^{1-\alpha}. \tag{2.41}
\]

\[
D_2 f(K, L) = (1 - \alpha) K^{\alpha} L^{-\alpha}. \tag{2.42}
\]

The partial derivative \( D_1 f(K, L) \) is called the marginal product of capital and the partial derivative \( D_2 f(K, L) \) is called the marginal product of labor.
Example 8: Negative Semi-definite Hessian  Let’s consider the second derivatives of a Cobb-Douglas production function. The Hessian matrix is given by:

\[ D^2 f (K, L) = \begin{bmatrix}
  D^2_{11} f (K, L) & D^2_{12} f (K, L) \\
  D^2_{21} f (K, L) & D^2_{22} f (K, L)
\end{bmatrix} = \begin{bmatrix}
  -\alpha (1 - \alpha) K^{\alpha-2} L^{1-\alpha} & \alpha (1 - \alpha) K^{\alpha-1} L^{-\alpha} \\
  \alpha (1 - \alpha) K^{\alpha-1} L^{-\alpha} & -\alpha (1 - \alpha) K^{\alpha} L^{-\alpha-1}
\end{bmatrix}. \]  

(2.43)

Since \( 0 < \alpha < 1 \), then \( D^2_{11} f (K, L) < 0 \) and \( D^2_{22} f (K, L) < 0 \), while \( D^2_{12} f (K, L) = D^2_{21} f (K, L) > 0 \). I claim that the matrix \( D^2 f (K, L) \) is negative semi-definite. Consider any \( a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \) and the quadratic form \( a^T D^2 f (K, L) a \), which is expanded out to:

\[ (a_1, a_2) \begin{bmatrix}
  -\alpha (1 - \alpha) K^{\alpha-2} L^{1-\alpha} & \alpha (1 - \alpha) K^{\alpha-1} L^{-\alpha} \\
  \alpha (1 - \alpha) K^{\alpha-1} L^{-\alpha} & -\alpha (1 - \alpha) K^{\alpha} L^{-\alpha-1}
\end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = (a_1)^2 (-\alpha (1 - \alpha) K^{\alpha-2} L^{1-\alpha}) + (a_2)^2 (-\alpha (1 - \alpha) K^{\alpha} L^{-\alpha-1}) + 2 a_1 a_2 (\alpha (1 - \alpha) K^{\alpha-1} L^{-\alpha}). \]  

(2.44)

I claim that \( a^T D^2 f (K, L) a \leq 0 \). In the inequality, \( a^T D^2 f (K, L) a \leq 0 \), all terms contain \( \alpha (1 - \alpha) K^{\alpha} L^{1-\alpha} \). Cancelling this term implies \( a^T D^2 f (K, L) a \leq 0 \) is equivalent to:

\[ (a_1)^2 \left( -\frac{1}{K^2} \right) + (a_2)^2 \left( -\frac{1}{L^2} \right) + 2 a_1 a_2 \left( \frac{1}{KL} \right) \leq 0. \]  

(2.45)

Bringing all terms to the right-hand side:

\[ (a_1)^2 \left( \frac{1}{K^2} \right) + (a_2)^2 \left( \frac{1}{L^2} \right) - 2 a_1 a_2 \left( \frac{1}{KL} \right) \geq 0. \]  

(2.46)

Factoring leads to:

\[ \left( \frac{a_1}{K} - \frac{a_2}{L} \right)^2 \geq 0. \]  

(2.47)

This weak inequality is always satisfied, by definition. Therefore, \( a^T D^2 f (K, L) a \leq 0 \) and \( D^2 f (K, L) \) is negative semi-definite.

In future sections, we will see that a negative semi-definite Hessian matrix is equivalent to a concave function, so a Cobb-Douglas production function is a concave function.
2.3 Optimization

2.3.1 Unconstrained optimization

In economics, we typically write optimization problems as maximization problems (maximize utility, maximize profit), but the same basic methods can be used to solve minimization problems. If we are tasked with minimizing a function, it is equivalent to maximizing the inverse of that same function (i.e., multiply all elements in the function by $-1$).

A maximization problem is of the form

$$\max_{x \in \mathbb{R}} f(x). \quad (2.48)$$

Here $x \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is a univariate function. The function $f : \mathbb{R} \to \mathbb{R}$ is referred to as the objective function as it is the object to be maximized. Since there are no constraints imposed on the choice of $x$, the maximization problem is said to be unconstrained.

Denote $x^*$ as the value that maximizes $f(x)$. The value $x^*$ is referred to as the maximizer and the image $f(x^*)$ is referred to as the maximum. By definition, $f(x^*) \geq f(x)$ for all $x \in \mathbb{R}$. A necessary condition for a maximizer $x^*$ is that it satisfies the first order condition:

$$Df(x^*) = 0. \quad (2.49)$$

In words, every maximizer must satisfy the first order condition. However, the first order condition is not sufficient, since $x^*$ may not be a maximizer just because $Df(x^*) = 0$.

The function $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing if $x > y$ implies $f(x) > f(y)$. An equivalent definition for strictly increasing is $Df(x) > 0$ for all $x \in \mathbb{R}$. If a function is strictly increasing, then an unconstrained maximization problem will not have a solution as there will not exist $x^*$ satisfying $Df(x^*) = 0$. This is a typical situation for consumers seeking to maximize utility as most utility functions are strictly increasing (more consumption leads to higher utility). Optimization in economics is only well-defined when the consumers face constraints, typically in the form of a budget constraint. We consider this more interesting case of constrained optimization in the following subsection.

Recall that the first order condition $Df(x^*) = 0$ is a necessary condition for $x^*$ to be a maximizer. In words, if $x^*$ is a maximizer, it must satisfy $Df(x^*) = 0$. However, this does not imply that any $x^*$ satisfying $Df(x^*) = 0$ must be a maximizer. In order to obtain this latter implication, we have to verify that the second order condition is satisfied. The second
order condition states

\[ D^2 f (x^*) \leq 0. \quad (2.50) \]

If the second order condition is satisfied, then the first order condition \( Df (x^*) = 0 \) is both necessary and sufficient for a maximizer \( x^* \); necessary as the implication "if \( x^* \) is a maximizer, then \( Df (x^*) = 0 \)" holds and sufficient as the implication "if \( Df (x^*) = 0 \), then \( x^* \) is a maximizer" holds.

The second order condition is related to the property of concavity. The function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is concave if

\[ f (\theta a + (1 - \theta)b) \geq \theta f(a) + (1 - \theta)f(b) \quad (2.51) \]

for any \( a, b \in \mathbb{R} \) and any \( \theta \in [0, 1] \). This definition is equivalent to a non-positive second derivative:

\[ D^2 f(x) \leq 0 \text{ for all } x. \quad (2.52) \]

Thus, a concave function satisfies the second order condition.

If we wish to assume a stronger form of concavity, called strict concavity, then all of the weak inequalities in the above expressions are replaced by strict inequalities. Specifically, the function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is strictly concave if

\[ f (\theta a + (1 - \theta)b) > \theta f(a) + (1 - \theta)f(b) \quad (2.53) \]

for any \( a, b \in \mathbb{R} \) with \( a \neq b \) and any \( \theta \in (0, 1) \). This is equivalent to a strictly negative second derivative:

\[ D^2 f(x) < 0 \text{ for all } x. \quad (2.54) \]

If \( f : \mathbb{R} \rightarrow \mathbb{R} \) is strictly concave, then at most 1 maximizer exists (an important result that we prove in the following chapter).

**Example 9: Profit Maximization**  Consider a production function only in terms of capital, \( f(K) = K^\alpha \), where \( 0 < \alpha < 1 \). Suppose the price of capital is given by \( R \) and the price of output is given by \( p \). Firm profit is defined by \( \pi (K) = pf(K) - RK \). The firm chooses the optimal capital choice \( K \) to maximize profit. This optimal choice is found by taking the first order condition:

\[ \max_K \pi (K) = \max_K pf(K) - RK. \quad (2.55) \]
The first order condition is \( \alpha p (K^*)^{\alpha - 1} - R = 0 \), which can be solved to find the optimal choice of capital: \( K^* = \left( \frac{R}{\alpha p} \right)^{\frac{1}{1-\alpha}} \). Since \( \alpha - 1 < 0 \), it is more appropriate to write the optimal capital expression as follows: \( K^* = \left( \frac{\alpha p}{R} \right)^{\frac{1}{1-\alpha}} \).

### 2.3.2 Constrained optimization (Lagrangean)

Constraints in economics can take a variety of forms. For example, we typically assume that commodity consumption is non-negative, meaning that strictly negative consumption values are not allowed. If we are solving a utility maximization problem for a consumer, we require that the consumer satisfies its budget constraint. If we are solving a firm profit maximization problem, we require that a firm produces no more than its production function allows.

There are two types of constraints in economics to consider: equality constraints and weak inequality constraints. Strong inequality constraints are not used in economics for technical reasons. In what follows, an inequality constraint always refers to a weak inequality constraint.

In this subsection, we will focus exclusively on equality constraints. We will see shortly that this is just a special case of inequality constraints in which the inequality constraints are binding.

Consider the constrained maximization problem for a consumer:

\[
\max_{x_1, x_2 \in \mathbb{R}^2} f(x_1, x_2) \quad \text{subj. to} \quad p_1 x_1 + p_2 x_2 = m.
\]  

(2.56)

Here, unlike before, \((x_1, x_2) \in \mathbb{R}^2\) and \(f : \mathbb{R}^2 \to \mathbb{R}\) is a multivariate function. Assign \(\lambda\) as the Lagrange multiplier for the constraint \(p_1 x_1 + p_2 x_2 = m\).

The multivariate function \(f : \mathbb{R}^2 \to \mathbb{R}\) is strictly increasing when both \(D_1 f(x_1, x_2) > 0\) and \(D_2 f(x_1, x_2) > 0\).

Recall that \(D^2 f(x_1, x_2) = \begin{bmatrix} D^2_{11} f(x_1, x_2) & D^2_{12} f(x_1, x_2) \\ D^2_{21} f(x_1, x_2) & D^2_{22} f(x_1, x_2) \end{bmatrix}\) is the Hessian matrix. Recall that the Hessian matrix is negative semi-definite if for any \(a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\), the quadratic form

\[
a^T D^2 f(x_1, x_2) a \leq 0.
\]  

(2.57)
The Hessian matrix is negative definite if for any nonzero \( a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \), the quadratic form

\[ a^T D^2 f(x_1, x_2) a < 0. \] (2.58)

By definition, the multivariate function \( f : \mathbb{R}^2 \to \mathbb{R} \) is concave when the Hessian matrix \( D^2 f(x_1, x_2) \) is negative semi-definite. For concave objective functions, the second order condition is satisfied, meaning that any solution \((x_1^*, x_2^*)\) to the first order conditions must be a maximizer.

By definition, the multivariate function \( f : \mathbb{R}^2 \to \mathbb{R} \) is strictly concave when the Hessian matrix \( D^2 f(x_1, x_2) \) is negative definite. For strictly concave objective functions, at most 1 maximizer exists.

The Lagrangean is a function of the original variables \((x_1, x_2)\) and the Lagrange multiplier \( \lambda \) and is written as:

\[ L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda (m - p_1 x_1 - p_2 x_2). \] (2.59)

Notice that I have rewritten the constraint such that

\[ m - p_1 x_1 - p_2 x_2 = 0. \] (2.60)

An optimal solution \((x_1^*, x_2^*, \lambda^*)\) to the constrained maximization problem must satisfy the first order conditions for the Lagrangean \( L(x_1, x_2, \lambda) :\)

\[ D_1 L(x_1^*, x_2^*, \lambda^*) = D_1 f(x_1^*, x_2^*) - \lambda^* p_1 = 0. \] (2.61)
\[ D_2 L(x_1^*, x_2^*, \lambda^*) = D_2 f(x_1^*, x_2^*) - \lambda^* p_2 = 0. \]
\[ D_3 L(x_1^*, x_2^*, \lambda^*) = m - p_1 x_1^* - p_2 x_2^* = 0. \]

Notice that I have evaluated the first order conditions for both the consumption variables \((x_1, x_2)\) and the Lagrange multiplier variable \( \lambda \).

As in the previous section, a solution \((x_1^*, x_2^*, \lambda^*)\) to the first order conditions is not necessarily a solution to the constrained maximization problem. If the function \( f : \mathbb{R}^2 \to \mathbb{R} \) is concave, however, then a solution \((x_1^*, x_2^*, \lambda^*)\) to the first order conditions must be a solution to the constrained maximization problem.
Example 10: Consumer Demand  Suppose a consumer has the utility function 
\[ U(x_1, x_2) = \theta \ln(x_1) + (1 - \theta) \ln(x_2), \]
where \(0 < \theta < 1\). Such a utility function is of the Cobb-Douglas form as it is a monotonic transformation of \(x_1^\theta x_2^{1-\theta}\) (natural log \(\ln\) is a monotonic function and \(\ln(x_1^\theta x_2^{1-\theta}) = \theta \ln(x_1) + (1 - \theta) \ln(x_2)\) using the well-known properties of natural logs). The utility maximization problem is given by:

\[
\begin{align*}
\text{maximize} & \quad \theta \ln(x_1) + (1 - \theta) \ln(x_2) \\
\text{subject to} & \quad m - p_1 x_1 - p_2 x_2 = 0
\end{align*}
\]  

(2.62)

The first order conditions are given by:

\[
\begin{align*}
\frac{\theta}{x_1^*} - \lambda^* p_1 &= 0. \\
\frac{1 - \theta}{x_2^*} - \lambda^* p_2 &= 0.
\end{align*}
\]  

(2.63)

Solve for \((x_1^*, x_2^*)\) as a function of \(\lambda^*\):

\[
\begin{align*}
x_1^* &= \frac{\theta}{\lambda^* p_1}.
\end{align*}
\]  

(2.64)

\[
\begin{align*}
x_2^* &= \frac{1 - \theta}{\lambda^* p_2}.
\end{align*}
\]

Plug these expression into the budget constraint \(p_1 x_1^* + p_2 x_2^* = m\) yields:

\[
\begin{align*}
\frac{\theta}{\lambda^*} + \frac{1 - \theta}{\lambda^*} &= m.
\end{align*}
\]  

(2.65)

This implies that \(\frac{1}{\lambda^*} = m\). The demand functions for the consumer are therefore given by:

\[
\begin{align*}
x_1^* &= \frac{\theta m}{p_1} \quad \text{and} \quad x_2^* = \frac{(1 - \theta) m}{p_2}.
\end{align*}
\]  

(2.66)

Example 11: Elasticities  This example continues with the same consumer from Example 10. The price elasticities are defined as \(\epsilon_1 = \frac{\Delta x_1^*/x_1^*}{\Delta p_1/p_1}\) and \(\epsilon_2 = \frac{\Delta x_2^*/x_2^*}{\Delta p_2/p_2}\), respectively. For commodity 1, the numerator is the relative change in commodity demand and the denominator is the relative change in price. We can re-express the elasticity as follows:

\[
\epsilon_1 = \frac{\Delta x_1^* p_1}{\Delta p_1 x_1^*} = \frac{\partial x_1^*}{\partial p_1} x_1^*,
\]  

(2.67)
where the ratio of changes \( \frac{\Delta x_1^*}{\Delta m} \) is replaced with the derivative \( \frac{\partial x_1^*}{\partial p_1} \). From the demand function in Example 10, \( \frac{\partial x_1^*}{\partial p_1} = \frac{\partial m}{p_1 x_1^*} \). This means that the elasticity is given by \( \epsilon_1 = \frac{\partial m}{p_1 x_1^*} \). Using the equation for \( x_1^* \), then \( \epsilon_1 = -1 \). According to the law of demand (higher price, lower demand), the elasticity is typically negative, so it is common to report the elasticity as the absolute value \( \epsilon_1 = 1 \). In similar fashion, \( \epsilon_2 = 1 \).

The income elasticities are defined as \( \eta_1 = \frac{\Delta x_1^*/\Delta m}{x_1^*} \) and \( \eta_2 = \frac{\Delta x_2^*/\Delta m}{x_2^*} \), respectively. For commodity 1, the numerator is the relative change in commodity demand and the denominator is the relative change in income. We can re-express the elasticity as follows:

\[
\eta_1 = \frac{\Delta x_1^* m}{\Delta m x_1^*} = \frac{\partial x_1^* m}{\partial m x_1^*}.
\] (2.68)

From the demand function in Example 10, \( \frac{\partial x_1^*}{\partial m} = \frac{\partial m}{p_1} \). This implies \( \eta_1 = \frac{\partial m}{p_1 x_1^*} = 1 \). The elasticity is typically positive (higher income, higher demand), so there is no need to take the absolute value. In similar fashion, \( \eta_2 = 1 \). Cobb-Douglas utility has the special property that both price elasticities and income elasticities equal 1, a property called unit elasticity.

**Example 12: Indirect Utility and Roy’s Identity**  Since we found a closed-form demand function in Example 10, we can define the indirect utility function and verify a result called Roy’s Identity. Define

\[
V(p_1, p_2, m) = \theta \ln (x_1^*) + (1 - \theta) \ln (x_2^*) .
\] (2.69)

This is the utility that the consumer receives from an optimal choice of commodity vector. Using the demand functions:

\[
V(p_1, p_2, m) = \theta \ln \left( \frac{\theta m}{p_1} \right) + (1 - \theta) \ln \left( \frac{(1 - \theta) m}{p_2} \right) .
\] (2.70)

Roy’s Identity states that

\[
x_1^* = -\frac{D_1 V(p_1, p_2, m)}{D_3 V(p_1, p_2, m)} \quad \text{and} \quad x_2^* = -\frac{D_2 V(p_1, p_2, m)}{D_3 V(p_1, p_2, m)} .
\] (2.71)

Using the well-known property of the natural log:

\[
V(p_1, p_2, m) = \theta \{\ln (\theta) + \ln (m) - \ln (p_1)\} + (1 - \theta) \{\ln (1 - \theta) + \ln (m) - \ln (p_2)\} .
\] (2.72)
Evaluating the derivatives:

\[ D_1 V (p_1, p_2, m) = -\frac{\theta}{p_1}. \]  
(2.73)

\[ D_2 V (p_1, p_2, m) = -\frac{1 - \theta}{p_2}. \]

\[ D_3 V (p_1, p_2, m) = \frac{\theta}{m} + \frac{1 - \theta}{m} = \frac{1}{m}. \]

For commodity 1,

\[ -\frac{D_1 V (p_1, p_2, m)}{D_3 V (p_1, p_2, m)} = \frac{\theta m}{p_1} = x^*_1. \]  
(2.74)

For commodity 2,

\[ -\frac{D_2 V (p_1, p_2, m)}{D_3 V (p_1, p_2, m)} = \frac{(1 - \theta) m}{p_2} = x^*_2. \]  
(2.75)

Roy’s Identity is verified. This identity holds for all utility functions and for any number of commodities.

### 2.3.3 Constrained optimization (Kuhn-Tucker)

This section will introduce an equivalent method to solve constrained optimization problems. First, we need to generalize the constraint so that it is a weak inequality. This allows the possibility for the constraint to hold with either equality or strict inequality. If the constraint holds with equality, it is said to be "binding", while if it holds with strict inequality, it is said to be "loose".

Consider the following constrained maximization problem:

\[
\begin{align*}
\text{maximize} & \quad f(x_1, x_2) \\
\text{subject to} & \quad g(x_1, x_2) \geq 0
\end{align*}
\]  
(2.76)

As before, \((x_1, x_2) \in \mathbb{R}^2\) and \(f : \mathbb{R}^2 \to \mathbb{R}\) is a multivariate function. We assume from the outset that \(f : \mathbb{R}^2 \to \mathbb{R}\) is concave. In this formulation, we consider a general constraint function \(g : \mathbb{R}^2 \to \mathbb{R}\). This general formulation allows for constraints of the form \(g(x_1, x_2) = m - p_1 x_1 - p_2 x_2\), for example. From the outset, we assume that \(g\) is concave. We typically work with linear constraints. Any linear function is concave, by definition.

As above, assign \(\lambda\) as the Lagrange multiplier for the constraint \(g(x_1, x_2) \geq 0\). Notice that I always write the inequalities as nonnegative inequalities. This is important to get the
correct sign for the Lagrange multiplier.

The Kuhn-Tucker Theorem can now be stated.

**Theorem 2.1** If \((x_1^*, x_2^*)\) is an optimal solution to the constrained maximization problem, there exists a Lagrange multiplier \(\lambda \geq 0\) such that the following Kuhn-Tucker conditions are satisfied:

- **First order conditions**
  
  
  \[
  D_1f(x_1^*, x_2^*) + \lambda D_1g(x_1^*, x_2^*) = 0. \tag{2.77}
  \]
  \[
  D_2f(x_1^*, x_2^*) + \lambda D_2g(x_1^*, x_2^*) = 0. \tag{2.78}
  \]

- **Complimentary slackness conditions**

  \[
  \lambda g(x_1^*, x_2^*) = 0, \text{ with } \lambda \geq 0 \text{ and } g(x_1^*, x_2^*) \geq 0. \tag{2.79}
  \]

In the opposite direction, if there exists a Lagrange multiplier \(\lambda \geq 0\) such that \((x_1^*, x_2^*)\) solves the Kuhn-Tucker conditions, then \((x_1^*, x_2^*)\) is an optimal solution to the constrained maximization problem.

Let’s see how this theorem compares to the Lagrangean method considered in the previous section, where we now allow for a weak inequality in the budget constraint, i.e., \(p_1x_1 + p_2x_2 \leq m\) instead of \(p_1x_1 + p_2x_2 = m\). The constrained maximization problem is written:

\[
\begin{align*}
\text{maximize} & \quad f(x_1, x_2) \\
\text{subject to} & \quad m - p_1x_1 - p_2x_2 \geq 0 \\
\end{align*}
\tag{2.80}
\]

The variable \(\lambda\) is the Lagrange multiplier. The first order conditions (the first half of the Kuhn-Tucker conditions) are given by:

\[
\begin{align*}
D_1f(x_1^*, x_2^*) + \lambda^* (-p_1) &= 0. \tag{2.81}
D_2f(x_1^*, x_2^*) + \lambda^* (-p_2) &= 0.
\end{align*}
\]

This is equivalent to the first two of the three first order conditions of the Lagrangean.

Now consider the complimentary slackness condition (the second half of the Kuhn-Tucker conditions):

\[
\lambda (m - p_1x_1 - p_2x_2) = 0. \tag{2.82}
\]
If the constraint is binding, then \( m - p_1 x_1 - p_2 x_2 = 0 \) and \( \lambda > 0 \). This is equivalent to the third of the three first order conditions of the Lagrangean.

The Kuhn-Tucker condition is more general as it allows for the case in which the constraint is loose, meaning \( m - p_1 x_1 - p_2 x_2 > 0 \). If this is the case, the complimentary slackness condition requires \( \lambda = 0 \). If the constraint is loose, the first order conditions are \( D_1 f(x^*_1, x^*_2) = 0 \) and \( D_2 f(x^*_1, x^*_2) = 0 \). If the constraint is loose, the constraint is not a factor and the problem reduces to an unconstrained maximization problem.

**Example 13: Consumer Demand II** For the consumer from Example 10, update the utility maximization problem to include a weak inequality in the budget constraint:

\[
\begin{align*}
\text{maximize} & \quad \theta \ln (x_1) + (1 - \theta) \ln (x_2) \\
\text{subject to} & \quad m - p_1 x_1 - p_2 x_2 \geq 0
\end{align*}
\]

(2.82)

This representation of the budget constraint allows for both a binding and non-binding constraint. The first order conditions are given by:

\[
\begin{align*}
\frac{\theta}{x^*_1} - \lambda^* p_1 &= 0, \\
\frac{1 - \theta}{x^*_2} - \lambda^* p_2 &= 0.
\end{align*}
\]

(2.83)

Since \((x^*_1, x^*_2) \geq 0 \) and \( 0 < \theta < 1 \), then \( \frac{\theta}{x^*_1} > 0 \) and \( \frac{1 - \theta}{x^*_2} > 0 \). From the first order conditions, this implies \( \lambda^* p_1 > 0 \) and \( \lambda^* p_2 > 0 \). Since \( \lambda^* \geq 0 \), \( p_1 \geq 0 \), and \( p_2 \geq 0 \), then all three variables must be strictly positive: \( \lambda^* > 0 \), \( p_1 > 0 \), and \( p_2 > 0 \). Since \( \lambda^* > 0 \), the complimentary slackness condition requires that the budget constraint is binding:

\[ m - p_1 x^*_1 - p_2 x^*_2 = 0. \]

(2.84)

The algebra follows exactly as for the Lagrangean method in Example 10.

In general, for any utility function that is strictly increasing, the budget constraint will be binding. Other constraints, however, may not bind even with a strictly increasing utility function.

\(^{1}\) Although it is possible that \( m - p_1 x_1 - p_2 x_2 = 0 \) and \( \lambda = 0 \), this degenerate case is atypical.
2.3.4 Multivariate constrained optimization

We now want to extend the dimensions of the Kuhn-Tucker conditions. We should be comfortable applying these conditions for any finite-dimensional problem. Consider the following constrained maximization problem:

\[
\begin{align*}
\text{maximize} & \quad f(x) \\
\text{subject to} & \quad g(x) \geq 0.
\end{align*}
\] (2.85)

Here, \( x \in \mathbb{R}^n \) and \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is a multivariate function. We sometimes write the vector \( x \) in terms of its elements \( x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \). The objective function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is assumed to be concave.

In this formulation, we consider a vector-valued constraint function \( g : \mathbb{R}^n \rightarrow \mathbb{R}^m \). This means that there are \( m \) constraints \( g_1 : \mathbb{R}^n \rightarrow \mathbb{R} \) through \( g_m : \mathbb{R}^n \rightarrow \mathbb{R} \). The vector-valued function \( g : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is typically expressed by stacking all \( m \) constraint functions on top of one another:

\[
g(x) = \begin{pmatrix} g_1(x) \\ \vdots \\ g_m(x) \end{pmatrix}.
\] (2.86)

We assume that \( g \) is concave. For constraint \( j \), assign \( \lambda_j \) as the Lagrange multiplier for the constraint \( g_j(x) \geq 0 \). The vector of Lagrange multipliers is written as \( \lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} \).

We are now prepared to state the Kuhn-Tucker conditions. We first introduce them in long-form notation and then show how matrix notation simplifies the expressions.

In long-form, the Kuhn-Tucker conditions are given by:

- **First order conditions**

\[
D_1 f(x_1^*, x_2^*, ..., x_n^*) + \sum_{j=1}^{m} \lambda_j^* D_1 g_j(x_1^*, x_2^*, ..., x_n^*) = 0. \] (2.87)

\[
D_n f(x_1^*, x_2^*, ..., x_n^*) + \sum_{j=1}^{m} \lambda_j^* D_n g_j(x_1^*, x_2^*, ..., x_n^*) = 0.
\]
• **Complimentary slackness conditions**

\[ \lambda_1^* g_1(x_1^*, x_2^*, ..., x_n^*) = 0, \text{ with } \lambda_1^* \geq 0 \text{ and } g_1(x_1^*, x_2^*, ..., x_n^*) \geq 0. \]  
\[ \vdots \]

\[ \lambda_m^* g_m(x_1^*, x_2^*, ..., x_n^*) = 0, \text{ with } \lambda_m^* \geq 0 \text{ and } g_m(x_1^*, x_2^*, ..., x_n^*) \geq 0. \]  

(2.89)

Recall that \( Df(x^*) \) is the \( 1 \times n \) Jacobian matrix (1 function and \( n \) variables). The Jacobian \( Dg(x^*) \) is a \( m \times n \) matrix (\( m \) functions and \( n \) variables). In matrix notation, the Kuhn-Tucker conditions are given by

• **First order conditions**

\[ Df(x^*) + (\lambda^*)^T Dg(x^*) = 0_{1 \times n}. \]  

(2.90)

• **Complimentary slackness conditions**

\[ (\lambda^*)^T g(x^*) = 0_{1 \times 1}, \text{ with } \lambda^* \geq 0_{m \times 1} \text{ and } g(x^*) \geq 0_{m \times 1}. \]  

I have included the dimensions for the zero vectors to indicate that we are working with vectors and a system of equations. On your own, write down the dimensions of all matrices in the above expressions and verify that the indicated matrix operations (additional and multiplication) are well-defined.

**Example 14: Consumer Demand III**  
Suppose a consumer has the utility function \( u(x_1, x_2) = \theta x_1 + (1 - \theta) x_2 \) such that \( 0 < \theta < 1 \). The utility maximization problem is given by:

\[
\begin{align*}
\text{maximize} & \quad \theta x_1 + (1 - \theta) x_2 \\
\text{subject to} & \quad m - p_1 x_1 - p_2 x_2 \geq 0 \\
& \quad x_1 \geq 0 \\
& \quad x_2 \geq 0
\end{align*}
\]  

(2.91)

Notice that with linear utility (the commodities are perfect substitutes), we must include the nonnegativity constraints for each of the two commodities. Naturally, the consumer is not permitted to consume a negative amount of either commodity. The budget constraint has the Lagrange multiplier \( \lambda_1^* \). The nonnegativity constraint \( x_1 \geq 0 \) has the Lagrange multiplier \( \lambda_2^* \) and the nonnegativity constraint \( x_2 \geq 0 \) has the Lagrange multiplier \( \lambda_3^* \). The first order
2.3. **OPTIMIZATION**

conditions are given by:

\[ \theta - \lambda_1^* p_1 + \lambda_2^* = 0. \]  
\[ 1 - \theta - \lambda_1^* p_2 + \lambda_3^* = 0. \]  

(2.92)

The complimentary slackness conditions are given by:

\[ \lambda_1^* (m - p_1 x_1^* - p_2 x_2^*) = 0. \]  
\[ \lambda_2^* x_1^* = 0. \]  
\[ \lambda_3^* x_2^* = 0. \]  

(2.93)

Multiply the first order condition for commodity 1 by \( x_1^* \):

\[ (\theta - \lambda_1^* p_1) x_1^* + \lambda_2^* x_1^* = 0. \]  

(2.94)

From the complimentary slackness condition for nonnegative commodity 1, \( \lambda_2^* x_1^* = 0 \), the above equation reduces to:

\[ (\theta - \lambda_1^* p_1) x_1^* = 0. \]  

(2.95)

If \( x_1^* > 0 \), then it must be that \( \theta - \lambda_1^* p_1 = 0 \), implying that \( \lambda_1^* = \frac{\theta}{p_1} \).

Multiply the first order condition for commodity 2 by \( x_2^* \):

\[ (1 - \theta - \lambda_1^* p_2) x_2^* + \lambda_3^* x_2^* = 0. \]  

(2.96)

From the complimentary slackness condition for nonnegative commodity 2, \( \lambda_3^* x_2^* = 0 \), the above equation reduces to:

\[ (1 - \theta - \lambda_1^* p_2) x_2^* = 0. \]  

(2.97)

If \( x_2^* > 0 \), then it must be that \( 1 - \theta - \lambda_1^* p_2 = 0 \), implying that \( \lambda_1^* = \frac{1-\theta}{p_2} \).

Since the utility function is strictly increasing, then \( \lambda_1^* > 0 \) and the budget constraint is binding:

\[ p_1 x_1^* + p_2 x_2^* = m. \]  

(2.98)

Since \( (\theta - \lambda_1^* p_1) x_1^* = 0 \), then \( p_1 x_1^* = \frac{\theta x_1^*}{\lambda_1^*} \). Similarly, \( p_2 x_2^* = \frac{(1-\theta)x_2^*}{\lambda_1^*} \). Plug these expression into the budget constraint:

\[ \frac{\theta x_1^* + (1 - \theta) x_2^*}{\lambda_1^*} = m. \]  

(2.99)
The objective is to maximize \( \theta x_1^* + (1 - \theta) x_2^* \). From the above expression, \( \theta x_1^* + (1 - \theta) x_2^* = m \lambda_1^* \). Since \( m \) is fixed, the maximum utility value is found by maximizing \( \lambda_1^* \).

There are two options. If \( x_1^* > 0 \), then \( \lambda_1^* = \frac{\theta}{p_1} \); while if \( x_2^* > 0 \), then \( \lambda_1^* = \frac{1 - \theta}{p_2} \). To maximize \( \lambda_1^* \), we select \( \lambda_1^* = \max \left\{ \frac{\theta}{p_1}, \frac{1 - \theta}{p_2} \right\} \). If \( \frac{\theta}{p_1} > \frac{1 - \theta}{p_2} \), then \( \lambda_1^* = \frac{\theta}{p_1} \) and \( (x_1^*, x_2^*) = \left( \frac{m}{p_1}, 0 \right) \). If \( \frac{1 - \theta}{p_2} > \frac{\theta}{p_1} \), then \( \lambda_1^* = \frac{1 - \theta}{p_2} \) and \( (x_1^*, x_2^*) = \left( 0, \frac{m}{p_2} \right) \). In the knife-edge case with \( \frac{\theta}{p_1} = \frac{1 - \theta}{p_2} \), then any \((x_1^*, x_2^*)\) satisfying the budget constraint is the optimal solution.

### 2.4 Advanced Topics

#### 2.4.1 Implicit Function Theorem

Let \( G(v, p) = 0 \) be a system of equations in terms of a vector of variables \( v \) and a vector of parameters \( p \). The number of equations in \( G \) must be equal to the number of variables. The system of equations is characterized by the function \( G : \mathbb{R}^{\#v + \#p} \to \mathbb{R}^{\#v} \), where \( \#v \) is the number of variables and \( \#p \) is the number of parameters. In order to satisfy \( G(v, p) = 0 \), a change in the parameters must lead to a change in the variables. We want to show that the variables required for \( G(v, p) = 0 \) can be written as an implicit function of the parameters, namely an implicit function \( f : \mathbb{R}^{\#p} \to \mathbb{R}^{\#v} \) of the form \( v = f(p) \).

Use the expression \( v = f(p) \) in the system of equations \( G(v, p) = 0 \). In the resulting equation \( G(f(p), p) = 0 \), take the derivative of both sides (using the chain rule):

\[
D_v G(f(p), p) Df(p) + D_p G(f(p), p) = 0. \tag{2.100}
\]

The Jacobian matrix \( D_v G(f(p), p) \) is a square matrix with the same number of rows as columns (since the number of equations in \( G \) is equal to the number of variables). This Jacobian matrix \( D_v G(f(p), p) \) must be invertible (meaning it has full rank) in order for the Implicit Function Theorem to be valid. If so, then we can solve the above equation for \( Df(p) \).

1. First, we bring the term \( D_p G(f(p), p) \) to the right-hand side of the equation:

\[
D_v G(f(p), p) Df(p) = -D_p G(f(p), p). \tag{2.101}
\]

2. We then pre-multiply both sides of the equation by \( [D_v G(f(p), p)]^{-1} \). We do this because \( [D_v G(f(p), p)]^{-1} D_v G(f(p), p) = I_{\#v} \), the identity matrix. The result is what
is commonly known as the conclusion of the Implicit Function Theorem (though this is not technically correct, it suffices for our purposes):

\[
Df(p) = - [D_v G(f(p), p)]^{-1} \cdot D_p G(f(p), p).
\] (2.102)

If \( G \) only contains one equation and both \( v \) and \( p \) have only one element, then the result reduces to:

\[
Df(p) = -\frac{D_p G(f(p), p)}{D_v G(f(p), p)}.
\] (2.103)

For example, let’s consider the equation for the unit circle

\[
x^2 + y^2 = 1.
\] (2.104)

The \( x \) variable is the value along the \( x \)-axis and the \( y \) variable is the value along the \( y \)-axis. Suppose I want to know the rate of change of \( y \) in terms of \( x \) as we move along the unit circle. Let’s specify the parameter \( p = x \) and the variable \( v = y \). The system of equations is

\[
G(x, y) = x^2 + y^2 - 1 = 0.
\] (2.105)

From the equation above, the derivative:

\[
Df(x) = -\frac{D_x G(f(x), x)}{D_y G(f(x), x)}
\] (2.106)

\[
= -\frac{2x}{2y} = -\frac{x}{y}.
\]

This says that at the point \((x, y) = (1, 0)\), the rate of change of \( y \) in terms of \( x \) is equal to \( Df(x) = -\frac{1}{0} = -\infty \). Similarly, at the point \((x, y) = (0, 1)\), the rate of change of \( y \) in terms of \( x \) is equal to \( Df(x) = -\frac{0}{1} = 0 \). When is the rate of change equal to \( Df(x) = -1 \)? This occurs when either \((x, y) = (\sqrt{2}, \sqrt{2})\) or \((x, y) = (-\sqrt{2}, -\sqrt{2})\). All of these make sense if we were to draw a circle and then draw the tangent lines at \((x, y) = (1, 0)\), at \((x, y) = (0, 1)\), and at either \((x, y) = (\sqrt{2}, \sqrt{2})\) or \((x, y) = (-\sqrt{2}, -\sqrt{2})\). The first tangent line is vertical (slope = \(\pm\infty\)), the second tangent line is horizontal (slope = 0), and the third tangent line has slope = -1.
Example 15: Comparative Statics with the Firm Problem  Consider the firm from Example 9. The first order condition is

\[ p\alpha (K^*)^{\alpha-1} - R = 0. \]  

(2.107)

The parameters in this economy are the prices \((p, R)\). For any parameters \((p, R)\), the variable \(K^*\) is the optimal solution for the consumer provided that \(G(K^*, p, R) = 0\), where the function \(G\) is defined by:

\[ G(K^*, p, R) = p\alpha (K^*)^{\alpha-1} - R. \]  

(2.108)

Since \(D_{K^*}G(\cdot) = -p\alpha (1 - \alpha) (K^*)^{\alpha-2} < 0\), then the full rank condition is satisfied. We can evaluate the derivative matrix for the parameters \((p, R)\):

\[ D_{p,R}G(\cdot) = \begin{bmatrix} \alpha (K^*)^{\alpha-1}, & -1 \end{bmatrix}. \]  

(2.109)

Applying the Implicit Function Theorem equation, we can express how a parameter change will impact the variables:

\[ Df(p, R) = \frac{\partial K^*}{\partial (p, R)} = -\frac{D_{p,R}G(\cdot)}{D_{K^*}G(\cdot)} = \left[ \frac{K^*}{p(1-\alpha)}, -\frac{1}{p\alpha(1-\alpha)(K^*)^{\alpha-2}} \right]. \]  

(2.110)

If \(p\) increases by 1 unit, then \(K^*\) increases by the amount \(\frac{K^*}{p(1-\alpha)}\) (a higher output price leads to more capital input). If \(R\) increases by 1 unit, then \(K^*\) decreases by the amount \(-\frac{1}{p\alpha(1-\alpha)(K^*)^{\alpha-2}}\) (a higher price for capital will decrease the capital input).

We can express these derivative only in terms of parameters by utilizing the expression for optimal capital, \(K^* = (\frac{\alpha p}{R})^{\frac{1}{1-\alpha}}\):

\[ Df(p, R) = \frac{\partial K^*}{\partial (p, R)} = -\frac{D_{p,R}G(\cdot)}{D_{K^*}G(\cdot)} = \left[ \frac{1}{1-\alpha} \left( \frac{\alpha}{R} \right)^{\frac{1}{1-\alpha}} p^{\frac{1}{1-\alpha}}, -\frac{1}{1-\alpha} \left( \frac{1}{R} \right)^{\frac{2-\alpha}{1-\alpha}} (\alpha p)^{\frac{1}{1-\alpha}} \right]. \]  

(2.111)

As a check to verify that the Implicit Function Theorem is doing what it should, we can evaluate the derivative \(\frac{\partial K^*}{\partial p}\) directly using the closed-form solution \(K^* = (\frac{\alpha p}{R})^{\frac{1}{1-\alpha}}\). Using the
2.4. ADVANCED TOPICS

chain rule,

$$\frac{\partial K^*}{\partial p} = \frac{1}{1 - \alpha} \left( \frac{ap}{R} \right)^{\frac{\alpha}{1-\alpha}} \frac{\alpha}{R}$$

$$= \frac{1}{1 - \alpha} \left( \frac{\alpha}{R} \right)^{\frac{1}{1-\alpha}} p^{\frac{\alpha}{1-\alpha}}.$$  

(2.112)

Rewriting the closed-form expression as $K^* = (ap)^{\frac{1}{1-\alpha}} R^{-\frac{1}{1-\alpha}}$, then we can evaluate the derivative $\frac{\partial K^*}{\partial R}$ directly. Using the chain rule,

$$\frac{\partial K^*}{\partial R} = -\frac{1}{1 - \alpha} \left( \frac{a}{R} \right)^{\frac{1}{1-\alpha}} R^{\frac{2}{1-\alpha}}$$

$$= -\frac{1}{1 - \alpha} \left( \frac{1}{R} \right)^{\frac{2}{1-\alpha}} (ap)^{\frac{1}{1-\alpha}}.$$

These are the same derivatives as found using the Implicit Function Theorem above.

**Example 16: Comparative Statics in the Consumer Problem**

Consider the consumer problem

$$\text{maximize} \quad U(x_1, x_2)$$
$$\text{subject to} \quad m - p_1 x_1 - p_2 x_2 \geq 0.$$  

Notice that the utility function is kept as a general function. Assume that $U$ is strictly increasing and strictly concave. This generality allows us to demonstrate the power of the Implicit Function Theorem. For utility functions in which a closed form solution exists and can be computed with ease, refer back to Examples 10 and 14. The equilibrium equations for this consumer consist of two first order conditions and one budget constraint. The equilibrium equations are given by:

$$D_1 U (x_1^*, x_2^*) - \lambda^* p_1 = 0.$$  

$$D_2 U (x_1^*, x_2^*) - \lambda^* p_2 = 0.$$  

$$m - p_1 x_1^* - p_2 x_2^* = 0.$$  

(2.113)

The parameters in this economy are the prices $(p_1, p_2)$ and the income $m$. For any parameters $(p_1, p_2, m)$, the variables $(x_1^*, x_2^*, \lambda^*)$ are optimal solutions for the consumer provided that
$G(x_1^*, x_2^*, \lambda^*, p_1, p_2, m) = 0$, where the vector-valued function $G$ is defined by:

$$
G(x_1^*, x_2^*, \lambda^*, p_1, p_2, m) = \begin{pmatrix}
D_1 U(x_1^*, x_2^*) - \lambda^* p_1 \\
D_2 U(x_1^*, x_2^*) - \lambda^* p_2 \\
m - p_1 x_1^* - p_2 x_2^*
\end{pmatrix}.
$$

(2.114)

The derivative matrix $D_{x_1^*, x_2^*, \lambda^*} G(\cdot)$ has dimension $3 \times 3$. It is referred to as the Jacobian matrix. The element in row $i$ and column $j$ of the matrix refers to the derivative of the $i$th equation in $G$ with respect to the $j$th variable in the vector of variables $(x_1^*, x_2^*, \lambda^*)$. Taking these derivatives yields:

$$
D_{x_1^*, x_2^*, \lambda^*} G(\cdot) = \begin{bmatrix}
D_{1,1}^2 U(x_1^*, x_2^*) & D_{1,2}^2 U(x_1^*, x_2^*) & -p_1 \\
D_{2,1}^2 U(x_1^*, x_2^*) & D_{2,2}^2 U(x_1^*, x_2^*) & -p_2 \\
-p_1 & -p_2 & 0
\end{bmatrix}.
$$

(2.115)

Use the shorthand that the submatrix $D^2 U(x_1^*, x_2^*) = \begin{bmatrix}
D_{1,1}^2 U(x_1^*, x_2^*) & D_{1,2}^2 U(x_1^*, x_2^*) \\
D_{2,1}^2 U(x_1^*, x_2^*) & D_{2,2}^2 U(x_1^*, x_2^*)
\end{bmatrix}$, which is the Hessian matrix for the multivariate function $U$. Since $U$ is strictly concave, the Hessian matrix $D^2 U(x_1^*, x_2^*)$ is negative definite.

Let us determine if the full rank condition is satisfied. The matrix $D_{x_1^*, x_2^*, \lambda^*} G(\cdot)$ has full rank if the equation $v^T D_{x_1^*, x_2^*, \lambda^*} G(\cdot) = 0$ implies that $v^T = (0, 0, 0)$, where $v^T = (v_1, v_2, v_3)$. The equations $v^T D_{x_1^*, x_2^*, \lambda^*} G(\cdot) = 0$ are given by:

$$
(v_1, v_2) D^2 U(x_1^*, x_2^*) + v_3 (-p_1, -p_2) = (0, 0),
$$

(2.116)

$$
v_1 (-p_1) + v_2 (-p_2) = 0.
$$

From the third equation (the first line $(v_1, v_2) D^2 U(x_1^*, x_2^*) + v_3 (-p_1, -p_2) = (0, 0)$ contains both the first and second equations), $p_1 v_1 + p_2 v_2 = 0$. Post-multiply the first and second equations by $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$:

$$
(v_1, v_2) D^2 U(x_1^*, x_2^*) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + v_3 (-p_1, -p_2) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0.
$$

(2.117)
Bringing the negative terms over to the right-hand side yields:

\[(v_1, v_2) D^2 U (x_1^*, x_2^*) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_3 (p_1 v_1 + p_2 v_2). \] (2.118)

From the third equation, \(p_1 v_1 + p_2 v_2 = 0\), so the post-multiplied equation reduces to:

\[(v_1, v_2) D^2 U (x_1^*, x_2^*) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0. \] (2.119)

Since \(D^2 U (x_1^*, x_2^*)\) is negative definite, then \((v_1, v_2) D^2 U (x_1^*, x_2^*) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0\) only if \((v_1, v_2) = (0, 0)\). Therefore, \((v_1, v_2) = (0, 0)\), the original first and second equations reduce to \(v_3 (-p_1, -p_2) = (0, 0)\). Since it is not possible for both prices to equal 0, then \(v_3 = 0\). We have just verified \((v_1, v_2, v_3) = (0, 0, 0)\), then \(D x_1^*, x_2^*, \lambda^* G (\cdot)\) has full rank.

We can evaluate the derivative matrix for the parameters \((p_1, p_2, m)\):

\[
D_{p_1, p_2, m} G (\cdot) = \begin{bmatrix} -\lambda^* & 0 & 0 \\ 0 & -\lambda^* & 0 \\ -x_1^* & -x_2^* & 1 \end{bmatrix}.
\] (2.120)

Applying the Implicit Function Theorem equation, we can express how a parameter change will impact the variables:

\[
Df (p_1, p_2, m) = \frac{\partial (x_1^*, x_2^*, \lambda^*)}{\partial (p_1, p_2, m)} = - \left[ D_{x_1^*, x_2^*, \lambda^*} G (\cdot) \right]^{-1} D_{p_1, p_2, m} G (\cdot).
\] (2.121)

The matrix \(D_{x_1^*, x_2^*, \lambda^*} G (\cdot)\) can be viewed as the matrix with blocks \(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\), where \(A = D^2 U (x_1^*, x_2^*), B = \begin{pmatrix} -p_1 \\ -p_2 \end{pmatrix}\), \(C = B^T = (-p_1, -p_2)\), and \(D = 0\). The inverse of this matrix is given by:

\[
\begin{bmatrix}
A^{-1} + A^{-1} B (D - C A^{-1} B)^{-1} C A^{-1} & -A^{-1} B (D - C A^{-1} B)^{-1} \\
-(D - C A^{-1} B)^{-1} C A^{-1} & (D - C A^{-1} B)^{-1}
\end{bmatrix},
\] (2.122)
which can be verified from any undergraduate textbook on matrix algebra. Given the fact that \( D = 0 \), the blocks in the inverse matrix \( [D_{x_1^*,x_2^*,\lambda^*}G(\cdot)]^{-1} \) are such that

\[
[D_{x_1^*,x_2^*,\lambda^*}G(\cdot)]^{-1} = \begin{bmatrix}
A^{-1} - A^{-1}B(CA^{-1}B)^{-1}CA^{-1} & A^{-1}B(CA^{-1}B)^{-1} \\
(CA^{-1}B)^{-1}CA^{-1} & -(CA^{-1}B)^{-1}
\end{bmatrix}.
\] (2.123)

Therefore,

\[
Df(p_1,p_2,m) = -[D_{x_1^*,x_2^*,\lambda^*}G(\cdot)]^{-1}D_{p_1,p_2,m}G(\cdot)
\] (2.124)

\[
= -\begin{bmatrix}
A^{-1} - A^{-1}B(CA^{-1}B)^{-1}CA^{-1} & A^{-1}B(CA^{-1}B)^{-1} \\
(CA^{-1}B)^{-1}CA^{-1} & -(CA^{-1}B)^{-1}
\end{bmatrix}D_{p_1,p_2,m}G(\cdot).
\]

We wish to derive the equation for the partial derivative \( \frac{\partial U}{\partial (p_1, p_2, m)} \). Since \( U \) is independent of parameters \((p_1, p_2, m)\), the effect of parameters on the optimal utility value \( U \) occurs only through changes in variables \((x_1^*, x_2^*, \lambda^*)\). From the chain rule:

\[
\frac{\partial U}{\partial (p_1, p_2, m)} = \frac{\partial U}{\partial (x_1^*, x_2^*, \lambda^*)} \frac{\partial (x_1^*, x_2^*, \lambda^*)}{\partial (p_1, p_2, m)}.
\] (2.125)

The derivative matrix \( Df(p_1,p_2,m) = \frac{\partial (x_1^*, x_2^*, \lambda^*)}{\partial (p_1,p_2,m)} \) has already been evaluated using the Implicit Function Theorem. The derivative vector

\[
\frac{\partial U}{\partial (x_1^*, x_2^*, \lambda^*)} = (D_1U(x_1^*, x_2^*), D_2U(x_1^*, x_2^*), 0),
\] (2.126)

where from the first order conditions

\[
(D_1U(x_1^*, x_2^*), D_2U(x_1^*, x_2^*), 0) = \lambda^*(p_1, p_2, 0).
\] (2.127)

By the definition of \( C \),

\[
(D_1U(x_1^*, x_2^*), D_2U(x_1^*, x_2^*), 0) = -\lambda^*(C, 0).
\] (2.128)
The derivative vector \( \frac{\partial U}{\partial (p_1, p_2, m)} \) is then given by:

\[
\frac{\partial U}{\partial (p_1, p_2, m)} = \lambda^* (C, 0) \begin{bmatrix} A^{-1} - A^{-1} B (CA^{-1} B)^{-1} C A^{-1} & A^{-1} B (CA^{-1} B)^{-1} \\ (CA^{-1} B)^{-1} C A^{-1} & -(CA^{-1} B)^{-1} \end{bmatrix} D_{p_1, p_2, m} G (\cdot). \tag{2.129}
\]

The product

\[
\lambda^* (C, 0) \begin{bmatrix} A^{-1} - A^{-1} B (CA^{-1} B)^{-1} C A^{-1} & A^{-1} B (CA^{-1} B)^{-1} \\ (CA^{-1} B)^{-1} C A^{-1} & -(CA^{-1} B)^{-1} \end{bmatrix} = \lambda^* \begin{bmatrix} CA^{-1} - CA^{-1} B (CA^{-1} B)^{-1} C A^{-1}, & CA^{-1} B (CA^{-1} B)^{-1} \end{bmatrix}.
\tag{2.130}
\]

Since \( CA^{-1} B (CA^{-1} B)^{-1} = 1 \) and \( CA^{-1} - CA^{-1} = (0, 0) \), the matrix reduces to:

\[
\lambda^* \begin{bmatrix} CA^{-1} - CA^{-1}, & 1 \end{bmatrix} = \lambda^* \begin{bmatrix} 0, & 0, & 1 \end{bmatrix}. \tag{2.131}
\]

Returning to the expression

\[
\frac{\partial U}{\partial (p_1, p_2, m)} = \lambda^* (C, 0) \begin{bmatrix} A^{-1} - A^{-1} B (CA^{-1} B)^{-1} C A^{-1} & A^{-1} B (CA^{-1} B)^{-1} \\ (CA^{-1} B)^{-1} C A^{-1} & -(CA^{-1} B)^{-1} \end{bmatrix} D_{p_1, p_2, m} G (\cdot) \tag{2.132}
\]

and inserting the derivative matrix for \( D_{p_1, p_2, m} G (\cdot) \), the derivative vector \( \frac{\partial U}{\partial (p_1, p_2, m)} \) is given by:

\[
\frac{\partial U}{\partial (p_1, p_2, m)} = \lambda^* \begin{bmatrix} 0, & 0, & 1 \end{bmatrix} \begin{bmatrix} -\lambda^* & 0 & 0 \\ 0 & -\lambda^* & 0 \\ -x_1^* & -x_2^* & 1 \end{bmatrix} = (-\lambda^* x_1^*, -\lambda^* x_2^*, \lambda^*).
\tag{2.133}
\]

The Lagrange multiplier is called the shadow value of utility. If \( m \) increases by 1 unit, then utility increases by \( \lambda^* \) units. If the price of commodity 1 increases by 1 unit, then the utility decreases by \( \lambda^* x_1^* \) units (price goes up, utility goes down). Similarly, if the price of commodity 2 increases by 1 unit, then the utility decreases by \( \lambda^* x_2^* \) units.

Recalling Example 12, with utility function

\[
U (x_1, x_2) = \theta \ln (x_1) + (1 - \theta) \ln (x_2), \tag{2.134}
\]

the optimal solutions are \( x_1^* = \frac{\theta m}{p_1} \) and \( x_2^* = \frac{(1-\theta)m}{p_2} \) and the optimal utility function is (using
2. MATHEMATICAL PRELIMINARIES

properties of the natural log):

\[ U(x_1^*, x_2^*) = \theta \ln \left( \frac{\theta m}{p_1} \right) + (1 - \theta) \ln \left( \frac{(1 - \theta) m}{p_2} \right) \]

\[ = \theta (\ln(\theta) + \ln(m) - \ln(p_1)) \]

\[ + (1 - \theta) (\ln(1 - \theta) + \ln(m) - \ln(p_2)) \].

Evaluating the derivative directly yields:

\[ \frac{\partial U(x_1^*, x_2^*)}{\partial (p_1, p_2, m)} = \left( -\frac{\theta}{p_1}, -\frac{1 - \theta}{p_2}, \frac{1}{m} \right). \]

Compare this result to the conclusion from the Implicit Function Theorem:

\[ \frac{\partial U}{\partial (p_1, p_2, m)} = (-\lambda^* x_1^*, -\lambda^* x_2^*, \lambda^*). \]

Since \( x_1^* = \frac{\theta}{\lambda^* p_1} \) and \( x_2^* = \frac{1 - \theta}{\lambda^* p_2} \), then \( \frac{\partial U}{\partial (p_1, p_2)} = \left( -\frac{\theta}{p_1}, -\frac{1 - \theta}{p_2} \right) \). From the budget constraint, \( m = p_1 x_1^* + p_2 x_2^* = \frac{1}{\lambda^*} \), so \( \frac{\partial U}{\partial m} = \frac{1}{m} \).

This verifies that our application of the Implicit Function Theorem works. When the demand function has a closed-form solution, it is easier to compute the partial derivatives directly. The method with the Implicit Function Theorem is appropriate for problems in which a closed-form solution cannot be found.

2.4.2 Envelope Theorem

Another general result that can be used no matter how complicated the problem (even if a closed-form solution does not exist) is the Envelope Theorem. The Envelope Theorem provides a formula for the marginal effects of changes in parameters on the utility value.

Define \( V(p) \) as the value for a particular maximization problem as a function of the parameters \( p \in \mathbb{R}^k \). The choice variables will be \( x \in \mathbb{R}^n \) as before. Using the same constrained maximization problem as in the previous section, the value \( V(p) \) will be defined as:

\[ V(p) = \max_{x \in \mathbb{R}^n} f(x, p) \]

subject to \( g(x, p) \geq 0 \).

Notice that the objective function \( f: \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R} \) and the constraint function \( g: \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R} \)
\( \mathbb{R}^m \) depend on both \( x \) and \( p \), but the maximization only occurs with respect to the variable \( x \) (consider \( p \) as a fixed parameter).

Define \( D_x f (x, p) \) as the partial derivatives with respect to the variables \( x \in \mathbb{R}^n \) and \( D_p f (x, p) \) as the partial derivatives with respect to the parameters \( p \). From the Implicit Function Theorem, define \( x^*(p) \) and \( \lambda^*(p) \) as the implicit functions that satisfy the Kuhn-Tucker conditions:

- **First order conditions**

\[
D_x f (x^*(p), p) + (\lambda^*(p))^T D_x g (x^*(p), p) = 0.
\] (2.139)

- **Complimentary slackness conditions**

\[
(\lambda^*(p))^T g (x^*(p), p) = 0, \text{ with } \lambda^*(p) \geq 0 \text{ and } g (x^*(p), p) \geq 0.
\] (2.140)

This means that \( V(p) \) can be written as:

\[
V(p) = f (x^*(p), p).
\] (2.141)

This leads us to a general result called the Envelope Theorem.

**Theorem 2.2 Envelope Theorem**

The derivative mapping

\[
DV(p) = D_p f (x^*(p), p) + (\lambda^*(p))^T D_p g (x^*(p), p).
\] (2.142)

To see why the Envelope Theorem is true, consider a problem without any constraints. As before,

\[
V(p) = f (x^*(p), p).
\] (2.143)

Evaluating the derivative with respect to \( p \) means using the Chain Rule:

\[
DV(p) = D_x f (x^*(p), p) D x^*(p) + D_p f (x^*(p), p).
\] (2.144)

From the first order conditions of the Kuhn-Tucker conditions

\[
D_x f (x^*(p), p) = 0.
\] (2.145)
Plugging this back into the derivative $DV(p)$ yields:

$$DV(p) = D_v f(x^*(p), p).$$

(2.146)

This is the statement of the Envelope Theorem for the unconstrained problems.

To see how the above argument generalizes, consider a problem with constraints. The complimentary slackness conditions from the Kuhn-Tucker Theorem can be used to rewrite the value function as:

$$V(p) = f(x^*(p), p) + (\lambda^*(p))^T g(x^*(p), p).$$

(2.147)

Evaluating the derivative with respect to $p$ means using the Chain Rule:

$$DV(p) = D_x f(x^*(p), p) Dx^*(p) + D_v f(x^*(p), p)$$

$$+ (\lambda^*(p))^T (D_x g(x^*(p), p) Dx^*(p) + D_v g(x^*(p), p)).$$

(2.148)

From the first order conditions of the Kuhn-Tucker conditions:

$$D_x f(x^*(p), p) + (\lambda^*(p))^T D_x g(x^*(p), p) = 0.$$ 

(2.149)

Plugging this back into the derivative $DV(p)$ yields:

$$DV(p) = D_v f(x^*(p), p) + (\lambda^*(p))^T D_v g(x^*(p), p).$$

(2.150)

**Example 17: Envelope Theorem applied to Consumer Problem**  This is a reinterpretation of Example 16. Recall that the Implicit Function Theorem result found that the effect of a change in parameters $(p_1, p_2, m)$ on the optimal choices $(x_1^*, x_2^*, \lambda^*)$ of the consumer is given by

$$D_f(p_1, p_2, m) = - \left[ \begin{array}{cc} A^{-1} - A^{-1}B(CA^{-1}B)^{-1}CA^{-1} & A^{-1}B(CA^{-1}B)^{-1} \\ (CA^{-1}B)^{-1}CA^{-1} & - (CA^{-1}B)^{-1} \end{array} \right] D_{p_1,p_2,m} G(\cdot)$$

(2.151)

and the effects on utility is given by:

$$\frac{\partial U}{\partial (p_1, p_2, m)} = (-\lambda^*x_1^*, -\lambda^*x_2^*, \lambda^*).$$

(2.152)
In terms of the Envelope Theorem, the maximization problem can be expressed as:

\[ V(p_1, p_2, m) = \max_{x_1, x_2} U(x_1, x_2) \]
subject to \[ g(x_1, x_2, p_1, p_2, m) = m - p_1 x_1 - p_2 x_2 \geq 0 \] .

(2.153)

According to the expression for the Envelope Theorem,

\[ DV(p_1, p_2, m) = D_{p_1,p_2,m} U(x_1, x_2) + \lambda^* D_{p_1,p_2,m} g(x_1^*, x_2^*, p_1, p_2, m) . \]

(2.154)

The derivative \( D_{p_1,p_2,m} U(x_1, x_2) = 0 \). The derivative

\[ D_{p_1,p_2,m} g(x_1^*, x_2^*, p_1, p_2, m) = (-x_1^*, -x_2^*, 1) . \]

(2.155)

Using the expression for the Envelope Theorem,

\[ DV(p_1, p_2, m) = (-\lambda^* x_1^*, -\lambda^* x_2^*, \lambda^*) . \]

(2.156)

This is exactly what we found (with far greater effort) for the derivative \( \frac{\partial U}{\partial (p_1, p_2, m)} \) from the Implicit Function Theorem.

### 2.4.3 Integration

Integration measures the area under a curve. It can be viewed as the sum of an uncountable number of weighted values, where the weights are proportional to the value of the function at each point. As the sum over an uncountable number of values is not well-defined mathematically, the notion of an integral must be introduced.

An integral is related to a derivative using the Fundamental Theorem of Calculus. We first work through a few problems to illustrate the rules of integration before stating the Fundamental Theorem of Calculus. The notation for an integral is

\[ \int_a^b f(x) \, dx . \]

(2.157)

The expression \( \int_a^b f(x) \, dx \) represents the integral of \( f(x) \) over the region \([a, b] \). The term \( 'dx' \) means that we are taking the derivative with respect to the variable \( x \). This is important when dealing with multivariate functions, but we won’t be integrating multivariate functions in this course.
Tying back to our first notion of an integral, the value of \( \int_a^b f(x)dx \) is equal to the area under the curve \( f(x) \) over the region \([a, b]\).

All of the integrals that we are interested in taking in economics fall into one of the following five categories:

1. If \( f(x) = k \) for a constant \( k \), then \( \int_a^b f(x)dx = k(b - a) \).

2. If \( f(x) = kx \) for a constant \( k \), then \( \int_a^b f(x)dx = k \left( \frac{b^2}{2} - \frac{a^2}{2} \right) \).

3. If \( f(x) = kx^p \) for constants \( k \) and \( p \neq -1 \), then \( \int_a^b f(x)dx = k \left( \frac{b^{p+1}}{p+1} - \frac{a^{p+1}}{p+1} \right) \).

4. If \( f(x) = \frac{k}{x} \) for constant \( k \), then \( \int_a^b f(x)dx = k (\ln(b) - \ln(a)) \).

5. If \( f(x) = ke^{px} \) for constants \( k \) and \( p \), then \( \int_a^b f(x)dx = k \left( e^{pb} - e^{pa} \right) \).

It is easy to see that the following additive property of integration holds:

- If \( f(x) = g(x) + h(x) \), then \( \int_a^b f(x)dx = \int_a^b g(x)dx + \int_a^b h(x)dx \).

The Fundamental Theorem of Calculus states the relation between integration and differentiation and can be used to evaluate all of the integrals evaluated above.

**Theorem 2.3** If \( f \) is a continuous function on \([a, b]\), then the following two statements hold:

1. If \( g(x) = \int_a^x f(t)dt \), then \( Dg(x) = f(x) \). This is the same as writing \( D_x (\int_a^x f(t)dt) = f(x) \).

2. If \( DF = f \), then \( \int_a^b f(x)dx = F(b) - F(a) \).

The notation is typically that \( \int_a^b f(x)dx = [F(x)]_a^b \), where \([F(x)]_a^b\) means the difference between the function \( F \) evaluated at the right endpoint \( b \) and the function \( F \) evaluated at the left endpoint \( a \):

\[
[F(x)]_a^b = F(b) - F(a). \tag{2.158}
\]

Using the Fundamental Theorem of Calculus, we can derive two important results:

1. The Fundamental Theorem of Calculus and the chain rule for differentiation lead to the substitution rule: if \( u = g(x) \), then \( \int_a^b f(g(x))Dg(x)dx = \int_{g(a)}^{g(b)} f(u)du \).

2. The Fundamental Theorem of Calculus and the product rule for differentiation lead to integration by parts: \( \int_a^b f(x)Dg(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x)Df(x)dx \).

The following two examples provide a little bit of practice with the substitution rule and integration by parts.
Example 18  Evaluate the integral $\int_0^1 \sqrt{2x+1} \, dx$.

Using the substitution $u = g(x) = 2x + 1$ and $f(u) = \sqrt{u}$, then $du = 2 \, dx$ and

$$\int_0^1 \sqrt{2x+1} \, dx = \int_1^3 \frac{u \, du}{2} = \left( \frac{1}{2} \right) \left( \frac{2}{3} \right) \left[ u^{\frac{3}{2}} \right]_1^3 = \frac{1}{3} \left( 3^{\frac{3}{2}} - 1 \right).$$

Example 19  Evaluate the integral $\int_0^1 \ln(x) \, dx$.

Here, let $f(x) = \ln(x)$ and $Dg(x) = dx$. Then $df(x) = \frac{1}{x} \, dx$ and $g(x) = x$. Using the integration by parts formula:

$$\int_0^1 \ln(x) \, dx = \ln(1) \times 1 - \ln(0) \times 0 - \int_0^1 dx = -[x]_0^1 = -1.$$  

Both of these calculations can be verified by plotting the area under the curves over the range $[0, 1]$. Example 18 depicts a concave function with value 1 at $x = 0$ and value $\sqrt{3}$ at $x = 1$. The area under the curve (the area between the curve and the x-axis) contains a square region with area 1 (with vertices $(0, 0), (0, 1), (1, 1)$, and $(1, 0)$) and a curved triangular region with vertices $(0, 1), (1, \sqrt{3})$, and $(1, 1)$. The summed area equals the integral value $\frac{1}{3} \left( 3^{\frac{3}{2}} - 1 \right) \approx 1.4$.

Negative values for the integral of a function correspond to an area whose upper bound is the x-axis and whose lower bound is the function itself. Example 19 depicts a concave function that diverges to a value of $-\infty$ at $x = 0$ and equals a value of 0 at $x = 1$. This curve lies entirely below the x-axis. The area between the x-axis and this curve equals 1, and the integral value equals $-1$, as we are evaluating an area below the x-axis.
2. MATHEMATICAL PRELIMINARIES
Bibliography


3

Microfoundations

3.1 Utility functions

3.1.1 Sneak peek

Summary

Economic choice is predicated on the fact that economic agents have preferences. A preference relation is a mathematical relation between one bundle of commodities and another. Strictly speaking, an individual may prefer the first bundle over the second, may prefer the second over the first, or may be indifferent between the two. By specifying such relations across all possible bundles of commodities, an individual has a complete preference relation. Notice that preference relations are only answers to the question "which bundle do you prefer?" They have nothing to do with the cost of a bundle. Cost is a market phenomenon and is only relevant when we consider budget constraints.

Though elegant and consistent with our understanding of psychology, preferences are incredibly difficult to work with. For that reason, economists use utility functions instead. A utility function is a representation of an individual’s preference relations. If an individual prefers bundle 1 over bundle 2, then the utility must be higher for bundle 1.

Working with utility functions that represent individual’s preference relations, we are able to determine which bundles maximize the function and how the maximizer changes when constraints are imposed.

This manuscript will use the term ‘household’ and ‘consumer’ to refer to the economic agent characterized by a utility function. A household is the fundamental unit of analysis.
in economics and is present in all microfounded models (equivalently called agent-based models). Since households are ubiquitous and all households have a utility function, it is essential to master the properties associated with utility functions.

**Notation**

The variables/parameters to be introduced in this section are given in the following table:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_i$</td>
<td>consumption of commodity $i$</td>
</tr>
<tr>
<td>$\alpha_i$</td>
<td>expected utility weights for commodity $i$</td>
</tr>
<tr>
<td>$u : \mathbb{R}_+ \to \mathbb{R}$</td>
<td>utility function</td>
</tr>
</tbody>
</table>

**Main takeaways**

After completing this section, you will be able to answer the following questions:

- What are the three main assumptions on utility functions in economics?
- Why do we make these assumptions?
- What are some examples of common utility functions?

### 3.1.2 Commodities and utility

An economic model consists of the set of agents, the set of choices available to the agents, and a description of the environment in which they interact.

The basic economic model introduced in this chapter will only contain households and markets. Agents such as firms and governments will be rolled out in the models in upcoming chapters. An economy refers to a certain set of characteristics or parameters for the households and markets: specifically, the number of households, the number of commodities, the household endowments, and the household utility functions.

Knowing their utility function, a household seeks to maximize utility by choosing a bundle of consumption. A consumption bundle consists of the amount that the household consumes of each commodity. Naturally, these consumption amounts must be nonnegative.

This chapter will only consider models of pure exchange, meaning that the models will not contain firms nor production. This chapter will also only consider economies with 2 households and 2 commodities. All of the methods for computing equilibria and all of the
theorems (in particular the basic welfare theorems) remain true for economies with firms and for economies with any finite number of households and commodities.

In this chapter, we consider utility functions that are of the expected utility form. With two commodities, the expected utility form is:

$$\alpha_1 u(x_1) + \alpha_2 u(x_2).$$

Here $x_1$ represents the consumption of commodity 1 and $x_2$ the consumption of commodity 2. These are variables of the model. The parameters are the utility weights $\alpha_1$ and $\alpha_2$ and the Bernoulli utility function $u : \mathbb{R}_+ \to \mathbb{R}$. For simplicity, I refer to $u$ simply as the utility function. The utility function satisfies 3 assumptions:

1. Smoothness assumption
2. Monotonicity assumption
3. Concavity assumption

Mathematically, the utility function will be assumed to be $C^2$ (smoothness assumption), strictly increasing (monotonicity assumption), and strictly concave (concavity assumption).

### 3.1.3 Smoothness assumption

The first assumption involves the smoothness of the utility function. The function $u : \mathbb{R}_+ \to \mathbb{R}$ is continuous (denoted $C^0$) if there are no jumps or breaks in the function. In other words, if you take a pencil to trace out the function and do not need to pick your pencil up at any time, then the function is continuous.

A stronger requirement is that the function $u : \mathbb{R}_+ \to \mathbb{R}$ is differentiable. This means that there are no kinks anywhere in the function and a derivative can be evaluated along the entire length of the function. Any function that is differentiable must also be continuous, but the reverse need not be true. For example, the function $f(x) = |x|$ is continuous, but is not differentiable (it has a kink, or point of non-differentiability at 0).

If a function is differentiable, then we can evaluate the derivative. The derivative is a new function obtained by taking a linear approximation of the original function at each point in the domain. Thankfully, for our sake, we have well-known rules for taking derivatives.

A function is continuously differentiable (denoted $C^1$) if its derivative is continuous. If a function’s derivative is also differentiable, then the function is twice differentiable and a
new function called the second derivative can be derived. The same rules of differentiation are applied to the first derivative in order to compute the second derivative.

A function is twice continuously differentiable (denoted $C^2$) if its second derivative is continuous. We could keep going, but in economics we only need to analyze first and second order conditions of functions, meaning that we only need to be able to take first and second derivatives.

### 3.1.4 Monotonicity assumption

The second assumption is monotonicity. This assumption captures the idea that more is better. The utility function $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is assumed to be strictly increasing, which means that if $a > b$, then $u(a) > u(b)$. This definition is equivalent to a strictly positive first derivative:

$$Du(x) > 0 \text{ for all } x. \quad (3.1)$$

### 3.1.5 Concavity assumption

The third assumption is concavity. The utility function $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is concave if

$$u(\theta a + (1 - \theta)b) \geq \theta u(a) + (1 - \theta)u(b) \quad (3.2)$$

for any $a, b \in \mathbb{R}$ and any $\theta \in [0, 1]$.

This definition is equivalent to a non-positive second derivative:

$$D^2u(x) \leq 0 \text{ for all } x.$$

If we wish to assume a strong form of concavity, called strict concavity, then all of the weak inequalities in the above expressions are replaced by strict inequalities. Specifically, the utility function $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is strictly concave if

$$u(\theta a + (1 - \theta)b) > \theta u(a) + (1 - \theta)u(b) \quad (3.3)$$

for any $a, b \in \mathbb{R}$ with $a \neq b$ and for any $\theta \in (0, 1)$. This is equivalent to a strictly negative second derivative:

$$D^2u(x) < 0 \text{ for all } x. \quad (3.4)$$
3.1. UTILITY FUNCTIONS

3.1.6 Examples

We will now consider several examples of utility functions that satisfy the 3 assumptions above. Use these examples as practice taking first and second derivatives (or as a refresher for those well-versed in such tasks).

1. Cobb-Douglas: \( u(x) = \ln(x) \). The first derivatives are strictly positive, \( Du(x) = \frac{1}{x} \), and the second derivatives are strictly negative, \( D^2u(x) = -\frac{1}{x^2} \). A Cobb-Douglas utility function is \( C^2 \), strictly increasing, and strictly concave.

2. Constant relative risk-aversion (CRRA): \( u(x) = \frac{x^{1-\rho}}{1-\rho} \) for \( \rho \neq 1 \). The first derivatives are strictly positive, \( Du(x) = x^{-\rho} \), and the second derivatives are strictly negative, \( D^2u(x) = -\rho x^{-\rho-1} \). A CRRA utility function is \( C^2 \), strictly increasing, and strictly concave.

Relative risk aversion, at any point \( x \), is defined as:

\[
RRA(x) = -x \frac{D^2u(x)}{Du(x)}.
\] (3.5)

For utility functions of this form, the relative risk aversion is constant:

\[
RRA(x) = -x \frac{(-\rho x^{-\rho-1})}{x^{-\rho}} = \rho.
\] (3.6)

3. Constant absolute risk-aversion (CARA): \( u(x) = -e^{-\kappa x} \) for \( \kappa > 0 \). The first derivatives are strictly positive, \( Du(x) = \kappa e^{-\kappa x} \), and the second derivatives are strictly negative, \( D^2u(x) = -\kappa^2 e^{-\kappa x} \). A CARA utility function is \( C^2 \), strictly increasing, and strictly concave.

Absolute risk aversion, at any point \( x \), is defined as

\[
ARA(x) = -\frac{D^2u(x)}{Du(x)}.
\] (3.7)

For utility functions of this form, the absolute risk aversion is constant:

\[
ARA(x) = -\frac{-\kappa^2 e^{-\kappa x}}{\kappa e^{-\kappa x}} = \kappa.
\] (3.8)

4. Quadratic utility: \( u(x) = -(A - x)^2 \) for \( A \) large. The first derivatives are strictly
positive, \( Du(x) = 2(A - x) \), provided that \( A \) is always larger than any consumption amount that the household could possibly choose. The second derivatives are non-positive, \( D^2u(x) = -2 \). The third derivative equals 0, \( D^3u(x) = 0 \). Quadratic utility is also referred to as mean-variance utility since the third derivative implies that the optimal choices of households only depend upon the mean and variance of a distribution. A quadratic utility function is \( C^2 \), strictly increasing, and strictly concave.

5. Linear utility: \( u(x) = ax \) for \( a > 0 \). The first derivatives are strictly positive, \( Du(x) = a \), and the second derivatives are non-positive, \( D^2u(x) = 0 \). A linear utility function is \( C^2 \), strictly increasing, and concave. Notice that the utility function is only concave, and not strictly concave, since the second derivative is non-positive (\( \leq 0 \)), but not strictly negative (\( < 0 \)).

### 3.2 Equilibrium

#### 3.2.1 Sneak peek

**Summary**

An equilibrium in economics can be described as a system at rest. Similar to a biological system, an economic system consists of many parts that each respond to stimuli and are mutually dependent. The setting considered in this chapter has markets that are perfectly competitive. In such a setting, households are price-takers, meaning that they take the prices as given and do not internalize how their own choices affect the values for the prices. Taking the prices as given, households maximize a utility function subject to budget constraints.

The budget constraint is measured in the unit of account (dollars when working with the US example). The budget constraint requires that the value of the expenditures is less than or equal to the value of income. In every period in which markets open, each household must satisfy its budget constraint in that period.

Though they are unaware of it, and only make decisions based upon the prices, households in the economy are mutually dependent. This is because the total resources that households demand cannot exceed the total resources available for consumption. This condition is equivalently called an aggregate consistency or market clearing condition.

A commodity market clearing condition is specified in units of the physical commodity. There is a market clearing condition for every market in the economy. Markets may exist for
the trading of commodities (as considered in this chapter), for the trading of capital, for the
trading of labor, for the trading of money, and for the trading of assets (all to be considered
in future chapters). For a commodity market, the market clearing condition requires that
the number of units of commodity consumed (totaled across all agents) must be equal to the
number of units of commodity available.

In equilibrium, the prices adjust so that the market clearing conditions are satisfied.
Households respond to the incentive of prices and typically decrease demand in response to
a price increase and increase demand in response to a price decrease. Using prices as a lever,
the market is able to ensure that the total equilibrium demand is exactly equal to the total
available resources.

**Notation**

The variables/parameters to be introduced in this section are given in the following table:

- $x^h_i$: the consumption of commodity $i$ by household $h$
- $e^h_i$: the endowment of commodity $i$ by household $h$
- $p_i$: price for commodity $i$

**Main takeaways**

After completing this section, you will be able to answer the following questions:

- What optimization problem do households solve in an equilibrium?
- What is a market clearing condition?
- How is an Arrow-Debreu equilibrium defined?

**3.2.2 The model setup**

The economy contains 2 households and 2 commodities. Households may have different
utility functions. The utility function for household 1 is given by:

$$
\alpha_1^1 u^1 (x^1_1) + \alpha_2^1 u^1 (x^1_2)
$$

The utility function for household 2 is given by:

$$
\alpha_1^2 u^2 (x^2_1) + \alpha_2^2 u^2 (x^2_2)
$$
Each household is endowed with $e^h_1$ units of commodity 1 and $e^h_2$ units of commodity 2. The endowments are parameters of the model. Endowments are strictly positive. Economies in which the total resources of the economy are specified by endowments are called pure-exchange economies. This is in contrast to production economies in which the total resources of the economy are determined by production decisions of firms.

Households, taking the prices as given, trade to reach a consumption bundle that maximizes utility. The commodity prices are $p_1$ and $p_2$. These are variables in the model. Prices are non-negative. Households take the prices as given when solving their optimization problems.

### 3.2.3 Budget constraint

The household budget constraint is:

$$p_1 x^h_1 + p_2 x^h_2 \leq p_1 e^h_1 + p_2 e^h_2.$$  
(3.9)

The left-hand side of the budget constraint is the value of the expenditure on the two commodities. The right-hand side of the budget constraint is the household income, which equals the value received from selling endowments.

Notice in the household budget constraint that if both prices $p_1$ and $p_2$ were to double, then both the expenditure and the income would double. This would not change the optimal decision made by the household about which consumption bundle maximizes utility. What truly matters for the household’s consumption decision is the relative price $\frac{p_2}{p_1}$ (or the relative price $\frac{p_1}{p_2}$; it does not matter which one is used). We typically make a price normalization in order to focus on the only price variable of interest: the relative price. Here, we will adopt the price normalization $p_2 = 1$.

### 3.2.4 Household maximization problem

The consumption bundle is chosen to solve the following utility maximization problem:

$$\begin{align*}
\text{maximize} & \quad \alpha^h_1 u^h (x^h_1) + \alpha^h_2 u^h (x^h_2) \\
\text{subject to} & \quad p_1 x^h_1 + x^h_2 \leq p_1 e^h_1 + e^h_2.
\end{align*}$$  
(3.10)
3.2. EQUILIBRIUM

Notice that we have included our price normalization $p_2 = 1$. The maximization problem is a constrained maximization problem.

The optimal solutions to household maximization problems are found using the Kuhn-Tucker conditions. Under our assumptions on the utility function ($C^2$, strictly increasing, and strictly concave), the Kuhn-Tucker conditions tell us that (i) the prices are strictly positive for all commodities and (ii) the budget constraint holds with equality.

3.2.5 Equilibrium

We are now prepared to define an Arrow-Debreu equilibrium, which is also called a Walrasian equilibrium or a competitive equilibrium. This is the most fundamental concept of equilibrium in all of economics as it represents the operation of competitive markets in a static environment. An Arrow-Debreu equilibrium consists of price $p_1$ and allocation $(x_1^1, x_1^2, x_2^1, x_2^2)$.

**Definition 3.1** An Arrow-Debreu equilibrium is price $p_1$ and allocation $(x_1^1, x_1^2, x_2^1, x_2^2)$ such that

1. For both households, taking as given the price $p_1$, the commodity bundle $(x_h^1, x_h^2)$ is an optimal solution to the household problem

$$
\begin{align*}
\text{maximize} & \quad \alpha_1^h u^h (x_1^h) + \alpha_2^h u^h (x_2^h) \\
\text{subject to} & \quad p_1 x_1^h + x_2^h \leq p_1 e_1^h + e_2^h
\end{align*}
$$

(3.11)

2. Markets clear

$$
\begin{align*}
x_1^1 + x_1^2 &= e_1^1 + e_1^2. \\
x_2^1 + x_2^2 &= e_2^1 + e_2^2.
\end{align*}
$$

(3.12) (3.13)

The first condition is household utility maximization. The second condition says that the equilibrium prices are determined so that the commodity markets clear. Specifically, the total consumption of commodity 1 must be equal to the total amount of commodity 1 available in the economy (equal to the total endowment of commodity 1). The same condition must hold for commodity 2.
In equilibrium, the budget constraints hold with equality (since utility is strictly increasing). Let’s sum the budget constraints over all households. The resulting equation is referred to as Walras’ Law:

\[
(p_1 x_1^1 + x_2^1) + (p_1 x_1^2 + x_2^2) = (p_1 e_1^1 + e_2^1) + (p_1 e_1^2 + e_2^2).
\] (3.14)

In words, Walras’ Law states that the total expenditures of the households must be equal to the total income of the households. This is a fundamental accounting property of economics.

Let’s rearrange the Walras’ Law equation:

\[
p_1 (x_1^1 + x_1^2) + (x_2^1 + x_2^2) = p_1 (e_1^1 + e_1^2) + (e_2^1 + e_2^2).
\] (3.15)

If market clearing holds for commodity 1, then Walras Law reduces to:

\[
(x_2^1 + x_2^2) = (e_2^1 + e_2^2).
\] (3.16)

This is identical to the market clearing for commodity 2. Thus, only one of the market clearing conditions is independent. If one of the market clearing conditions holds, the other one holds automatically.

### 3.3 Pareto efficiency

#### 3.3.1 Sneak peek

Summary

An equilibrium results in a market-based allocation of available resources. In each market, there must be a market price. There are numerous different settings that we will consider, meaning that there are many ways in which the market price is determined in equilibrium. In the model we have seen thus far, the market price is set in an abstract manner by the faceless market whose only desire is to satisfy market clearing conditions. This is called the competitive markets setting.

In contrast to a market-based allocation of resources, we can consider the allocation of resources as chosen by a central planner. The allocation problem of a central planner is referred to as the planner’s problem. The planner does not need markets in order to allocate goods. For this reason, the planner’s problem never includes prices.
3.3. PARETO EFFICIENCY

An aggregate resource constraint is a materials balance equation that is measured in units of the physical commodity. For each commodity, there is a separate aggregate resource constraint. The constraint requires that the total consumption of a particular commodity is equal to the total amount of that commodity available. This is identical to a market clearing condition, as previously introduced, but we refrain from using the terminology market clearing condition as the planner resource allocation problem operates without the use of the market mechanism.

The planner in our setting is benevolent and omnipotent. Economists use the notion of a central planner not in homage to centrally planned economies (which don’t work), but rather as a baseline with which to compare the market-based allocation. The solution of the planner’s problem is an efficient allocation of resources. This allocation would be obtained if all participants in the economy were to gather together and make decisions for the "common good."

There are many notions of efficiency considered by economists, but by far the most common is the concept of Pareto efficiency. An allocation satisfies Pareto efficiency if it is not possible to make one household strictly better off without making some other household strictly worse off.

Notation

The variables to be introduced in this section are given in the following table:

\[ \mu^h \]  the Pareto weight for household \( h \)

Main takeaways

After completing this section, you will be able to answer the following questions:

- What does it mean to be a feasible allocation?
- What does it mean to be a Pareto efficient allocation?
- What problem does the planner solve in order to determine a Pareto efficient allocation?

3.3.2 Defining Pareto efficiency

To begin, I must first define what is meant by a feasible allocation.
**Definition 3.2** A feasible allocation is the consumption vectors \((x_1^1, x_2^1, x_1^2, x_2^2)\) such that

1. (consumption is non-negative)

\[
x_1^1 \geq 0 \quad x_2^1 \geq 0 \\
x_1^2 \geq 0 \quad x_2^2 \geq 0
\]  
(3.17)

2. (aggregate resource constraint)

\[
x_1^1 + x_2^1 = e_1^1 + e_1^2. \\
x_1^2 + x_2^2 = e_2^1 + e_2^2.
\]  
(3.18)

(3.19)

**Definition 3.3** A Pareto efficient allocation is a feasible allocation \((x_1^1, x_2^1, x_1^2, x_2^2)\) such that there does not exist another feasible allocation \((\hat{x}_1^1, \hat{x}_2^1, \hat{x}_1^2, \hat{x}_2^2)\) where either:

\[
\alpha_1^1 u^1(\hat{x}_1^1) + \alpha_2^1 u^1(\hat{x}_2^1) > \alpha_1^1 u^1(x_1^1) + \alpha_2^1 u^1(x_2^1) \quad \text{and} \quad \alpha_1^2 u^2(\hat{x}_1^2) + \alpha_2^2 u^2(\hat{x}_2^2) \geq \alpha_1^2 u^2(x_1^2) + \alpha_2^2 u^2(x_2^2)
\]

or

\[
\alpha_1^1 u^1(\hat{x}_1^1) + \alpha_2^1 u^1(\hat{x}_2^1) \geq \alpha_1^1 u^1(x_1^1) + \alpha_2^1 u^1(x_2^1) \quad \text{and} \quad \alpha_1^2 u^2(\hat{x}_1^2) + \alpha_2^2 u^2(\hat{x}_2^2) > \alpha_1^2 u^2(x_1^2) + \alpha_2^2 u^2(x_2^2)
\]

In words, an allocation is Pareto efficient if it satisfies two requirements: (i) it is feasible and (ii) there does not exist another feasible allocation that makes all households at least as well off and some households strictly better off.

Colloquially, a Pareto efficient allocation is such that it is not possible to make one household strictly better off without making some other household strictly worse off.

### 3.3.3 Optimization problem

I denote the Pareto set as the set of Pareto efficient allocations. There are many Pareto efficient allocations. To see some examples of Pareto efficient allocations, observe that the allocation that gives all resources to household \(h = 1\) is Pareto efficient. This is Pareto efficient, because it is not possible to make \(h = 2\) better off without making \(h = 1\) worse off. Likewise, it is Pareto efficient to give all resources to household \(h = 2\).

We are really interested in more equitable allocations of resources in which each of the households receives some of the resources. A Pareto efficient allocation is one that is an
optimal solution to the planner’s problem:

\[
\text{maximize} \quad \mu^1 \left[ \alpha_1^1 u_1^1 (x_1^1) + \alpha_2^1 u_1^2 (x_1^2) \right] + \mu^2 \left[ \alpha_1^2 u_2^1 (x_2^1) + \alpha_2^2 u_2^2 (x_2^2) \right],
\]

subject to \( (x_1^1, x_1^2, x_2^1, x_2^2) \) is a feasible allocation \((3.20)\).

Moreover, any solution to the above maximization problem must be a Pareto efficient allocation.

The objective function of the planner is the weighted sum of household utilities. The parameters \((\mu^1, \mu^2)\) are called the Pareto weights. They must satisfy \(\mu^1 + \mu^2 = 1\) and both \(\mu^1 \geq 0\) and \(\mu^2 \geq 0\). If the planner increases the weight \(\mu^1\) assigned to household 1, then household 1 will receive higher utility from the Pareto efficient allocation. If \((\mu^1, \mu^2) = (1, 0)\) (as an extreme example), then all resources are allocated to household 1:

\[
(x_1^1, x_2^1) = (e_1^1 + e_1^2, e_2^1 + e_2^2).
\]

Each different value for \(\mu^1\) (under the normalization, \(\mu^2 = 1 - \mu^1\)) corresponds to a different Pareto efficient allocation.

## 3.4 Equilibrium equations

### 3.4.1 Sneak peek

**Summary**

The definition of equilibrium was very precise. Our analysis of economic models must be very precise as we will translate these models into mathematical systems of equations in order to derive testable predictions about economic behavior.

If you recall, the equilibrium definition could be broken down into three types of conditions: (i) households maximize utility, (ii) household budget constraints, and (iii) market clearing conditions. The latter two types of conditions are already in the form of equations that we can use in our system of equilibrium equations.

What we need are equations that are necessary and sufficient conditions for households to solve a constrained maximization problem. This is where we utilize our knowledge of optimization. Specifically, the Kuhn-Tucker conditions are necessary and sufficient conditions for an optimal solution to a constrained maximization problem, provided that certain
assumptions on the objective function and constraints functions are met.

In the model we will be working with in this chapter, those certain assumptions are met. Thus, to derive our system of equilibrium equations, we will need to find the Kuhn-Tucker conditions associated with the household utility maximization problems.

I emphasize that the equilibrium system of equations includes: (i) Kuhn-Tucker conditions associated with household optimization, (ii) household budget constraints that hold with equality, and (iii) market clearing conditions.

### Notation

The variables to be introduced in this section are given in the following table:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda^h$</td>
<td>Lagrange multiplier for the household budget constraint</td>
</tr>
</tbody>
</table>

### Main takeaways

After completing this section, you will be able to answer the following questions:

- What conditions must be satisfied in an Arrow-Debreu equilibrium?
- What are the steps to solve for an Arrow-Debreu equilibrium (price and allocation)?

### 3.4.2 Kuhn-Tucker conditions

An Arrow-Debreu equilibrium is defined as a vector that satisfies both the household optimization problems and the market clearing conditions. The solutions to the optimization problems must satisfy the Kuhn-Tucker conditions. For all utility functions we will consider, the equilibrium choices $(x_1^h, x_2^h)$ are always interior solutions, meaning $x_1^h > 0$ and $x_2^h > 0$. We do not need to include the non-negativity constraints $x_1^h \geq 0$ and $x_2^h \geq 0$ as constraints in our constrained maximization problem.

Consider the household utility maximization problem

\[
\text{maximize} \quad \alpha_1^h u^h(x_1^h) + \alpha_2^h u^h(x_2^h) \\
\text{subject to} \quad p_1 e_1^h + e_2^h - p_1 x_1^h - x_2^h \geq 0. 
\]  

(3.21)

The Kuhn-Tucker conditions for this problem are:
3.4. EQUILIBRIUM EQUATIONS

- **First order conditions**

\[ \alpha_1^h Du^h (x_1^h) - \lambda^h p_1 = 0. \]  
\[ \alpha_2^h Du^h (x_2^h) - \lambda = 0. \]

- **Complimentary slackness conditions**

\[ \lambda^h (p_1 e_1^h + e_2^h - p_1 x_1^h - x_2^h) = 0. \]

The Lagrange multiplier \( \lambda^h \geq 0 \) corresponds to the budget constraint in the household \( h \) optimization problem.

Since \( u^h \) is strictly increasing, then \( \alpha_2^h Du^h (x_2^h) > 0 \) from the second equation. This implies that \( \lambda^h > 0 \). From the third equation, it must be that

\[ p_1 e_1^h + e_2^h - p_1 x_1^h - x_2^h = 0, \]

or in other words, that the budget constraint holds with equality. Also, since \( u^h \) is strictly increasing, then \( \alpha_1^h Du^h (x_1^h) > 0 \) from the first equation. This implies that \( p_1 > 0 \).

The market clearing conditions are:

\[ x_1^1 + x_2^2 = e_1^1 + e_1^2. \]  
\[ x_2^1 + x_2^2 = e_2^1 + e_2^2. \]

3.4.3 Solving for an equilibrium

Here are the steps to solve for an equilibrium of a two household and two commodity economy.

1. Normalize \( p_2 = 1 \).

2. Solve for the consumption choices of household \( h \) as a function of \( p_1 \). To do this, we must solve the first order conditions and the budget constraint. The typical approach is to use the first order conditions in order to derive equations for \( (x_1^h, x_2^h) \). We can then take these equations for \( (x_1^h, x_2^h) \) and plug them into the budget constraint to obtain an equation for \( \lambda^h \). Finally, this equation for \( \lambda^h \) is plugged back into the original equations for \( (x_1^h, x_2^h) \). We should end up with \( (x_1^h, x_2^h) \) only as a function of \( p_1 \) and the parameters \( (\alpha_1^h, \alpha_2^h, e_1^h, e_2^h) \).
3. We repeat the same procedure for all households.

4. What we have are the consumption choices for all households as a function of \( p_1 \). These are called demand functions. We now use the market clearing condition. From the discussion of Walras’ Law, we know that one of the market clearing conditions is redundant. It does not matter which one we use, but it is typically easier to use the market clearing condition for commodity 2:

\[
x_2^1 + x_2^2 = e_2^1 + e_2^2. \tag{3.26}
\]

Using this equation, we can solve for the equilibrium price \( p_1 \) as a function of the parameters \( (\alpha_1^h, \alpha_2^h, e_1^h, e_2^h)_{h \in \{1,2\}} \).

5. Using this expression for the price, we can then plug this back into our equations for \( (x_1^h, x_2^h)_{h \in \{1,2\}} \) and obtain expressions for \( (x_1^h, x_2^h)_{h \in \{1,2\}} \) only in terms of parameters \( (\alpha_1^h, \alpha_2^h, e_1^h, e_2^h)_{h \in \{1,2\}} \).

The following subsection walks through these steps for a particular economy.

### 3.4.4 Example: Solving for an equilibrium

Consider an economy with two households and two commodities. The utility function for household 1 is:

\[
\alpha_1^1 u^1 (x_1^1) + \alpha_2^1 u^1 (x_2^1) = \frac{1}{2} \sqrt{x_1^1} + \frac{1}{2} \sqrt{x_2^1} \tag{3.27}
\]

and the household 1 endowment is \( (e_1^1, e_2^1) = (1, 2) \). The utility function for household 2 is

\[
\alpha_1^2 u^2 (x_1^2) + \alpha_2^2 u^2 (x_2^2) = \frac{1}{2} \ln (x_1^2) + \frac{1}{2} \ln (x_2^2) \tag{3.28}
\]

and the household 2 endowment is \( (e_1^2, e_2^2) = (2, 1) \).

**First step**

The first step is to normalize \( p_2 = 1 \).

**Second step**

The second step is to find the optimal choices for household \( h = 1 \). Denote \( \lambda^1 \) as the Lagrange multiplier associated with the budget constraint in the household \( h = 1 \) utility maximization.
3.4. EQUILIBRIUM EQUATIONS

problem

\[
\begin{aligned}
\text{maximize} & \quad \frac{1}{2} \sqrt{x_1^1} + \frac{1}{2} \sqrt{x_2^1} \\
\text{subject to} & \quad p_1 + 2 - p_1 x_1^1 - x_2^1 \geq 0.
\end{aligned}
\]  (3.29)

The first order conditions are given by:

\[
\begin{aligned}
0.25 (x_1^1)^{-0.5} - \lambda^1 p_1 &= 0. \\
0.25 (x_2^1)^{-0.5} - \lambda^1 &= 0.
\end{aligned}
\]  (3.30)

Solve the first order conditions for \((x_1^1, x_2^1)\):

\[
\begin{aligned}
x_1^1 &= \frac{1}{(4\lambda^1 p_1)^2}. \\
x_2^1 &= \frac{1}{(4\lambda^1)^2}.
\end{aligned}
\]  (3.31)

The budget constraint states that \(p_1 x_1^1 + x_2^1 = p_1 + 2\), where the left-hand side is given by:

\[
p_1 \left( \frac{1}{(4\lambda^1 p_1)^2} + \frac{1}{(4\lambda^1)^2} \right) = \frac{1}{p_1 (4\lambda^1)^2} + \frac{1}{(4\lambda^1)^2}.
\]  (3.32)

From the budget constraint:

\[
\frac{1}{p_1} \frac{1}{(4\lambda^1)^2} + \frac{1}{(4\lambda^1)^2} = p_1 + 2,
\]  (3.33)

so the term \(\frac{1}{(4\lambda^1)^2}\) is given by:

\[
\frac{1}{(4\lambda^1)^2} \left( \frac{1}{p_1} + 1 \right) = p_1 + 2
\]  (3.34)

Demand functions are, by definition, the equations for the consumptions as a function of
the price. The demand functions are given by:

\begin{align*}
x_1^1 &= \frac{1}{(p_1)^2} \frac{1}{(4 \lambda_1)^2} = \frac{(p_1 + 2)}{p_1 (1 + p_1)} \\
x_2^1 &= \frac{1}{(4 \lambda_1)^2} = \frac{p_1 (p_1 + 2)}{1 + p_1}.
\end{align*}

Third step

The third step is to find the optimal choices for household $h = 2$. Denote $\lambda^2$ as the Lagrange multiplier associated with the budget constraint in the household $h = 2$ utility maximization problem

\begin{equation}
\begin{aligned}
\text{maximize} & \quad \frac{1}{2} \ln (x_1^2) + \frac{1}{2} \ln (x_2^2) \\
\text{subject to} & \quad 2p_1 + 1 - p_1 x_1^2 - x_2^2 \geq 0.
\end{aligned}
\end{equation}

The first order conditions for household $h = 2$ are given by:

\begin{align*}
0.5 \frac{x_1^2}{x_1^2} - \lambda^2 p_1 &= 0. \quad (3.37) \\
0.5 \frac{x_2^2}{x_2^2} - \lambda^2 &= 0.
\end{align*}

Solving for $x^2$ as a function of $\lambda^2$ and plugging into the budget constraint $p_1 x_1^2 + x_2^2 = 2p_1 + 1$ yields:

\begin{equation}
\frac{0.5}{\lambda^2} + \frac{0.5}{\lambda^2} = 2p_1 + 1. \quad (3.38)
\end{equation}

This implies that $\frac{1}{\lambda^2} = 2p_1 + 1$. The demand functions for household $h = 2$ are:

\begin{equation}
x_1^2 = \frac{p_1 + 0.5}{p_1} \quad \text{and} \quad x_2^2 = p_1 + 0.5. \quad (3.39)
\end{equation}

Fourth step

The fourth step is to determine the value for the variable $p_1$ from one of the market clearing conditions. Recall that one of the market clearing conditions is always redundant, so it does not matter which one we pick. It is typically easier to use the market clearing condition for the second commodity:

\begin{equation}
x_1^2 + x_2^2 = e_1^2 + e_2^2 = 3. \quad (3.40)
\end{equation}
Using the demand functions:

\[
\frac{p_1 (p_1 + 2)}{1 + p_1} + p_1 + 0.5 = 3. \tag{3.41}
\]

This equation can be solved for \( p_1 \):

\[
p_1 (p_1 + 2) + (p_1 + 0.5) (p_1 + 1) = 3 (p_1 + 1). \tag{3.42}
\]

The equation is of the quadratic form:

\[
(p_1)^2 + 2p_1 + (p_1)^2 + 1.5p_1 + 0.5 = 3p_1 + 3. \tag{3.43}
\]

\[
2 (p_1)^2 + 0.5p_1 - 2.5 = 0.
\]

\[
4 (p_1)^2 + p_1 - 5 = 0.
\]

The quadratic equation can be factored:

\[
4 (p_1)^2 + p_1 - 5 = (4p_1 + 5) (p_1 - 1). \tag{3.44}
\]

This says that

\[
(4p_1 + 5) (p_1 - 1) = 0, \tag{3.45}
\]

meaning that \( p_1 = -\frac{5}{4} \) and \( p_1 = 1 \). Obviously, the only economic answer to this problem is \( p_1 = 1 \) as prices are always positive.

**Fifth step**

We can then go back and find the Arrow-Debreu equilibrium allocations:

\[
x_1^1 = \frac{(p_1 + 2)}{p_1 (1 + p_1)} = \frac{3}{2},
\]

\[
x_2^1 = \frac{p_1 (p_1 + 2)}{1 + p_1} = \frac{3}{2},
\]

\[
x_1^2 = \frac{p_1 + 0.5}{p_1} = \frac{3}{2},
\]

\[
x_2^2 = p_1 + 0.5 = \frac{3}{2}.
\]
3.5 Pareto efficiency equations

3.5.1 Sneak peek

Summary

A Pareto efficient allocation is a solution to the planner’s problem. The reason that we introduced such a constrained maximization problem is that we are now able to follow a similar methodology as the previous section and use the Kuhn-Tucker conditions associated with this constrained maximization problem to determine a Pareto efficient allocation.

As before, the Kuhn-Tucker conditions are necessary and sufficient conditions for an optimal solution to the planner’s problem, provided that certain assumptions on the objective function and the constraint functions are satisfied. For the planner’s problem, those assumptions are satisfied.

Keep in mind throughout that an equilibrium solution and the solution to the planner’s problem are the outcomes of two very different concepts of resource allocation. It is only when we introduce the basis welfare theorems that we are able to understand the relation between the outcomes of these two different concepts.

I emphasize that the system of equations for a Pareto efficient allocation includes: (i) the Kuhn-Tucker conditions of the planner’s problem and (ii) the aggregate resource constraints for the planner.

Notation

The variables to be introduced in this section are given in the following table:

| \( \nu_i \) | Lagrange multiplier for commodity \( i \) resource constraint |

Main takeaways

After completing this section, you will be able to answer the following questions:

- What conditions must be satisfied by a Pareto efficient allocation?

- Is there only one Pareto efficient allocation, or an entire set?
3.5.2 Kuhn-Tucker conditions

The planner’s problem is used to find the set of all Pareto efficient allocations:

\[
\begin{align*}
\text{maximize} & \quad \mu^1 [\alpha_1^1 u^1 (x_1^1) + \alpha_2^1 u^1 (x_1^2)] + \mu^2 [\alpha_1^2 u^2 (x_2^1) + \alpha_2^2 u^2 (x_2^2)] \\
\text{subject to} & \quad e_1^1 + e_1^2 - x_1^1 - x_2^1 \geq 0. \\
& \quad e_2^1 + e_2^2 - x_1^2 - x_2^2 \geq 0.
\end{align*}
\] (3.47)

Notice here that I have replaced the requirement ‘\( (x_1^1, x_2^1, x_1^2, x_2^2) \) is a feasible allocation’ with the explicit equations from the definition of feasibility. The nonnegativity constraints can be omitted since we know that \( x_1^1 > 0, x_2^1 > 0, x_1^2 > 0, \) and \( x_2^2 > 0 \).

Assign the Lagrange multiplier \( \nu_1 \) to the constraint \( e_1^1 + e_1^2 - x_1^1 - x_2^1 \geq 0 \) and the Lagrange multiplier \( \nu_2 \) to the constraint \( e_2^1 + e_2^2 - x_1^2 - x_2^2 \geq 0 \) (I choose to use different variables than \( \lambda \) in order to avoid confusion).

The Kuhn-Tucker conditions for the planner’s problem are given by:

- **First order conditions**

\[
\begin{align*}
\mu^1 \alpha_1^1 Du^1 (x_1^1) - \nu_1 & = 0. \\
\mu^1 \alpha_2^1 Du^1 (x_1^2) - \nu_2 & = 0. \\
\mu^2 \alpha_1^2 Du^2 (x_2^1) - \nu_1 & = 0. \\
\mu^2 \alpha_2^2 Du^2 (x_2^2) - \nu_2 & = 0.
\end{align*}
\] (3.48)

- **Complimentary slackness conditions**

\[
\begin{align*}
\nu_1 (e_1^1 + e_1^2 - x_1^1 - x_2^1) & = 0, \\
\nu_2 (e_2^1 + e_2^2 - x_1^2 - x_2^2) & = 0.
\end{align*}
\] (3.49)

3.5.3 Pareto efficiency condition

There are 2 first order conditions for household 1 and 2 first order conditions for household 2. If we divide the first of the two first order conditions for household 1 by the second, we arrive at:

\[
\frac{\nu_1}{\nu_2} = \frac{\alpha_1^1 Du^1 (x_1^1)}{\alpha_2^1 Du^1 (x_2^1)}.
\] (3.50)
If we do the same thing for household 2, we arrive at

\[
\frac{\nu_1}{\nu_2} = \frac{\alpha_1^2 Du^2(x_1^2)}{\alpha_2^2 Du^2(x_2^2)}.
\]  

(3.51)

Setting these 2 equations equal yields the Pareto efficiency condition:

\[
\frac{\alpha_1^1 Du^1(x_1^1)}{\alpha_2^1 Du^1(x_2^1)} = \frac{\alpha_2^2 Du^2(x_2^1)}{\alpha_2^2 Du^2(x_2^2)}.
\]  

(3.52)

A Pareto efficient allocation must satisfy this Pareto efficiency condition.

### 3.5.4 Example: Pareto efficiency condition

Let’s verify that the Arrow-Debreu equilibrium allocation found in the previous example satisfies the Pareto efficiency condition. The Arrow-Debreu equilibrium allocation was previously found to be \((x_1^1, x_1^2) = (\frac{3}{2}, \frac{3}{2})\) and \((x_2^1, x_2^2) = (\frac{3}{2}, \frac{3}{2})\).

Consider the Pareto efficiency condition. The left-hand side (the ratio of marginal utilities for household 1) is given by:

\[
\frac{\alpha_1^1 Du^1(x_1^1)}{\alpha_2^1 Du^1(x_2^1)} = \frac{0.25 (x_1^1)^{-0.5}}{0.25 (x_2^1)^{-0.5}} = \frac{x_2^1}{x_1^1}.
\]  

(3.53)

The right-hand side is given by:

\[
\frac{\alpha_2^2 Du^2(x_2^1)}{\alpha_2^2 Du^2(x_2^2)} = \frac{0.5 (x_2^1)^{-1}}{0.5 (x_2^2)^{-1}} = \frac{x_2^2}{x_1^2}.
\]  

(3.54)

We can now plug in the Arrow-Debreu equilibrium allocation and verify that the Pareto efficiency condition is satisfied:

\[
\sqrt{\frac{x_2^1}{x_1^1}} = \sqrt{\frac{1.5}{1.5}} = \frac{x_2^2}{x_1^2} = \frac{1.5}{1.5}.
\]  

(3.55)

Thus, the Arrow-Debreu equilibrium allocation is a Pareto efficient allocation.
3.6 Basic Welfare Theorems

3.6.1 Sneak peek

Summary

Recall the two mechanisms by which resources are allocated in the economy. In an equilibrium, the market mechanism allocates resources. The equilibrium solution is found such that households maximize utility subject to their budget constraints and the markets clear. In a Pareto efficient allocation, the planner controls the allocation of resources without the use of markets. This is equivalent to the planner dictating to the households how much to consume of each commodity. The planner maximizes a weighted sum of household utility subject to an aggregate resource constraint.

There are two facts that are true for pretty much all problem-solving in economics: (i) our objective is to solve for the equilibrium solution and (ii) it is easier to solve for the planner’s solution than the equilibrium solution. If the welfare theorems are valid, the allocation is identical in both the equilibrium solution and the planner’s solution. This means that we can adopt the following two-step approach for economic problem-solving: (i) solve for the allocation using the planner’s problem and (ii) once the allocation is known, solve for the prices in the markets.

Formally, the First Basic Welfare Theorem states that the equilibrium allocation is Pareto efficient.

The Second Basic Welfare Theorem states nearly the converse, namely that any Pareto efficient allocation can be supported as an equilibrium allocation with appropriate transfers between households.

Since economists assess the performance of the economy under the market mechanism and Pareto efficiency is a desirable benchmark, we conclude that the market mechanism cannot be outperformed when the Basic Welfare Theorems are satisfied.

The outcome of the planner’s problem is said to be the centralized outcome as it is centrally determined by the planner. The equilibrium allocation is identical to the planner allocation and is said to be decentralized as it is the outcome of a decentralized mechanism by which markets determine the distribution of resources.

Main takeaways

After completing this section, you will be able to answer the following questions:
• What does the First Basic Welfare Theorem state and what are its assumptions?
• How can we use the Kuhn-Tucker conditions to prove the First Basic Welfare Theorem?
• What does the Second Basic Welfare Theorem state and what are its assumptions?

3.6.2 First Basic Welfare Theorem

The First Basic Welfare Theorem (abbreviated FBWT) states that all Arrow-Debreu equilibrium allocations are Pareto efficient.

**Theorem 3.1 First Basic Welfare Theorem**

All Arrow-Debreu equilibrium allocations are Pareto efficient.

The First Basic Welfare Theorem is a mathematical statement of Adam Smith’s theory of the "invisible hand", in which Smith theorised (olde English spelling) that each individual working to solve its own maximization problem benefits the common good.

3.6.3 Classical proof of the FBWT

Let’s consider the classical proof of the First Basic Welfare Theorem.

The first order conditions of the household optimization problem of an Arrow-Debreu equilibrium are:

\[ \alpha_h^i D u^h(x^h_i) - \lambda^h p_i = 0 \text{ for both } h \text{ and both } i. \quad (3.56) \]

The Pareto efficiency condition is given by:

\[ \frac{\alpha_1^1 D u^1(x^1_1)}{\alpha_2^1 D u^1(x^1_2)} = \frac{\alpha_1^2 D u^2(x^2_1)}{\alpha_2^2 D u^2(x^2_2)} \quad (3.57) \]

Is this Pareto efficiency condition satisfied? Using the first order conditions above, the left-hand side of the Pareto efficiency condition (3.57) is given by:

\[ \frac{\alpha_1^1 D u^1(x^1_1)}{\alpha_2^1 D u^1(x^1_2)} = \frac{\lambda^1 p_1}{\lambda^1 p_2} = \frac{p_1}{p_2}. \quad (3.58) \]

Similarly, using the first order conditions, the right-hand side of the Pareto efficiency condition (3.57) is given by:

\[ \frac{\alpha_1^2 D u^2(x^2_1)}{\alpha_2^2 D u^2(x^2_2)} = \frac{\lambda^2 p_1}{\lambda^2 p_2} = \frac{p_1}{p_2}. \quad (3.59) \]
This means that the Arrow-Debreu equilibrium allocation satisfies the Pareto efficiency condition (3.57). Therefore, the First Basic Welfare Theorem has been verified.

This verification requires that the utility function is differentiable and strictly increasing, but the statement of the theorem and the general proof require no assumptions whatsoever.

### 3.6.4 Second Basic Welfare Theorem

The Second Basic Welfare Theorem (abbreviated SBWT) serves as a partial converse to the FBWT. The SBWT states that for any Pareto efficient allocation, equilibrium prices can be found so that the allocation is an Arrow-Debreu equilibrium allocation.

In the statement of the theorem, the equilibrium notion allows for transfers to the households. The transfers enter the budget constraints, meaning that the budget constraints are updated as:

\[ p_1 x_1^h + x_2^h \leq p_1 e_1^h + e_2^h + \tau^h. \]

The transfer \( \tau^h \) can be either positive or negative. A transfer \( \tau^h > 0 \) refers to a subsidy and a transfer \( \tau^h < 0 \) refers to a tax. The definition of an Arrow-Debreu equilibrium with transfers is an extension of Arrow-Debreu equilibrium in which households’ budget constraints include the transfers.

**Theorem 3.2 Second Basic Welfare Theorem**

*If the allocation \((x_1^1, x_2^1, x_1^2, x_2^2)\) is a Pareto efficient allocation, then there exists a price \(p_1\) and a tax/subsidy transfer scheme \((\tau^1, \tau^2)\) with \(\tau^1 + \tau^2 = 0\) such that \(p_1\) and \((x_1^1, x_2^1, x_1^2, x_2^2)\) is an Arrow-Debreu equilibrium with transfers.*

The SBWT requires a few assumptions, namely that the utility functions are continuous and concave. Such technical details lie outside the scope of this text.

### 3.7 Edgeworth Box

#### 3.7.1 Sneak peek

**Summary**

The previous sections have introduced some of the most important material in neoclassical economic theory (also called general equilibrium theory). The concepts and mechanisms have been abstract. For those requiring a geometric point of view, an Edgeworth box is a
very useful device. An Edgeworth box can be used for an economy with two households and two commodities. Though not ideal as part of an algorithm to evaluate numerical solutions, the Edgeworth box is able to capture all of the features and economic mechanisms listed as "Main takeaways".

Additionally, the Edgeworth box will be a trusted friend in all of the remaining sections of this chapter, and it is good to have such a friend as these sections will investigate increasingly complex aspects of the equilibrium model.

Main takeaways

After completing this section, you will be able to answer the following questions:

- How is household choice determined using budget constraints and indifference curves?
- How do households respond to a price change?
- How are the equilibrium prices determined, i.e., how are the prices set such that the optimal household choices satisfy the market clearing conditions?
- How are the transfers prescribed in the SBWT used in order to support a particular Pareto efficient allocation as an equilibrium allocation?

### 3.7.2 Indifference curves

A useful device to illustrate the market mechanism in an economy with two households and two commodities is the Edgeworth box. For household 1, we can draw a graph containing the indifference curves. This graph is shown in Figure 3.7.1. An indifference curve is the set of all consumption bundles that provide the household with the same utility value.

Under the assumption that the utility function is strictly increasing, the arrow indicates that the indifference curves to the upper-right of the graph provide higher utility values.

Under the assumption that the utility function is strictly concave, the indifference curves have the curvature as drawn in the graph. The indifference curves themselves are convex functions.

We do the exact same thing for household 2, but then rotate the entire graph 180 degrees. This is shown in Figure 3.7.2.
3.7. **EDGEWORTH BOX**

### 3.7.3 Edgeworth box

The Edgeworth box is the combination of the household 1 graph and the household 2 graph rotated 180 degrees. The Edgeworth box is shown in Figure 3.7.3. The Edgeworth box is $e_1^1 + e_1^2$ units wide and $e_2^1 + e_2^2$ units tall. The width of the Edgeworth box is the aggregate endowments of commodity 1 and the height of the Edgeworth box is the aggregate endowments of commodity 2.

If household 1 receives the consumption bundle $(x_1^1, x_2^1)$, then the aggregate resource constraint dictates that household 2 receives the consumption bundle

\[
(x_2^1, x_2^2) = (e_1^1 + e_2^2 - x_1^1, e_1^2 + e_2^2 - x_1^2). \tag{3.60}
\]

Consequently, any point in the Edgeworth box corresponds both to the household 1 allocation and the household 2 allocation. The household 1 allocation values $(x_1^1, x_2^1)$ are found by counting up and right from the lower-left corner, while the household 2 allocation values $(x_2^1, x_2^2)$ are found by counting left and down from the upper-right corner.

For an equilibrium, the vector of prices $p = (p_1, p_2)$ determines a price line or a budget line in the Edgeworth box. The slope of the price line is equal to $-\frac{p_1}{p_2}$. Why is this the case? Let’s determine this from the vantage point of the household 1. Consider a change in commodity 1 consumption by $\Delta x_1$ units. At a price of $p_1$, the change in the household expenditure is $p_1 \Delta x_1$. As the household income has not changed, the total change in the household expenditure across both commodities must sum to 0:

\[
p_1 \Delta x_1 + p_2 \Delta x_2 = 0. \tag{3.61}
\]

This implies that the change in commodity 2 consumption is equal to:

\[
\Delta x_2 = -\frac{p_1}{p_2} \Delta x_1. \tag{3.62}
\]

So if commodity 1 consumption changes by 1 unit, then commodity 2 consumption changes by $-\frac{p_1}{p_2}$ units. This is the slope of the price line.

Households take the price line as given. For any given price line, households solve their utility maximization problem. For any household, an optimal solution to the utility maximization problem (found from the first order conditions of the Kuhn-Tucker conditions) is such that the price line is tangent to the highest indifference curve that the household can
reach (highest utility value). This intersection of the indifference curve and the price line is
the optimal household consumption bundle for the given price line.

Household 1 chooses an optimal bundle (indifference curve tangent to the price line) and
household 2 chooses an optimal bundle (indifference curve tangent to the price line). In
the Edgeworth box, equilibrium requires that the consumption bundle chosen by household
1 must be at exactly the same point as the consumption bundle chosen by household 2. Recall
that household 1 consumption is measured up and right from the lower-left corner
and household 2 consumption is measured down and left from the upper-right corner. The
requirement that both of the optimal solutions occur at the same point in the Edgeworth
box comes from the market clearing conditions.

### 3.7.4 Equilibrium

It is easy to depict an Arrow-Debreu equilibrium in the Edgeworth box (Figure 3.7.4 provides
one example). The equilibrium price line is shown as the straight line that is downward slop-
ing and has slope equal to \(-\frac{p_1}{p_2}\). The price line is said to support the equilibrium allocation,
meaning that the price line passes through the equilibrium allocation.

An optimal solution to the household 1 optimization problem occurs where the indi-
ference curve for household 1 is tangent to the price line. Similarly, an optimal solution to
the household 2 optimization problem occurs where the indifference curve for household 2 is
tangent to the price line. The equilibrium price line must be such that the optimal solution
for household 1 coincides with the optimal solution for household 2 in the Edgeworth box.
This means that the equilibrium allocation occurs at the point of triple tangency (i.e., the
indifference curve for household 1 is tangent to the price line and the indifference curve for
household 2 is tangent to the price line).

The slope of the equilibrium price line is equal to \(-\frac{p_1}{p_2}\). We typically normalize
\(p_2 = 1\), meaning that the slope of the equilibrium price line is simply equal to \(-p_1\). If the equilibrium
price \(p_1\) increases, then the budget line becomes steeper, while if the equilibrium price \(p_1\)
decreases, then the budget line becomes shallower.

### 3.7.5 Pareto efficiency condition

Recall the Pareto efficiency condition:

\[
\frac{\alpha_1^1 Du^1 (x_1^1)}{\alpha_2^1 Du^1 (x_2^1)} = \frac{\alpha_1^2 Du^2 (x_1^2)}{\alpha_2^2 Du^2 (x_2^2)}.
\]

(3.63)
In words, this says that the ratio of marginal utilities for household 1 must be equal to the ratio of marginal utilities for household 2. This is equivalent to saying that the indifference curve for household 1 must be tangent to the indifference curve for household 2.

Consider an economy with the following utility functions:

\[ u^1(x) = u^2(x) = \ln(x). \]  

(3.64)

The total endowment of commodity 1 is \( e_1 = e_1^1 + e_1^2 \) and the total endowment of commodity 2 is \( e_2 = e_2^1 + e_2^2 \). Under the utility functions given above, the Pareto efficiency condition is given by:

\[ \frac{\alpha_1^1 x_2^1}{\alpha_2^1 x_1^1} = \frac{\alpha_1^2 x_2^2}{\alpha_2^2 x_1^2}. \]  

(3.65)

Let’s use the market clearing condition to express \( x_1^2 = e_1 - x_1^1 \) and \( x_2^2 = e_2 - x_2^1 \):

\[ \frac{\alpha_1^1 x_2^1}{\alpha_2^1 x_1^1} = \frac{\alpha_1^2 e_2 - x_2^1}{\alpha_2^2 e_1 - x_1^1}. \]  

(3.66)

We can solve this equation for \( x_2^1 \) in terms of \( x_1^1 \):

\[ \frac{\alpha_1^1}{\alpha_2^1} x_2^1 (e_1 - x_1^1) = \frac{\alpha_1^2}{\alpha_2^2} x_1^1 (e_2 - x_2^1). \]  

(3.67)

Collect the terms involving \( x_2^1 \) on the left-hand side:

\[ x_2^1 \left( e_1 \frac{\alpha_1^1}{\alpha_2^2} + x_1^1 \left( \frac{\alpha_1^2}{\alpha_2^2} - \frac{\alpha_1^1}{\alpha_2^1} \right) \right) = e_2 \frac{\alpha_1^2}{\alpha_2^2} x_1^1. \]  

(3.68)

Then divide through by the coefficient in front of \( x_2^1 \) in order to obtain an equation for \( x_2^1 \):

\[ x_2^1 = \frac{e_2 \frac{\alpha_1^2}{\alpha_2^2} x_1^1}{e_1 \frac{\alpha_1^1}{\alpha_2^2} + x_1^1 \left( \frac{\alpha_1^2}{\alpha_2^2} - \frac{\alpha_1^1}{\alpha_2^1} \right)}. \]  

(3.69)

This is the equation for the entire set of Pareto efficient allocations. This set is typically called the Pareto set and the equation is called the Pareto curve.
If $\frac{\alpha_1^2}{\alpha_2^2} = \frac{\alpha_1^1}{\alpha_2^1}$, the Pareto curve is given by:

$$x_2^1 = \frac{e_2^2 \alpha_1^2 x_1^1}{e_1^1 \alpha_1^1 + 0} = \frac{e_2^2 x_1^1}{e_1^1}.$$  \hfill (3.70)

This is the equation for a line. If $\frac{\alpha_1^2}{\alpha_2^2} > \frac{\alpha_1^1}{\alpha_2^1}$, the Pareto curve is strictly concave. If $\frac{\alpha_1^2}{\alpha_2^2} < \frac{\alpha_1^1}{\alpha_2^1}$, the Pareto curve is strictly convex.

Consider an economy with $e_1 = e_2 = 3$ and $\frac{\alpha_1^2}{\alpha_2^2} > \frac{\alpha_1^1}{\alpha_2^1}$. Figure 3.7.5 depicts the Pareto set, or the set of Pareto efficient allocations. Notice that each point on that curve is a Pareto efficient allocation as it occurs at a tangency of the indifference curves for both households.

### 3.7.6 Second Basic Welfare Theorem

Recall the statement of the Second Basic Welfare Theorem

**Theorem 3.3** Second Basic Welfare Theorem

If the allocation $(x_1^1, x_1^2, x_1^2, x_2^2)$ is a Pareto efficient allocation, then there exists a price $p_1$ and a tax/subsidy transfer scheme $(\tau^1, \tau^2)$ with $\tau^1 + \tau^2 = 0$ defined by $\tau^h = p_1 x_1^h + x_2^h - p_1 e_1^h - e_2^h$ (a transfer $\tau^h > 0$ denotes a subsidy and a transfer $\tau^h < 0$ denotes a tax) such that $p_1$ and $(x_1^1, x_1^2, x_2^1, x_2^2)$ is an Arrow-Debreu equilibrium with transfers.

Figure 3.7.6 illustrates the mechanism for the Second Basic Welfare Theorem. As Figure 3.7.6 shows, the equilibrium price line is tangent to the indifference curves of both households at the Pareto efficient allocation $x$. The initial endowment is $e = (e_1^1, e_1^2, e_2^1, e_2^2)$. The transfers $\tau^1 > 0$ and $\tau^2 = -\tau^1$ are set to increase the income of household 1 until he/she is able to afford the allocation $x^1$. Upon reaching the correct budget line, the households trade commodities until optimization for both households occurs at $x = (x_1^1, x_1^2, x_2^1, x_2^2)$.

### 3.8 Failures of the FBWT

#### 3.8.1 Sneak peek

**Summary**

Though remarkably concise, the FBWT contains some implicit assumptions that are indispensable in the static model of perfect competition. This section considers the consequences
3.8. FAILURES OF THE FBWT

if some of these implicit assumptions are violated. Does the FBWT continue to hold, namely does the market mechanism continue to deliver an efficient allocation of resources (according to the Pareto criterion)?

The failures discussed in this section are useful both for a full appreciation of what the FBWT entails (and what it doesn’t) and for practice with economic problem-solving.

Main takeaways

After completing this section, you will be able to answer the following questions:

- How is the performance of the markets affected by consumption externalities?
- How is the performance of the markets affected by the absence of markets?
- How is the performance of the markets affected if households deviate from competitive behavior?

3.8.2 Listing the types of failures

Recall the statement of the First Basic Welfare Theorem.

**Theorem 3.4 First Basic Welfare Theorem**

*The Arrow-Debreu equilibrium allocations are Pareto efficient.*

There are a number of implicit assumptions for the FBWT. These implicit assumptions are a product of the way that a competitive equilibrium is modeled. I list the most important of these implicit assumptions:

- Economies contain a finite number of households and commodities.
- Household preferences only depend upon its own consumption.
- Markets exist for the trading of all commodities.
- Competitive hypothesis:
  - All agents (households and firms) believe their behavior does not affect prices.
  - All agents (households and firms) believe they can transact any quantities of the commodities.

This section considers the consequences if any one of the final three implicit assumptions is violated.
3.8.3 Consumption externalities

We first address the issue of consumption externalities. Under consumption externalities, a household’s utility does not only depend upon its own consumption, but can also depend upon the consumption of other households in the economy.

Consider an example in which the utility function for household 1 is \( \frac{1}{2} \ln (x_1^1 + x_1^2) + \frac{1}{2} \ln (x_1^2) \), the utility function for household 2 is \( \frac{1}{2} \ln (x_2^1) + \frac{1}{2} \ln (x_2^2) \), and the endowments are \((e_1^1, e_1^2) = (1, 2)\) and \((e_2^1, e_2^2) = (2, 1)\). With these utility functions, household 1 cares about household 2’s consumption of the first good \((x_2^1)\).

We will solve for the set of Pareto efficient allocations and then solve for the equilibrium and compare.

The Pareto efficient allocations are \( x^1 = (0, \theta) \) and \( x^2 = (3, 3 - \theta) \) for any \( \theta \in [0, 3] \). Only household 2 cares about individual consumption of commodity 2. Household 1 only cares about the sum \( x_1^1 + x_1^2 \). Giving all of commodity 1 to household 2 is ideal for both households.

First step

To find the equilibrium, the first step is to normalize \( p_2 = 1 \).

Second step

Denote \( \lambda^1 \) as the Lagrange multiplier associated with the budget constraint in the household 1 utility maximization problem

\[
\begin{align*}
\text{maximize} & \quad \frac{1}{2} \ln (x_1^1 + x_1^2) + \frac{1}{2} \ln (x_1^2) \\
\text{subject to} & \quad p_1 + 2 - p_1 x_1^1 - x_2^1 \geq 0
\end{align*}
\] (3.71)

The first order conditions for household 1 are given by:

\[
\begin{align*}
\frac{0.5}{x_1^1 + x_1^2} - \lambda^1 p_1 &= 0. \\
\frac{0.5}{x_1^2} - \lambda^1 &= 0.
\end{align*}
\] (3.72)

This means that \( x_2^1 = \frac{0.5}{\lambda^1} \) and \( x_1^1 + x_1^2 = \frac{0.5}{x_1^1 p_1} \).
3.8. FAILURES OF THE FBWT

Third step

Denote $\lambda^2$ as the Lagrange multiplier associated with the budget constraint in the household 2 utility maximization problem

$$\begin{align*}
\text{maximize} & \quad \frac{1}{2} \ln (x_1^2) + \frac{1}{2} \ln (x_2^2) \\
\text{subject to} & \quad 2p_1 + 1 - p_1 x_1^2 - x_2^2 \geq 0.
\end{align*}$$

(3.73)

The first order conditions for household 2 are given by:

$$\begin{align*}
\frac{0.5}{x_1^2} - \lambda^2 p_1 &= 0, \\
\frac{0.5}{x_2^2} - \lambda^2 &= 0.
\end{align*}$$

(3.74)

Solving for $x^2$ as a function of $\lambda^2$ and plugging into the budget constraint $p_1 x_1^2 + x_2^2 = 2p_1 + 1$ yields:

$$\frac{0.5}{\lambda^2} + \frac{0.5}{\lambda^2} = 2p_1 + 1.$$ 

(3.75)

This implies that $\frac{1}{\lambda^2} = 2p_1 + 1$. The demand functions for household 2 are:

$$x_1^2 = \frac{p_1 + 0.5}{p_1} \quad \text{and} \quad x_2^2 = p_1 + 0.5.$$ 

(3.76)

Fourth step

Market clearing for commodity 1 states:

$$x_1^1 + x_1^2 = e_1^1 + e_1^2 = 3.$$ 

(3.77)

Since the household 1 first order condition states

$$x_1^1 + x_1^2 = \frac{0.5}{\lambda^2 p_1},$$

(3.78)

then market clearing dictates that $\frac{0.5}{\lambda^2 p_1} = 3$. Incorporating the household 1 first order condition for commodity 2, $x_2^1 = \frac{0.5}{\lambda^1} = 3p_1$.

Market clearing for commodity 2 states:

$$x_2^1 + x_2^2 = e_2^1 + e_2^2 = 3.$$ 

(3.79)
We are free to use both market clearing conditions in this solution since we are not using the budget constraint for household 1 in our solution method.

Since \( x_1^2 = 3p_1 \) and \( x_2^2 = p_1 + 0.5 \), market clearing implies that:

\[
(3p_1) + (p_1 + 0.5) = 3. 
\]

(3.80)

Solving for \( p_1 \) yields \( p_1 = \frac{5}{8} \).

**Fifth step**

Under this price, the equilibrium consumption variables can be easily calculated:

\[
x^1 = (1.2, 1.875). 
\]

(3.81)

\[
x^2 = (1.8, 1.125). 
\]

This equilibrium allocation is plotted in the Edgeworth box in Figure 3.8.1. As discussed in the Edgeworth box section, the slope of the budget line is always equal to \(-\frac{p_1}{p_2}\), which is equal to \(-\frac{5}{8}\) in this example.

The Pareto set is plotted as the vertical line on the left-side of the Edgeworth box (i.e., giving all of commodity 1 to household 2). The equilibrium allocation is not Pareto efficient. The FBWT does not hold with consumption externalities.

### 3.8.4 Missing markets

We next address the issue of what the equilibrium allocation will be in the presence of missing markets for commodity trade.

**Two commodity economy**

Household utility functions are \( C^2 \), strictly increasing, and strictly concave, but there is a missing market for commodity 2. That is, each household can consume no more than its initial endowment of commodity 2.

The Edgeworth box in Figure 3.8.2 shows that the equilibrium allocation occurs at the endowment point. The budget set is the set of all non-negative consumption bundles that satisfy both the budget constraint and the inequality \( x_2^h \leq e_2^h \) imposed by the missing commodity market. As the utility functions are strictly increasing, equilibrium prices must
satisfy $p_1 > 0$ and $p_2 > 0$. For any such prices, the only allocation that lies in both the budget set for household 1 and the budget set for household 2 is the endowment point $(e_1^1, e_2^1, e_1^2, e_2^2)$.

The term autarchy refers to the outcome in which households consume their endowments and no trade occurs.

The Pareto efficiency condition specifies that a Pareto efficient allocation must be such that the indifference curves for both households are tangent. The set of all instances in which the indifference curves are tangent will be the Pareto set, which is the set of Pareto efficient allocations. As shown in Figure 3.7.5, the Pareto set is a curve in the 2-dimensional Edgeworth box. It is extremely unlikely that the endowment $(e_1^1, e_2^1, e_1^2, e_2^2)$ happens to lie on this Pareto curve. The typical outcome is that the equilibrium allocation is not Pareto efficient.

### Three commodity economy

Let’s take the analysis one step further and consider an economy with two households and three commodities. Maintain the assumption that both utility functions are $C^2$, strictly increasing, and strictly concave. Suppose that there is a missing market for the third commodity. This is an important economy to analyze, since the missing market for commodity 3 does not prevent the trading of commodities 1 and 2.

The Pareto efficiency conditions for the case of 3 commodities state that a Pareto efficient allocation $(x_1^1, x_2^1, x_3^1, x_1^2, x_2^2, x_3^2)$ must satisfy:

$$\frac{\alpha_1^1 Du^1 (x_1^1)}{\alpha_2^1 Du^1 (x_2^1)} = \frac{\alpha_1^2 Du^2 (x_1^2)}{\alpha_2^2 Du^2 (x_2^2)}$$  \text{and}  $$\frac{\alpha_1^3 Du^3 (x_1^3)}{\alpha_2^3 Du^3 (x_2^3)} = \frac{\alpha_3^1 Du^1 (x_1^1)}{\alpha_3^2 Du^2 (x_3^2)}.$$  \(3.82\)

For the equilibrium, we normalize the price of the last commodity: $p_3 = 1$. The following conditions characterize the equilibrium with a missing commodity market for the third commodity:

$$\alpha_1^1 Du^1 (x_1^1) - \lambda_1 p_1 = 0 \quad \alpha_1^1 Du^2 (x_1^2) - \lambda_2 p_1 = 0$$
$$\alpha_2^1 Du^1 (x_2^1) - \lambda_1 p_2 = 0 \quad \alpha_2^2 Du^2 (x_2^2) - \lambda_2 p_2 = 0$$
$$\alpha_3^1 Du^3 (x_3^1) - \lambda_1 = 0 \quad \alpha_3^2 Du^2 (x_3^2) - \lambda_2 = 0.$$  \(3.83\)

There is only one Pareto efficient allocation with $(x_3^1, x_3^2) = (e_3^1, e_3^2)$ (see Figure 3.8.3). This Pareto efficient allocation is labeled $((x_1^1, x_2^1, e_3^1), (x_1^2, x_2^2, e_3^2))$. If this allocation satisfies
the household budget constraints

\[ p_1 (e_1^1 - x_1^1) + p_2 (e_1^2 - x_2^1) = 0 \]  \hspace{1cm} (3.84) \\
\[ p_1 (e_1^2 - x_1^2) + p_2 (e_2^2 - x_2^2) = 0, \]

then the allocation is an equilibrium allocation. The prices in the budget constraints must satisfy the first order conditions:

\[ p_1 = \frac{\alpha_1^1 Du^1 (x_1^1)}{\alpha_3^1 Du^1 (e_3^1)} = \frac{\alpha_2^1 Du^2 (x_2^1)}{\alpha_3^1 Du^2 (e_3^2)} \text{ and } \]
\[ p_2 = \frac{\alpha_1^2 Du^1 (x_2^1)}{\alpha_3^2 Du^1 (e_3^1)} = \frac{\alpha_2^2 Du^2 (x_2^1)}{\alpha_3^2 Du^2 (e_3^2)}. \]  \hspace{1cm} (3.85)

It is possible that the equilibrium allocation is Pareto efficient even if the endowment is not.

Figure 3.8.3 illustrates the 3-D Edgeworth box, or Edgeworth cube. The x-axis is for commodity 1, the y-axis is for commodity 2, and the vertical z-axis is for commodity 3. The Pareto set is a 1-dimensional curve within this 3-dimensional cube. Figure 3.8.3 illustrates the Pareto set and the restriction to the plane \((x_1^3, x_2^3) = (e_1^3, e_2^3)\).

Figure 3.8.4 looks only at the plane \((x_1^3, x_2^3) = (e_1^3, e_2^3)\) and illustrates an example in which the Arrow-Debreu equilibrium allocation is Pareto efficient. Any cross-section of the 3-D Edgeworth cube (with the commodity 3 fixed) is simply a 2-D Edgeworth box over commodities 1 and 2. Such an example, in which the Arrow-Debreu equilibrium allocation is Pareto efficient with a missing market and three commodities, though possible, is not typical. The FBWT does not hold with missing markets.

### 3.8.5 Noncompetitive behavior

We lastly address the implications of failures of the competitive hypothesis. Recall that the competitive hypothesis contains two components:

- All agents (households and firms) believe their behavior does not affect prices.
- All agents (households and firms) believe they can transact any quantities of the commodities.
Strategic price setting

Consider an economy with two households and two commodities. Household 1 has utility function \(\frac{1}{2} \ln (x_1^1) + \frac{1}{2} \ln (x_2^1)\) and household 2 has utility function \(\frac{1}{2} \ln (x_1^2) + \frac{1}{2} \ln (x_2^2)\). The endowments are \((e_1^1, e_2^1) = (1, 2)\) and \((e_1^2, e_2^2) = (2, 1)\). In order to set a baseline for comparison, let us first solve for the Arrow-Debreu equilibrium.

**Competitive equilibrium**  The Arrow-Debreu equilibrium will serve as the benchmark to which I compare the non-competitive equilibrium.

The reader is invited to verify that the Arrow-Debreu equilibrium is \(p_1 = 1\) (after normalizing \(p_2 = 1\)) and with allocation

\[
(x_1^1, x_2^1) = \left(3, \frac{3}{2}\right).
\]

\[
(x_1^2, x_2^2) = \left(3, \frac{3}{2}\right).
\]

**Non-competitive equilibrium**  Now assume that household 1 sets the price, household 2 chooses optimal demand at this price, and household 1 consumes the remaining amount such that markets clear. We can compute this non-competitive equilibrium and show that it is not Pareto efficient.

Since household 2 has strictly increasing utility, it must be that \(p_1 > 0\) and \(p_2 > 0\). Normalize \(p_2 = 1\).

Household 1 will correctly anticipate household 2’s demand, as given by the demand functions \((x_1^2, x_2^2) = \left(\frac{2p_1 + 1}{2p_1}, \frac{2p_1 + 1}{2}\right)\). Household 1 consumption is whatever remains of the aggregate endowment:

\[
x_1^1 = 3 - x_1^2 = 3 - \frac{2p_1 + 1}{2p_1}.
\]

\[
x_1^2 = 3 - x_2^2 = 3 - \frac{2p_1 + 1}{2}.
\]
3. MICROFOUNDATIONS

Household 1 is therefore choosing $p_1$ to maximize his/her indirect utility function:

$$ V^1(p_1) = \frac{1}{2} \ln \left( 3 - \frac{2p_1 + 1}{2p_1} \right) + \frac{1}{2} \ln \left( 3 - \frac{2p_1 + 1}{2} \right) $$

$$ = \frac{1}{2} \ln \left( \frac{4p_1 - 1}{2p_1} \right) + \frac{1}{2} \ln \left( \frac{-2p_1 + 5}{2} \right) $$

$$ = \frac{1}{2} \ln (4p_1 - 1) - \frac{1}{2} \ln(2p_1) + \frac{1}{2} \ln (-2p_1 + 5) - \frac{1}{2} \ln(2). \tag{3.88} $$

I have repeatedly used the ln property that $\ln \left( \frac{a}{b} \right) = \ln(a) - \ln(b)$. We can cancel the $\frac{1}{2}$ from all terms when finding the optimal solution. The optimal solution of this indirect utility function is found from the first order condition:

$$ DV^1(p_1) = \frac{4}{4p_1 - 1} - \frac{1}{p_1} - \frac{2}{-2p_1 + 5} = 0. \tag{3.89} $$

Let’s find the common denominator:

$$ -8 (p_1)^2 + 20p_1 \quad (4p_1 - 1) (-2p_1 + 5) p_1 + 8 (p_1)^2 - 22p_1 + 5 \quad (4p_1 - 1) (-2p_1 + 5) p_1 + \frac{-8 (p_1)^2 + 2p_1}{(4p_1 - 1) (-2p_1 + 5) p_1} = 0. \tag{3.90} $$

The optimal solution can only be between $p_1 = \frac{1}{4}$ and $p_1 = \frac{5}{2}$. Values outside this range yield negative consumption for household 1, which violates feasibility. Simplifying:

$$ \frac{-8 (p_1)^2 + 5}{(4p_1 - 1) (-2p_1 + 5) p_1} = 0. \tag{3.91} $$

Solving for $p_1$:

$$ -8 (p_1)^2 + 5 = 0 \tag{3.92} $$

$$ (p_1)^2 = \frac{5}{8}, $$

which yields

$$ p_1 = \sqrt{\frac{5}{8}} = \frac{1}{2} \sqrt{\frac{5}{2}}. \tag{3.93} $$
The non-competitive equilibrium is given by $p_1 = \sqrt{\frac{5}{8}}$ and
\[
x^1 = (x^1_1, x^1_2) = (2 - \sqrt{\frac{25}{8}}, \frac{5}{2} - \sqrt{\frac{5}{8}}) \approx (1.37, 1.71).
\]
\[
x^2 = (x^2_1, x^2_2) = (1 + \sqrt{\frac{25}{8}}, \frac{1}{2} + \sqrt{\frac{5}{8}}) \approx (1.63, 1.29).
\]

In the Edgeworth box in Figure 3.8.5, you can see the indifference curves for each agent at the non-competitive equilibrium. The non-competitive equilibrium lies at the upper-left intersection of the two indifference curves. The lens between the two indifference curves contains a set of allocations that Pareto dominate the non-competitive equilibrium. Any allocation in that lens provides strictly higher utility to both households. The FBWT is violated under strategic price setting.

Notice in this economy that household 1 (the price setter) has a strictly higher utility compared to the Arrow-Debreu equilibrium allocation.

\[
\begin{align*}
\text{Arrow-Debreu:} & \quad \frac{1}{2} \ln \left(\frac{3}{2}\right) + \frac{1}{2} \ln \left(\frac{3}{2}\right) \approx 0.405. \\
\text{Noncompetitive:} & \quad \frac{1}{2} \ln (1.37) + \frac{1}{2} \ln (1.71) \approx 0.426. 
\end{align*}
\]

This must be the case as household 1 is capable of choosing the price $p_1 = 1$ (the Arrow-Debreu equilibrium price), but finds that he/she can do better by choosing $p_1 = \sqrt{\frac{5}{8}} < 1$.

### Transaction constraints

Suppose that household 2 behaves competitively, but household 1 takes the market prices as given and also believes that he/she can only sell $\bar{z}_1^1$ units of commodity 1. We call this a sales constraint. Mathematically, the constraint is stated as:
\[
e^1_1 - x^1_1 \leq \bar{z}_1^1.
\]

Using an Edgeworth box, we can depict an economy in which the equilibrium allocation with the sales constraint is still Pareto efficient. See Figure 3.8.6.

Alternatively, using an Edgeworth box, we can depict an economy in which the equilibrium allocation with the sales constraint is not Pareto efficient. As can be seen in Figure 3.8.7, the price line is steeper (compared to the price line for a competitive equilibrium) in order to satisfy the constraint
\[
x^1_1 \geq e^1_1 - \bar{z}_1^1.
\]
If the price line is steeper (meaning the absolute value of the slope is larger), where the slope equals \(-\frac{p_1}{p_2}\), the price \(p_1\) must be higher in the equilibrium with the constraints compared to the competitive equilibrium. Since \(p_1\) is higher, household 1 has more income and purchases more units of commodity 2. Household 1 is solving its optimization problem by choosing consumption to satisfy the constraint

\[ x_1^1 \geq e_1^1 - z_1^1. \tag{3.98} \]

This means that the household 1 indifference curve is not tangent to the price line. Household 1 can receive higher utility by moving in the direction of Pareto improvement indicated in Figure 3.8.7.

Household 2 doesn’t face a constraint in its optimization problem, meaning that its indifference curve is tangent to the price line. However, due to the distortion leading to the high price \(p_1\), household 2 is buying fewer units of commodity 1 than it otherwise would have (at the competitive equilibrium). Additionally, as household 2 is paying a higher price for commodity 1, it has less income left over for the purchase of commodity 2. By moving in the direction of Pareto improvement, household 2 can receive higher utility.

The result is that sometimes transaction constraints prevent Pareto efficiency and other times the constraints have no impact. We say that the FBWT does not hold for transaction constraints as Pareto efficiency is not guaranteed for all economies.

### 3.9 Household demand properties

#### 3.9.1 Sneak peek

**Summary**

This section collects some key properties of household demand, importantly making use of the main assumptions on the utility functions that we studied at the beginning of this chapter.

Since household demand is one component of equilibrium, a mastery of the properties of household demand allows for a better understanding of the equilibrium system as a whole, and how changes in parameters affect the underlying equilibrium variables.

A popular exercise for economists is to conduct comparative statics, meaning that we seek to understand the effect of a change in parameter values on the underlying equilibrium
variables. Equilibrium variables include household demand and prices, which are mathematical objects that change values such that our equilibrium system of equations is satisfied. Economic parameters are fixed objects, such as household utility functions and endowments. By understanding comparative statics, we, the economists, can understand how policy can best be implemented to achieve desired changes in the equilibrium variables.

**Main takeaways**

After completing this section, you will be able to answer the following questions:

- Under what conditions is household demand uniquely determined?
- How can the Implicit Function Theorem be used to calculate the effects of a change in the price on household demand?
- Since economic welfare is measured in terms of household utility, what determines the effects of price changes on welfare?

### 3.9.2 Condition for a unique solution

Assume that the utility function $u^h$ is strictly concave.

Recall that the household optimization problem is given by

\[
\begin{align*}
\text{maximize} & \quad \alpha_1^h u^h (x_1^h) + \alpha_2^h u^h (x_2^h) \\
\text{subject to} & \quad p_1 e_1^h + e_2^h - p_1 x_1^h - x_2^h \geq 0 
\end{align*}
\] (3.99)

Consider what happens if the optimal solution is not unique. Then there are two bundles, let’s call them $(x_1^h, x_2^h)$ and $(y_1^h, y_2^h)$, that satisfy the budget constraint and yield the highest utility

\[
\alpha_1^h u^h (x_1^h) + \alpha_2^h u^h (x_2^h) = \alpha_1^h u^h (y_1^h) + \alpha_2^h u^h (y_2^h). 
\] (3.100)

A convex combination of the two bundles

\[
(\theta x_1^h + (1 - \theta)y_1^h, \theta x_2^h + (1 - \theta)y_2^h) 
\] (3.101)

for any $\theta \in (0, 1)$ will also satisfy the budget constraint and will result in strictly higher
utility, since the definition of strict concavity states

\[ u^h (\theta x^h_1 + (1 - \theta) y^h_1) > \theta u^h (x^h_1) + (1 - \theta) u^h (y^h_1). \tag{3.102} \]

and \( u^h (\theta x^h_2 + (1 - \theta) y^h_2) > \theta u^h (x^h_2) + (1 - \theta) u^h (y^h_2). \)

So it can only be that a unique optimal solution exists.

#### 3.9.3 Effects of a parameter change

**Implicit Function Theorem**

Recall the Kuhn-Tucker conditions for the household problem are:

\[ \alpha^h_1 Du^h (x^h_1) - \lambda^h p_1 = 0. \tag{3.103} \]
\[ \alpha^h_2 Du^h (x^h_2) - \lambda^h = 0. \]
\[ \lambda^h (p_1 e^h_1 + e^h_2 - p_1 x^h_1 - x^h_2) = 0. \]

The variable \( \lambda^h \geq 0 \) is the Lagrange multiplier associated with the budget constraint.

Since \( u^h \) is strictly increasing, \( \alpha^h_2 Du^h (x^h_2) > 0 \) from the second equation. This implies that \( \lambda^h > 0 \). From the third equation, it must be that

\[ p_1 e^h_1 + e^h_2 - p_1 x^h_1 - x^h_2 = 0, \tag{3.104} \]

or in other words, that the budget constraint holds with equality. Also since \( u^h \) is strictly increasing, \( \alpha^h_1 Du^h (x^h_1) > 0 \) from the first equation. This implies that \( p_1 > 0 \). This means that the optimal solutions are given by:

\[ \alpha^h_1 Du^h (x^h_1) - \lambda^h p_1 = 0. \tag{3.105} \]
\[ \alpha^h_2 Du^h (x^h_2) - \lambda^h = 0. \]
\[ p_1 e^h_1 + e^h_2 - p_1 x^h_1 - x^h_2 = 0. \]

The parameters in this setting will be the price \( p_1 \) and the endowments \( e^h_1 \) and \( e^h_2 \). We think of \( p_1 \) as a parameter since it is taken as given by the households in their optimization problem. Endowments \( e^h_1 \) and \( e^h_2 \) are always parameters in economic models. For any vector of parameters \((p_1, e^h_1, e^h_2)\), the variables \((x^h_1, x^h_2, \lambda^h)\) are optimal solutions for the household.
provided that $G \left( x_1^h, x_2^h, \lambda^h, p_1, e_1^h, e_2^h \right) = 0$, where the vector-valued function $G$ is defined by:

$$G \left( x_1^h, x_2^h, \lambda^h, p_1, e_1^h, e_2^h \right) = \begin{pmatrix} \alpha_1^h Du^h (x_1^h) - \lambda^h p_1 \\ \alpha_2^h Du^h (x_2^h) - \lambda^h \\ p_1 e_1^h + e_2^h - p_1 x_1^h - x_2^h \end{pmatrix}. \quad (3.106)$$

The multi-valued function $G$ is a system of 3 equations (there are 2 first order conditions and 1 budget constraint) in terms of 3 variables (2 consumption variables $x^h$ and 1 Lagrange multiplier $\lambda^h$) and 3 parameters (the price $p_1$ and the endowments $e_1^h$ and $e_2^h$).

Recall the Implicit Function Theorem from the Mathematical Preliminaries chapter.

**Theorem 3.5 Implicit Function Theorem**

Let $G(v, p) = 0$ be a system of equations in terms of a vector of variables $v$ and a vector of parameters $p$. The number of equations in $G$ must be equal to the number of variables. In order to satisfy $G(v, p) = 0$, a change in the parameters must lead to a change in the variables. So the variables can be written as an implicit function of the parameters: $v = f(p)$.

For the equation $G(f(p), p) = 0$, we take the derivative of both sides (using the chain rule):

$$D_v G(f(p), p) Df(p) + D_p G(f(p), p) = 0. \quad (3.107)$$

The derivative matrix $D_v G(f(p), p)$ has the same number of rows as columns (since the number of equations in $G$ is equal to the number of variables). This derivative matrix $D_v G(f(p), p)$ must be invertible. If so, then we can solve the above equation for $Df(p)$:

$$Df(p) = - [D_v G(f(p), p)]^{-1} \cdot D_p G(f(p), p). \quad (3.108)$$

**Verifying the full rank condition**

In the present application of the Implicit Function Theorem, the variables are $v = (x_1^h, x_2^h, \lambda^h) \in \mathbb{R}_+^3$, and the parameters are $p = (p_1, e_1^h, e_2^h) \in \mathbb{R}_+^3$. Let us determine if the full rank condition is satisfied:

$$D_v G(v, p) = \begin{bmatrix} \alpha_1^h D^2 u^h (x_1^h) & 0 & -p_1 \\ 0 & \alpha_2^h D^2 u^h (x_2^h) & -1 \\ -p_1 & -1 & 0 \end{bmatrix}. \quad (3.109)$$

The derivative matrix $D_v G(v, p)$ has dimension $3 \times 3$. It is referred to as the Jacobian matrix. The element in row $i$ and column $j$ of the matrix refers to the derivative of the $i$th
equation in \( G \) with respect to the \( j \)th variable in \( v \).

A matrix \( M \) has full rank if the equation \( v^T M = 0 \) implies that \( v = \overrightarrow{0} \). Define the Jacobian matrix as

\[
M = D_v G (v, p). \tag{3.110}
\]

The vector \( v \in \mathbb{R}^3 \) is a column vector \( v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \). The transpose is given by \( v^T = (v_1, v_2, v_3) \). The equations \( v^T M = 0 \) are given by:

\[
\begin{align*}
v_1 \left( \alpha_1^h D^2 u^h \left( x_1^h \right) \right) + v_3 (-p_1) &= 0. \\
v_2 \left( \alpha_2^h D^2 u^h \left( x_2^h \right) \right) + v_3 (-1) &= 0. \\
v_1 (-p_1) + v_2 (-1) &= 0.
\end{align*} \tag{3.111}
\]

By the assumption that \( u^h : \mathbb{R}_+ \to \mathbb{R} \) is strictly concave, \( \alpha_1^h D^2 u^h \left( x_1^h \right) < 0 \). From the first equation of (3.111), this implies that \( v_1 \times v_3 \leq 0 \). This is because both terms in parentheses are strictly negative (both \( \alpha_1^h D^2 u^h \left( x_1^h \right) < 0 \) and \(-p_1 < 0\)) and the only way for the sum to equal 0 is if the coefficients \((v_1, v_3)\) have opposite sign.

Again using the assumption of strict concavity, \( \alpha_2^h D^2 u^h \left( x_2^h \right) < 0 \). This implies that \( v_2 \times v_3 \leq 0 \) from the second equation of (3.111).

Finally, the third equation of (3.111), as both the terms in parentheses are strictly negative, implies that \( v_1 \times v_2 \leq 0 \).

Gather all of the facts:

1. \( v_1 \times v_3 \leq 0 \).
2. \( v_2 \times v_3 \leq 0 \).
3. \( v_1 \times v_2 \leq 0 \).

We see that the only way for all three inequalities to be satisfied is if \( (v_1, v_2, v_3) = (0, 0, 0) \). Since \( v^T M = 0 \) implies that \( v = \overrightarrow{0} \), then \( M \) has full rank.
3.9. **HOUSEHOLD DEMAND PROPERTIES**

**Consumption effects of a parameter change**

We can evaluate the derivative matrix

\[
D_pG(v, p) = \begin{bmatrix}
-\lambda^h & 0 & 0 \\
0 & 0 & 0 \\
e^h_1 - x^h_1 & p_1 & 1
\end{bmatrix}.
\]  (3.112)

Applying the Implicit Function Theorem equation (3.108), we can express how a parameter change will impact the household choice variables:

\[
Df(p) = \frac{\partial (x^h_1, x^h_2, \lambda^h)}{\partial (p_1, e^h_1, e^h_2)} = -[D_vG(v, p)]^{-1} D_pG(v, p).
\]  (3.113)

The matrix \(D_vG(v, p)\) can be viewed as the \(3 \times 3\) matrix with elements \(\begin{bmatrix} a & b & c \\
d & e & f \\
g & h & i \end{bmatrix}\). The inverse of this matrix is given by:

\[
\frac{1}{\text{det}} \begin{bmatrix}
ei - fh & ch - bi & bf - ce \\
f - di & ai - cg & cd - af \\
dh - eg & bg - ah & ae - bd
\end{bmatrix},
\]  (3.114)

which can be verified from any undergraduate textbook on matrix algebra. The determinant of the matrix is given by:

\[
\text{det} = a(ei - fh) + b(fg - id) + c(dh - eg)
= \alpha^h_1 D^2u^h(x^h_1) (0 - 1) + 0 - p_1\left(0 + p_1 \alpha^h_2 D^2u^h(x^h_2)\right)
= -\alpha^h_1 D^2u^h(x^h_1) - (p_1)^2 \alpha^h_2 D^2u^h(x^h_2).
\]  (3.115)

We can evaluate the 9 elements for the inverse matrix:

\[
\frac{1}{\text{det}} \begin{bmatrix}
-1 & p_1 & p_1 \alpha^h_2 D^2u^h(x^h_2) \\
p_1 & -(p_1)^2 & \alpha^h_1 D^2u^h(x^h_1) \\
p_1 \alpha^h_2 D^2u^h(x^h_2) & \alpha^h_1 D^2u^h(x^h_1) & \alpha^h_1 \alpha^h_2 D^2u^h(x^h_1) D^2u^h(x^h_2)
\end{bmatrix}.
\]  (3.116)
The derivatives \( \frac{\partial \nu}{\partial p} = \left( \frac{\partial \nu}{\partial p_1}, \frac{\partial \nu}{\partial e^h_1}, \frac{\partial \nu}{\partial e^h_2} \right) \) can be expressed as follows:

\[
\frac{\partial \nu}{\partial p_1} = -\frac{1}{\det} \begin{bmatrix}
-1 & p_1 & p_1 \alpha^h_2 D^2 u^h (x^h_2) \\
p_1 & -(p_1)^2 & \alpha^h_1 D^2 u^h (x^h_1) \\
p_1 \alpha^h_2 D^2 u^h (x^h_2) & \alpha^h_1 D^2 u^h (x^h_1) & \alpha^h_1 \alpha^h_2 D^2 u^h (x^h_1) D^2 u^h (x^h_2)
\end{bmatrix} \begin{bmatrix}
-\lambda^h \\
0 \\
e^h_1 - x^h_1
\end{bmatrix}
\]

\[
= \frac{1}{\det} \begin{bmatrix}
-\lambda^h - p_1 \alpha^h_2 D^2 u^h (x^h_2) (e^h_1 - x^h_1) \\
p_1 \lambda^h - \alpha^h_1 D^2 u^h (x^h_1) (e^h_1 - x^h_1) \\
\lambda^h p_1 \alpha^h_2 D^2 u^h (x^h_2) - \alpha^h_1 \alpha^h_2 D^2 u^h (x^h_1) D^2 u^h (x^h_2) (e^h_1 - x^h_1)
\end{bmatrix}
\]

\[
\frac{\partial \nu}{\partial e^h_1} = -\frac{1}{\det} \begin{bmatrix}
-1 & p_1 & p_1 \alpha^h_2 D^2 u^h (x^h_2) \\
p_1 & -(p_1)^2 & \alpha^h_1 D^2 u^h (x^h_1) \\
p_1 \alpha^h_2 D^2 u^h (x^h_2) & \alpha^h_1 D^2 u^h (x^h_1) & \alpha^h_1 \alpha^h_2 D^2 u^h (x^h_1) D^2 u^h (x^h_2)
\end{bmatrix} \begin{bmatrix}
0 \\
0 \\
p_1
\end{bmatrix}
\]

\[
= \frac{1}{\det} \begin{bmatrix}
-(p_1)^2 \alpha^h_2 D^2 u^h (x^h_2) \\
-p_1 \alpha^h_1 D^2 u^h (x^h_1) \\
-\alpha^h_1 \alpha^h_2 p_1 D^2 u^h (x^h_1) D^2 u^h (x^h_2)
\end{bmatrix}
\]

\[
\frac{\partial \nu}{\partial e^h_2} = -\frac{1}{\det} \begin{bmatrix}
-1 & p_1 & p_1 \alpha^h_2 D^2 u^h (x^h_2) \\
p_1 & -(p_1)^2 & \alpha^h_1 D^2 u^h (x^h_1) \\
p_1 \alpha^h_2 D^2 u^h (x^h_2) & \alpha^h_1 D^2 u^h (x^h_1) & \alpha^h_1 \alpha^h_2 D^2 u^h (x^h_1) D^2 u^h (x^h_2)
\end{bmatrix} \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]

\[
= \frac{1}{\det} \begin{bmatrix}
-p_1 \alpha^h_2 D^2 u^h (x^h_2) \\
-\alpha^h_1 D^2 u^h (x^h_1) \\
-\alpha^h_1 \alpha^h_2 D^2 u^h (x^h_1) D^2 u^h (x^h_2)
\end{bmatrix}
\]

Utility effects of a parameter change

We wish to derive the equation for the partial derivative \( \frac{\partial U^h (x^h_1, x^h_2, \lambda^h)}{\partial (p_1, e^h_1, e^h_2)} \). To do this, we can use the chain rule, which implies that

\[
\frac{\partial U^h (x^h_1, x^h_2, \lambda^h)}{\partial (p_1, e^h_1, e^h_2)} = \frac{\partial U^h (x^h_1, x^h_2, \lambda^h)}{\partial (x^h_1, x^h_2, \lambda^h)} \frac{\partial (x^h_1, x^h_2, \lambda^h)}{\partial (p_1, e^h_1, e^h_2)}.
\]
The derivative matrix $\frac{\partial (x^h, x^h, \lambda^h)}{\partial (p^h, e^h_1, e^h_2)}$ has already been evaluated using the Implicit Function Theorem. The derivative vector

$$\frac{\partial U^h (x^h_1, x^h_2, \lambda^h)}{\partial (x^h_1, x^h_2, \lambda^h)} = (\alpha^h_1 Du^h (x^h_1), \alpha^h_2 Du^h (x^h_2), 0),$$

(3.121)

where from the first order conditions

$$(\alpha^h_1 Du^h (x^h_1), \alpha^h_2 Du^h (x^h_2), 0) = \lambda^h (p_1, 1, 0).$$

(3.122)

The products

$$\frac{\partial U^h (x^h_1, x^h_2, \lambda^h)}{\partial p_1} \overset{\text{det}}{=} \lambda^h (p_1, 1, 0) \begin{bmatrix}
-p_1 \alpha^h_1 D^2 u^h (x^h_1) (e^h_1 - x^h_1) \\
\alpha^h_2 D^2 u^h (x^h_2) - \alpha^h_1 \alpha^h_2 D^2 u^h (x^h_1) D^2 u^h (x^h_2) (e^h_1 - x^h_1)
\end{bmatrix}$$

(3.123)

$$= \frac{1}{\text{det}} (-\alpha^h_1 D^2 u^h (x^h_1) - (p_1)^2 \alpha^h_2 D^2 u^h (x^h_2)) \lambda^h (e^h_1 - x^h_1).$$

$$\frac{\partial U^h (x^h_1, x^h_2, \lambda^h)}{\partial e^h_1} \overset{\text{det}}{=} \lambda^h (p_1, 1, 0) \begin{bmatrix}
-p_1 \alpha^h_1 D^2 u^h (x^h_1) \\
-\alpha^h_1 \alpha^h_2 p_1 D^2 u^h (x^h_1) D^2 u^h (x^h_2)
\end{bmatrix}$$

(3.124)

$$= \frac{p_1}{\text{det}} (- (p_1)^2 \alpha^h_2 D^2 u^h (x^h_2) - \alpha^h_1 D^2 u^h (x^h_1)) \lambda^h.$$  

$$\frac{\partial U^h (x^h_1, x^h_2, \lambda^h)}{\partial e^h_2} \overset{\text{det}}{=} \lambda^h (p_1, 1, 0) \begin{bmatrix}
-p_1 \alpha^h_1 D^2 u^h (x^h_1) \\
-\alpha^h_1 \alpha^h_2 D^2 u^h (x^h_1) D^2 u^h (x^h_2)
\end{bmatrix}$$

(3.125)

$$= \frac{1}{\text{det}} (- (p_1)^2 \alpha^h_2 D^2 u^h (x^h_2) - \alpha^h_1 D^2 u^h (x^h_1)) \lambda^h.$$  

Since the determinant was previously found to be equal to

$$\text{det} = -\alpha^h_1 D^2 u^h (x^h_1) - (p_1)^2 \alpha^h_2 D^2 u^h (x^h_2),$$

(3.126)
then the derivatives
\[
\frac{\partial U^h(x^h_1, x^h_2)}{\partial p_1} = \lambda^h (e^h_1 - x^h_1).
\]
(3.127)
\[
\frac{\partial U^h(x^h_1, x^h_2, \lambda^h)}{\partial e^h_1} = p_1 \lambda^h.
\]
(3.128)
\[
\frac{\partial U^h(x^h_1, x^h_2, \lambda^h)}{\partial e^h_2} = \lambda^h.
\]
(3.129)

Consider the derivative \( \frac{\partial U^h(x^h_1, x^h_2)}{\partial p_1} = \lambda^h (e^h_1 - x^h_1) \). If \( e^h_1 - x^h_1 > 0 \) (i.e., the household is a net-seller of commodity 1), an increase in \( p_1 \) increases utility. If \( e^h_1 - x^h_1 < 0 \) (i.e., the household is a net-buyer of commodity 1), an increase in \( p_1 \) decreases utility. This matches our economic intuition: households that sell a commodity are better off when the price of that commodity is high, and vice-versa.

Consider the derivatives \( \frac{\partial U^h(x^h_1, x^h_2, \lambda^h)}{\partial e^h_1} = p_1 \lambda^h \) and \( \frac{\partial U^h(x^h_1, x^h_2, \lambda^h)}{\partial e^h_2} = \lambda^h \). An increase in either \( e^h_1 \) or \( e^h_2 \) will increase household income. The derivatives \( \frac{\partial m}{\partial e^h_1} = p_1 \) and \( \frac{\partial m}{\partial e^h_2} = 1 \), where \( m \) is total household income. In each case, the marginal effect of an increase in household income on optimal utility equals \( \lambda^h \). The Lagrange multiplier captures the shadow value of utility and is instrumental in calculating the effects of parameter changes on household utility.

### 3.10 Exercises

1. **Equilibrium equations**

   Consider an economy with two households and two commodities. The utility function for household 1 is:
   \[
   \alpha^1_1 u^1(x^1_1) + \alpha^1_2 u^1(x^1_2) = \frac{1}{4} \sqrt{x^1_1} + \frac{3}{4} \sqrt{x^1_2}
   \]
   and the household 1 endowment is \((e^1_1, e^1_2) = (1, 3)\). The utility function for household 2 is
   \[
   \alpha^2_1 u^2(x^2_1) + \alpha^2_2 u^2(x^2_2) = \frac{3}{4} \ln(x^2_1) + \frac{1}{4} \ln(x^2_2)
   \]
   and the household 2 endowment is \((e^2_1, e^2_2) = (1, 1)\). Solve for the equilibrium allocation and price.

2. **Equilibrium equations**

   Consider an economy with two households and two commodities. The utility function
for household 1 is:
\[ \alpha_1^1 u^1 (x_1^1) + \alpha_2^1 u^1 (x_2^1) = \frac{2}{3} \sqrt{x_1^1} + \frac{1}{3} \sqrt{x_2^1} \]
and the household 1 endowment is \((e_1^1, e_2^1) = (1, 1)\). The utility function for household 2 is
\[ \alpha_1^2 u^2 (x_1^2) + \alpha_2^2 u^2 (x_2^2) = \frac{1}{3} \ln (x_1^2) + \frac{2}{3} \ln (x_2^2) \]
and the household 2 endowment is \((e_1^2, e_2^2) = (3, 1)\). Solve for the equilibrium allocation and price.
3. MICROFOUNDATIONS
Bibliography


Part II

Growth Theory
Neoclassical Growth Model

4.1 Introducing the model

4.1.1 Sneak peek

Summary

The history of economic thought on the theory of growth is nearly as interesting as the history of economic growth itself. All theories are based on the recognition that growth in economic output is algebraically caused by growth in the stock of productive inputs into the production processes of firms. The decision about how much to increase the stock of productive inputs is made by the agents in the economy, meaning that the growth of the aggregate economy is determined by the choices made by the individual participants. The nature of the factors of production and how they change over time are the key features distinguishing different growth models.

The simplest model of economic growth is the Solow growth model (Solow, 1956). In this model, the factors of production are capital and labor. Though labor supply is in principle a choice of the households (how much to supply) and the firms (how much to hire), the model is stylized by assuming an inelastic labor supply, namely that the households’ supply of labor is constant. The remaining factor of production is the stock of capital. The stock of capital evolves over time and is augmented by investment made by households. Again, in principle, the investment made by households would be a choice of the households, but the model is stylized by assuming a fixed savings rate, namely that all households save the same constant fraction of their income every period. The households save in the form of capital investment...
and are compensated for their savings next period by receiving an interest payment.

As with all growth models, the initial capital stock is fixed. With a fixed savings rate and a fixed labor supply, households don’t have any interesting decisions to make in the Solow growth model. While they do choose consumption, this is determined directly from the budget constraint in each period. The model captures the dynamics of household consumption, investment, and output. These dynamics are functions of the parameters in the model, especially the savings rate and the labor supply.

A steady state is defined as a situation in which the capital stock remains unchanged for all future periods. The growth models in this chapter do not contain shocks, meaning that the capital stock will converge to a steady state (in the limit as time approaches infinity). Furthermore, the steady state is unique. The dynamics of the capital stock are straightforward: if the initial capital stock is higher than the steady state value, then the stock will decrease exponentially until it reaches the steady state, and the opposite occurs if the initial capital stock is lower than the steady state value. The same dynamics hold for the output.

Mathematically, the only interesting equations for the economist to solve are: (i) what is the value of steady state capital stock and output? and (ii) at what rate does the capital stock and the output converge to the steady state? Neither outcome offers groundbreaking economic insight. If the fixed savings rate is higher, then (i) the steady state capital stock and output are higher and (ii) the rate of convergence to the steady state is faster.

What we, as economists and policymakers, would love to understand is what determines the savings rate of households in a more robust model in which households actually have interesting choices to make. This more robust model is the neoclassical growth model and will be the focus of this chapter. The neoclassical growth model was conceived by Frank Ramsey (1928) and formalized independently by Tjalling Koopmans (1965) and David Cass (1965). In recognition of its originators, the neoclassical growth model is sometimes referred to as the Ramsey-Cass-Koopmans growth model.

In the neoclassical growth model, the two factors of production are the capital stock and the labor supply. The capital stock evolves over time according to the household’s optimal decisions about how much to invest. The labor supply is optimally chosen by the households in each period. As with the Solow growth model, there will exist a steady state, but the variable that determines the steady state and the rate of convergence is the endogenous savings rate. From the neoclassical growth model, we can learn what factors affect the savings rate and how policy can change the savings rate.

The neoclassical growth model leaves something to be desired about the root causes,
or what economists term the key determinants, of economic growth. While an increase in the capital stock causes growth in output, this growth is capped by the steady state previously mentioned unless (i) the productivity of the production process increases (typically via technological improvements) or (ii) the efficiency of the labor input increases. It is possible in the neoclassical growth model to include an exogenous growth factor for these two characteristics, but then growth is determined exogenously and not endogenously by decisions made inside the economy by economic agents.

To capture the fact that output grows without bound, it is essential to model the economic processes by which (i) productivity or technology increases and/or (ii) labor efficiency increases. This forms the foundation for the class of endogenous growth models. The following chapter will consider one type of endogenous growth models in which research and development investments can be made to increase the technology.

It should be emphasized that the growth models introduced at this point in the text are deterministic models with homogeneous households and firms. A deterministic model is one in which there are no shocks. Homogeneous households means that all households are identical and make identical savings and labor supply decisions. Homogeneous firms means that all firms are identical and make identical production decisions. With these features, the model predicts a "smooth" time path for output. This is desirable when discussing economic growth, but unacceptable when studying business cycles.

Economic interactions in reality are quite complex, so it is the duty of the economist to only introduce the machinery necessary to be able to extract the key predictions and insights for a particular topic. While future chapters studying business cycles will introduce stochastic models, deterministic models with homogeneous households and firms suffice to understand the theory of economic growth.

This section introduces the basic model of neoclassical growth and discusses the difference between the planner solution and the equilibrium solution. The model contains two different agents: households and firms. An equilibrium consists of the allocation and prices such that households are solving their problem, firms are solving their problem, and markets clear. The household problem is a dynamic problem with a budget constraint in every time period. The budget constraint requires that total expenditure is less than total income. The firm is solving a static profit maximization problem in each and every period. Profit equals revenue minus cost. Both the household problem and the firm problem capture the fact that economic agents respond to changes in market prices.

Now consider a central planner. The planner controls the household and the firm. The
planner tells the household how much to consume and invest and tells the firm how much to produce. The planner does not use markets, only these direct mandates about what the household and the firm should do. The planner maximizes household utility subject to an aggregate resource constraint. The resource constraint requires that the total amount produced must be split between consumption and investment.

There are two facts that are true for pretty much all problem-solving in economics: (i) our objective is to solve for the equilibrium solution and (ii) it is easier to solve for the planner’s solution than the equilibrium solution. If the welfare theorems are valid, the allocation is identical in both the equilibrium solution and the planner’s solution. In the initial sections of this chapter, the welfare theorems are valid, meaning that we adopt the following two-step approach: (i) solve for the allocation using the planner’s problem and (ii) once the allocation is known, solve for the prices in the markets.

**Notation**

The variables/parameters to be introduced in this section are given in the following table:

- $\beta$ : discount factor
- $c_t$ : household consumption in period $t$
- $k_t$ : capital stock held by household at beginning of $t$
- $K_t$ : capital stock rented by the firm in $t$
- $n_t$ : labor supply provided by household in $t$
- $N_t$ : labor supply hired by the firm in $t$
- $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ : utility function for the household
- $f : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ : production function for the firm
- $\theta$ : capital share under Cobb-Douglas production function
- $A$ : total factor productivity
- $R_t$ : rate of return on capital
- $w_t$ : wage rate
- $\pi_t$ : profit for the firm

**Main takeaways**

After completing this section, you will be able to answer the following questions:

- What is the difference between the planner’s problem and an equilibrium?
4.1. INTRODUCING THE MODEL

- What is the firm’s profit maximization problem?
- What is the household’s utility maximization problem?
- What is an equilibrium?

4.1.2 Model basics

Macroeconomics considers models with an infinite number of discrete time periods \( t \in \{0, 1, 2, \ldots\} \). The models in this chapter consider households that are infinite-lived, and governments that are infinite-lived. If you are uncomfortable with the immortality of households, you are welcome to think of households as dynasties, in which each living member of the dynasty cares equally about its own consumption and the consumption of its progeny.

In each time period, a single physical commodity is traded and consumed. It is possible to consider multiple commodities traded and consumed in each period, but macroeconomics typically analyzes decisions made across periods, rather than multivariate consumption decisions made within a period (as studied in microeconomics).

Households have preferences defined over the infinite time horizon. The utility function is of the form:

\[
U(c) = \sum_{t=0}^{\infty} \beta^t u(c_t).
\]  

The variable \( c_t \) is the consumption of the commodity in period \( t \). The utility function \( u : \mathbb{R}_+ \to \mathbb{R} \) satisfies the standard assumptions of \( C^2 \), strictly increasing, and strictly concave. Additionally, \( u : \mathbb{R}_+ \to \mathbb{R} \) satisfies the Inada condition:

\[
\lim_{c \to 0} Du(c) = +\infty.
\]  

The Inada condition states that the marginal utility diverges to infinity as consumption approaches 0. The purpose of the Inada condition is to ensure that households never find it optimal to choose zero consumption in any period.

4.1.3 Discounting

The other parameter is \( \beta \in (0, 1) \), the discount factor. The discount factor will be identical for all households and in all periods. This factor captures the economic intuition that a household prefers consumption sooner rather than later. Additionally, discounting is required in order to solve the problem using the recursive techniques of dynamic programming.
The parameter $\beta$ has no counterpart in macroeconomic data. Households do not report their discount factor in surveys or when making investment decisions. What macroeconomists can do is estimate the value of $\beta$ using the observed savings decisions of households in the data.

Suppose that a household decides to consume a constant amount in each time period: $c_t = c$ in all time periods. The per-period utility of this consumption is $u(c)$. What is the value of the infinite discounted utility $\sum_{t=0}^{\infty} \beta^t u(c)$? We can use the property of an infinite sum to evaluate this infinite discounted utility. Define $\Psi = \sum_{t=0}^{\infty} \beta^t u(c)$. Beginning in period $t = 0$, the discounted infinite sum

$$\Psi = \sum_{t=0}^{\infty} \beta^t u(c) = u(c) + \beta u(c) + \beta^2 u(c) + \ldots. \tag{4.3}$$

If we factor out $\beta$ from every element in the sum beginning in period $t = 1$, then

$$\Psi = \sum_{t=0}^{\infty} \beta^t u(c) = u(c) + \beta \{u(c) + \beta u(c) + \ldots\}. \tag{4.4}$$

The term inside the brackets is also equal to $\Psi$. This allows us to write the infinite sum in a recursive form:

$$\Psi = \sum_{t=0}^{\infty} \beta^t u(c) = u(c) + \beta \Psi. \tag{4.5}$$

We can solve this equation for $\Psi$:

$$(1 - \beta) \Psi = u(c) \tag{4.6}$$

$$\Psi = \frac{u(c)}{1 - \beta}.$$

Thus, the infinite sum

$$\Psi = \sum_{t=0}^{\infty} \beta^t u(c) = \frac{u(c)}{1 - \beta}. \tag{4.7}$$

### 4.1.4 Planner’s problem

As with the previous chapter’s static model, and any model in economics for that matter, there are two main concepts for how resources are allocated: via the market mechanism and via planner allocation. We begin our study of the neoclassical growth model by looking at the planner allocation. Solutions of the planner’s problem are Pareto efficient allocations.
If the welfare theorems are satisfied, the solution of the planner’s problem is identical to the equilibrium solution. A main goal of economists is to characterize the equilibrium solution. By citing the welfare theorems, economists can make their computations easier by dividing the equilibrium characterization into two manageable steps: (i) compute the solution to the planner’s problem and (ii) find the prices (whose existence is guaranteed by the Second Basic Welfare Theorem) for which the Pareto efficient allocation can be supported as an equilibrium allocation.

The planner has a resource allocation problem. The constraint for the planner is called the aggregate resource constraint, which requires that consumption plus investment equals output:

\[ c_t + i_t = f(k_t, 1) \quad \text{for all periods.} \quad (4.8) \]

Recall that GDP is defined as the total output of the economy, so we can think of \( f(k_t, 1) \) as GDP. From the expenditure definition of GDP, the total expenditure equals the expenditure on consumption and the expenditure on investment. In the neoclassical growth model, the households own the capital in the economy and make the investment decisions. It is perhaps more reasonable to write down a model in which the firms own the capital and make the investment decisions. But firms are publicly held and are owned by households (the only other agent in the model). If households own and manage the firms, then it is simpler and equivalent to just have the households own the capital directly.

The variable \( c_t \) is the household consumption, the variable \( k_t \) is the amount of capital stock held by a household in the beginning of period \( t \), and \( i_t \) is the amount of investment by a household in period \( t \). The capital stock \( k_t \) is owned by the households and rented to the firms for production. The labor is supplied inelastically, meaning that it is fixed at \( n_t = 1 \) in all periods.

The law of motion for capital provides a relation between the current and future capital stocks. Define \( \delta \in [0, 1] \) as the depreciation rate. The law of motion for capital is given by:

\[ k_{t+1} = (1 - \delta) k_t + i_t. \quad (4.9) \]

The law of motion says that the new capital stock is equal to the amount remaining after depreciation plus the new investment.

In the model, we typically use the law of motion for capital for investment to update the
aggregate resource constraint:

\[ c_t + k_{t+1} = f (k_t, 1) + (1 - \delta) k_t \text{ for all periods.} \tag{4.10} \]

The production function is given by \( f (K_t, N_t) \), where \( f : \mathbb{R}_+^2 \to \mathbb{R}_+ \).

The production function \( f \) is \( C^2 \), strictly increasing, and concave. Examples include:

- Linear production (AK technology): \( f (K_t, N_t) = AK_t \) for the total factor of productivity \( A > 0 \). Linear functions are concave, by definition.

- Decreasing returns: \( f (K_t, N_t) = A (k_t)^\theta (N_t)^{1-\theta} \) for the total factor of productivity \( A > 0 \) and some capital share \( \theta \in (0, 1) \).

A production function of the form \( f (K_t, N_t) = A (K_t)^\theta (N_t)^{1-\theta} \) is called a Cobb-Douglas production function. These production functions satisfy constant returns to scale, meaning that \( f (\alpha K_t, \alpha N_t) = \alpha f (K_t, N_t) \) for any scaling factor \( \alpha \geq 0 \).

The planner is benevolent, meaning that it has the same objective function as the households. The planner’s problem is given by (where the initial capital stock \( k_0 \) is a given parameter):

\[
\begin{align*}
\text{maximize} & \quad \sum_{t=0}^{\infty} \beta^t u (c_t) \\
\text{subject to} & \quad c_t + k_{t+1} = f (k_t, 1) + (1 - \delta) k_t \text{ in all periods} \\
& \quad k_0 \text{ given}
\end{align*}
\tag{4.11}
\]

The notation \( \{c_t, k_{t+1}\}_{t \in \mathbb{N}} \) means that the planner is choosing an infinite sequence of variables, one pair for each period \( t \in \mathbb{N} \). The set of natural numbers is \( \mathbb{N} = \{0, 1, ..., \} \).

### 4.1.5 Equilibrium

An equilibrium consists of the optimal choices of the firm and the optimal choices of the household, each of whom take the prices as given. In the case of the neoclassical growth model, the competitive prices will be the rate of return on capital and the wage rate. In addition to firm and household optimization, an equilibrium requires that all markets clear, i.e., that supply equals demand.
Firm problem

In an equilibrium, the firm chooses its capital and labor inputs in each period, taking as given the prices \((R_t, w_t)\). The price \(R_t\) is the rate of return on capital and the price \(w_t\) is the wage rate.

The firm problem is to maximize profits in all periods:

\[
\max_{K_t, N_t} \pi (K_t, N_t) = f (K_t, N_t) - R_t K_t - w_t N_t \quad \text{in all periods.} \tag{4.12}
\]

The firm maximizes profit by choosing the capital input \(K_t\) and the labor input \(N_t\). The first term \(f (K_t, N_t)\) is the output of the firm. The price of the commodity is normalized to 1, so firm revenue equals \(f (K_t, N_t)\). The term \(R_t K_t\) is the cost for the capital input and the term \(w_t N_t\) is the cost of the labor input. The entire term \(R_t K_t + w_t N_t\) is the cost for the firm. As always, profit is equal to revenue minus cost.

The firm maximizes profit by taking the first order conditions with respect to both capital and labor (the two choices of the firm). The first order condition with respect to capital is given by:

\[
D_1 f (K_t, N_t) - R_t = 0. \tag{4.13}
\]

This provides an equation for the rate of return on capital. The first order condition with respect to labor is given by:

\[
D_2 f (K_t, N_t) - w_t = 0. \tag{4.14}
\]

This provides an expression for the labor return.

One of the end-of-chapter exercises asks you to show that for a Cobb-Douglas production function \(f (K_t, N_t) = A (K_t)^\theta (N_t)^{1-\theta}\), the expressions for the rate of return on capital and the wage rate imply that the firm profit \(\pi (K_t, N_t) = 0\) in equilibrium.

Household problem

The household problem is a dynamic problem, given by:

\[
\max_{\{c_t, k_{t+1}\}_{t \in \mathbb{N}}} \sum_{t=0}^{\infty} \beta^t u (c_t)
\]

subject to \(c_t + i_t \leq R_t k_t + w_t + \pi_t\) in all periods \(k_{t+1} = (1 - \delta) k_t + i_t\). \(k_0\) given \(\quad \text{in all periods}. \tag{4.15}\)
The budget constraint for the household requires that expenditure is less than or equal to income. Expenditure consists of the sum of consumption plus investment. Household income consists of the capital income (principal plus interest earned on the capital stock brought into the period), labor income, and stock dividend. Since the household is the sole shareholder of the firm, the stock dividend is equal to the profit that the firm earns in the period (all profits are distributed as dividends).

In this manuscript, only Cobb-Douglas production functions are considered. Under such a production function, an end-of-chapter exercise asks you to show that \( \pi_t = 0 \) in all time periods.

**Market clearing**

In this setting, households inelastically supply labor, meaning that the labor is fixed at \( n_t = 1 \). In equilibrium, the capital choice of the household (the amount rented out to the firm for production) must be equal to the amount of capital that the firm chooses as an input in its production function:

\[
k_t = K_t.
\]

(4.16)

Additionally, in equilibrium, the labor supply choice of the household must be equal to the amount of labor that the firm is willing to hire for production:

\[
n_t = 1 = N_t.
\]

(4.17)

### 4.2 Euler equation

**4.2.1 Sneak peek**

**Summary**

Like most models in macroeconomics, the neoclassical growth model is a dynamic model with an infinite number of time periods. The neoclassical growth model includes a dynamic trade-off between more consumption in the current period and more investment (which allows for more consumption in future periods).

The mathematical machinery to analyze dynamic models in economics is borrowed from engineering. An Euler equation is a relation that captures the optimal trade-off between the present and the future.
4.2. **Euler Equation**

In any dynamic setting, rather than comparing the present period with all future periods (an infinite number), it is most convenient to analyze a pair of successive time periods. The agent recognizes the trade-off between more consumption in the current period and more investment that leads to a higher return in the next period. Using this methodology, all future time periods are presumed to be optimally solved, so the agent is only worried about the trade-off between the present period and the next period. There is an optimal balance between consumption and investment in the current period that allows the household to optimize consumption across both periods. This is captured by the Euler equation.

With an infinite number of time periods, an Euler equation must be satisfied between periods \( t = 0 \) and \( t = 1 \), between periods \( t = 1 \) and \( t = 2 \), and so forth ad infinitum.

Bear in mind the logic that we only seek to capture the optimal tradeoff between two successive periods. With this stored in your memory bank, we will soon introduce a useful set of tools to solve dynamic problems. But, before we can do that, we have to characterize the optimal tradeoff between two successive periods with an Euler equation.

**Notation**

The variables to be introduced in this section are given in the following table:

| \( \lambda_t \) | Lagrange multiplier for budget constraint in \( t \) |
| \( \mu_t \) | Lagrange multiplier for non-negativity constraint in \( t \) |

**Main takeaways**

After completing this section, you will be able to answer the following questions:

- How is the Euler equation derived?
- How is the transversality condition derived?
- If an infinite sequence \( \{c_t, k_{t+1}\} \) satisfies the Euler equations and the transversality condition, is it a solution to the planner’s problem? What about in the other direction? (Yes! and Yes!)

### 4.2.2 Formatting the planner’s problem

To find a Pareto efficient allocation, we solve the planner’s problem. The planner’s problem is in the form of a constrained maximization problem and we can apply the Kuhn-Tucker
4. NEOCLASSICAL GROWTH MODEL

conditions.

Necessary and sufficient conditions for an optimal solution of the planner’s problem include the Euler equations and the transversality condition, both of which are derived from the Kuhn-Tucker conditions.

Let’s consider the Kuhn-Tucker conditions associated with the planner’s problem

\[
\max_{\{c_t, k_{t+1}\}_{t \in \mathbb{N}}} \sum_{t=0}^{\infty} \beta^t u(c_t)
\]
subject to \( c_t + k_{t+1} = f(k_t, 1) + (1 - \delta) k_t \) in all periods \( k_0 \) given \( (4.18) \)

The aggregate resource constraint will be written as:

\[
f(k_t, 1) + (1 - \delta) k_t - c_t - k_{t+1} \geq 0. \quad (4.19)
\]

4.2.3 Euler derivation 1

Denote \( \lambda_t \) as the Lagrange multiplier associated with the aggregate resource constraint in time period \( t \). The Kuhn-Tucker conditions are given by:

- **First order conditions**
  
  - Derivative with respect to \( c_t \):
    \[
    \beta^t Du(c_t) - \lambda_t = 0. \quad (4.20)
    \]
  
  - Derivative with respect to \( k_{t+1} \):
    \[
    \lambda_{t+1} (D_1 f(k_{t+1}, 1) + (1 - \delta)) - \lambda_t = 0. \quad (4.21)
    \]

- **Complimentary slackness conditions (for the period \( t \) constraint):**
  
  \[
  \lambda_t (f(k_t, 1) + (1 - \delta) k_t - c_t - k_{t+1}) = 0. \quad (4.22)
  \]

Solve the first order condition with respect to \( c_t \) for the Lagrange multiplier \( \lambda_t \):

\[
\lambda_t = \beta^t Du(c_t). \quad (4.23)
\]
4.2. **EULER EQUATION**

The first order condition with respect to consumption $c_{t+1}$ is given by:

$$\beta^{t+1} Du (c_{t+1}) - \lambda_{t+1} = 0. \quad (4.24)$$

Solve the first order condition with respect to $c_{t+1}$ for the Lagrange multiplier $\lambda_{t+1}$:

$$\lambda_{t+1} = \beta^{t+1} Du (c_{t+1}). \quad (4.25)$$

Using these two expressions for the Lagrange multipliers, let’s re-consider the first order condition with respect to $k_{t+1}$:

$$\beta^t Du (c_t) = \beta^{t+1} Du (c_{t+1}) (D_1 f (k_{t+1}, 1) + 1 - \delta). \quad (4.26)$$

Dividing through by $\beta^t$ yields the Euler equation:

$$Du (c_t) = \beta Du (c_{t+1}) (D_1 f (k_{t+1}, 1) + 1 - \delta). \quad (4.27)$$

### 4.2.4 Euler derivation 2

Equivalently, we can obtain the Euler equation simply by inserting the aggregate resource constraint into the utility function and taking the derivative with respect to $k_{t+1}$. From the aggregate resource constraint

$$c_t = f (k_t, 1) + (1 - \delta) k_t - k_{t+1}. \quad (4.28)$$

This means that the maximization problem can be equivalently expressed as:

$$\maximize_{\{c_t, k_{t+1}\}_{t \in \mathbb{N}}} \sum_{t=0}^{\infty} \beta^t u (f (k_t, 1) + (1 - \delta) k_t - k_{t+1}), \quad (4.29)$$

where the infinite sum is:

$$\sum_{t=0}^{\infty} \beta^t u (f (k_t, 1) + (1 - \delta) k_t - k_{t+1}) = \ldots + \beta^t u (f (k_t, 1) + (1 - \delta) k_t - k_{t+1}) + \beta^{t+1} u (f (k_{t+1}, 1) + (1 - \delta) k_{t+1} - k_{t+2}) + \ldots \quad (4.30)$$
Take the derivative of this objective function with respect to $k_{t+1}$:

$$-\beta^t Du (f (k_t, 1) + (1 - \delta) k_t - k_{t+1})$$

$$+ \beta^{t+1} Du (f (k_{t+1}, 1) + (1 - \delta) k_{t+1} - k_{t+2}) \left( D_1 f (k_{t+1}, 1) + 1 - \delta \right) = 0.$$  \hspace{1cm} (4.31)

Rearranging and using the equations $c_t = f (k_t, 1) + (1 - \delta) k_t - k_{t+1}$ and $c_{t+1} = f (k_{t+1}, 1) + (1 - \delta) k_{t+1} - k_{t+2}$, we arrive at the Euler equation found above:

$$Du (c_t) = \beta Du (c_{t+1}) \left( D_1 f (k_{t+1}, 1) + 1 - \delta \right).$$  \hspace{1cm} (4.32)

4.2.5 **Step-by-step process for finding Euler equation**

This procedure is appropriate for finding the Euler equation in any macroeconomic model.

1. First, you want to take the constraint (here, the constraint is the aggregate resource constraint) and solve the constraint for current consumption. In this case, you have:

$$c_t = f (k_t, 1) + (1 - \delta) k_t - k_{t+1}.$$  \hspace{1cm} (4.33)

2. The expression for current consumption is in terms of $(k_t, k_{t+1})$ plus parameters. Define the right-hand side of the consumption function as $\gamma (k_t, k_{t+1})$ (the Greek letter $\gamma$ is the third letter of the Greek alphabet and corresponds to the English letter $c$).

3. The partial derivative $D_2 \gamma (k_t, k_{t+1})$ is the derivative of the function $\gamma (k_t, k_{t+1})$ with respect to the second term $k_{t+1}$. The partial derivative $D_1 \gamma (k_{t+1}, k_{t+2})$ is the derivative of the function $\gamma (k_{t+1}, k_{t+2}) = f (k_{t+1}, 1) + (1 - \delta) k_{t+1} - k_{t+2}$ with respect to the first term $k_{t+1}$. Notice that $\gamma (k_{t+1}, k_{t+2})$ is the equation for next-period consumption.

4. The Euler equation is always of the following form:

$$Du (c_t) D_2 \gamma (k_t, k_{t+1}) + \beta Du (c_{t+1}) D_1 \gamma (k_{t+1}, k_{t+2}) = 0.$$  \hspace{1cm} (4.34)

For the case of the planner’s problem, $D_2 \gamma (k_t, k_{t+1}) = -1$ and $D_1 \gamma (k_{t+1}, k_{t+2}) = D_1 f (k_{t+1}, 1) + 1 - \delta$. Putting these together, then the above equation is given by:

$$Du (c_t) = \beta Du (c_{t+1}) \left( D_1 f (k_{t+1}, 1) + 1 - \delta \right).$$  \hspace{1cm} (4.35)
4.2.6 Steady state

Once the Euler equation is found, we can solve for a "steady state". A steady state occurs when all of the variables are constant, meaning that the consumption value is \( c_t = c_{ss} \), the capital stock is \( k_t = k_{ss} \), the investment is \( i_t = i_{ss} \), and the output is \( y_t = f (k_t, 1) = y_{ss} \) in all periods. The steady state is a good initial benchmark to consider. We will eventually determine the equilibrium dynamics of the system, but most systems converge to the steady state in the limit as \( t \to \infty \), so it is important to know how the steady state is determined as a function of parameters.

1. Recall the Euler equation and consider that the Euler equation must hold for the steady state values:

\[
Du (c_{ss}) = \beta Du (c_{ss}) (D_1 f (k_{ss}, 1) + 1 - \delta). \tag{4.36}
\]

The marginal utility terms \( Du (c_{ss}) \) cancel and we can solve for \( k_{ss} \). If \( f (k_{ss}, 1) = k_{ss}^\theta \), then \( D_1 f (k_{ss}, 1) = \theta k_{ss}^{\theta - 1} \) and

\[
k_{ss} = \left( \frac{\theta}{1/\beta - (1 - \delta)} \right)^{\frac{1}{1-\theta}}. \tag{4.37}
\]

2. From the production function, \( y_{ss} = k_{ss}^\theta = \left( \frac{\theta}{1/\beta - (1 - \delta)} \right)^{\frac{\theta}{1-\theta}}. \)

3. Recall the law of motion for capital and consider this law of motion at steady state values:

\[
k_{ss} = (1 - \delta) k_{ss} + i_{ss}. \tag{4.38}
\]

This implies that \( i_{ss} = \delta k_{ss} = \delta \left( \frac{\theta}{1/\beta - (1 - \delta)} \right)^{\frac{1}{1-\theta}}. \)

4. Finally, recall the aggregate resource constraint and consider this constraint at steady state values:

\[
c_{ss} = y_{ss} - i_{ss} = \left( \frac{\theta}{1/\beta - (1 - \delta)} \right)^{\frac{\theta}{1-\theta}} - \delta \left( \frac{\theta}{1/\beta - (1 - \delta)} \right)^{\frac{1}{1-\theta}}. \tag{4.39}
\]
4.2.7 Transversality condition

With an optimization problem over an infinite time horizon, the optimal solutions must satisfy the Euler equations in all time periods plus the transversality condition:

\[
\lim_{t \to \infty} \beta^t Du(c_t) k_{t+1} = 0.
\] (4.40)

In an infinite time horizon model, the transversality condition is necessary and sufficient for an optimal solution. It serves as the ‘final period’ condition.

To see this, consider the truncated problem with a finite number of periods. Denote the final period as \( T \). In this final period \( T \), the household capital holdings cannot be negative: \( k_{T+1} \geq 0 \). These capital holdings are chosen in period \( T \) and are carried over into period \( T+1 \). Since there does not exist a period \( T + 1 \), we need to include the constraint \( k_{T+1} \geq 0 \) in the optimization problem. This basically says that households cannot hold debt when they exit the economy (or exit life). The truncated planner’s problem is given by:

\[
\begin{align*}
\text{maximize} & \quad \sum_{t=0}^{T} \beta^t u(c_t) \\
\text{subject to} & \quad c_t + k_{t+1} = f(k_t, 1) + (1 - \delta) k_t \text{ in all time periods } 0 \leq t \leq T \\
& \quad k_0 \text{ given} \\
& \quad k_{T+1} \geq 0
\end{align*}
\] (4.41)

Denote \( \mu_{T+1} \) as the Lagrange multiplier associated with the constraint \( k_{T+1} \geq 0 \). The Kuhn-Tucker conditions are given by:

- **First order conditions**
  
  - Derivative with respect to \( c_T \):
    
    \[
    \beta^T Du(c_T) - \lambda_T = 0.
    \] (4.42)
  
  - Derivative with respect to \( k_{T+1} \):
    
    \[-\lambda_T + \mu_{T+1} = 0.\] (4.43)

- **Complimentary slackness conditions** (for the constraint \( k_{T+1} \geq 0 \)):
4.3. DYNAMIC PROGRAMMING

\[ \mu_{T+1} k_{T+1} = 0. \]  

(4.44)

From the first order condition with respect to consumption \( c_T \), we can solve for the Lagrange multiplier \( \lambda_T \):

\[ \lambda_T = \beta^T Du(c_T). \]  

(4.45)

Making this substitution into the first order condition with respect to \( k_{T+1} \) yields:

\[ \mu_{T+1} = \beta^T Du(c_T). \]  

(4.46)

Inserting this expression into the complimentary slackness condition yields:

\[ \beta^T Du(c_T) k_{T+1} = 0. \]  

(4.47)

The infinite time horizon problem can be viewed as the limit of the truncated problem as \( T \to \infty \). Taking the limit yields the transversality condition:

\[ \lim_{T \to \infty} \beta^T Du(c_T) k_{T+1} = 0. \]  

(4.48)

4.3 Dynamic programming

4.3.1 Sneak peek

Summary

This section introduces the tools of dynamic programming. The Kuhn-Tucker conditions form the cornerstone of the theory of nonlinear programming. Nonlinear programming is restricted to Euclidean space, meaning that the vector of choice variables needs to be finite.

In a dynamic model with an infinite number of periods, the number of consumption variables is equal to the number of periods (infinite). As we have seen in the previous section, the optimal solution for the entire problem (the infinite time horizon) requires that the transversality condition is satisfied together with an infinite number of Euler equations. Euler equations form the basis for the discussion in this section, but we should immediately recognize that solving an infinite number of equations for an infinite number of unknowns is a fool’s errand.
The theory of dynamic programming exploits the fact that there are an infinite number of future periods by positing that the decision horizon looks the same in period $t = 1$ as it does in period $t = 100$. Suppose in either period that an agent begins with 100 units of capital stock. There are an infinite number of periods that follow period $t = 1$, but there are also an infinite number of periods that follow period $t = 100$. If the capital stock is identical in both periods, the optimal investment choices going forward should be identical as well.

With dynamic programming, rather than viewing the problem as one giant problem over an infinite number of time periods, we view it as an infinite number of problems over two successive periods. We solve for the optimal choice of an agent over periods $t$ and $t + 1$ about how much to consume and how much to invest in period $t$ (with investment in period $t$ leading to increased consumption in period $t + 1$).

Once we have solved for the optimal choice, then we know that the same choice will be optimal in the problem involving periods $t + 1$ and $t + 2$ and the one involving periods $t - 1$ and $t$, and so forth.

In order to capture the dynamics of variables, we recognize that periods $t$ and $t + 1$ are rarely identical as the capital stock is rarely the same in these two periods. If the capital stock is the same, then naturally the optimal choices going forward must be the same as well. Unless we have reached a steady state, the capital stock will not be the same. What we are solving in dynamic programming is not an optimal vector of values, but an optimal function. The optimal consumption and investment choices in period $t$ are functions of the capital stock $k_t$ brought into period $t$.

By solving for these optimal functions (one for consumption and one for investment), which mathematicians call the policy functions, we are able to determine the optimal sequences of consumption and investment.

This section provides a set of very general properties about dynamic programming and the optimal functions for consumption and investment. Future sections will provide the steps to apply such theory to solve problems.
4.3. **DYNAMIC PROGRAMMING**

Notation

The variables to be introduced in this section are given in the following table:

- \(c\) \hspace{1cm} \text{current period consumption} \\
- \(c'\) \hspace{1cm} \text{next period consumption} \\
- \(k\) \hspace{1cm} \text{capital stock brought into the current period} \\
- \(k'\) \hspace{1cm} \text{capital stock carried over to the next period} \\
- \(V : \mathbb{R}_+ \to \mathbb{R}\) \hspace{1cm} \text{value function} \\
- \(g : \mathbb{R}_+ \to \mathbb{R}_+\) \hspace{1cm} \text{policy function}

Main takeaways

After completing this section, you will be able to answer the following questions:

- What is the Bellman equation?
- What are the properties of the value function?
- Is solving the Bellman equation (finding the value function) equivalent to solving the infinite-time horizon planner’s problem? (Yes!)

4.3.2 Bellman equation

We derive in this section the Bellman equation associated with the planner’s problem. Define the state variable as \(k\). This is the amount of capital that a household holds at the beginning of the current period. The state variable contains all the information that the planner needs to know (about past decisions in the economy) in order to solve its problem in the current period. The choices variables are the current period consumption \(c\) and the amount of capital \(k'\) to be carried forward into the next period. The convention is that all current period variables are unprimed (such as \(k\)), while all variables in the next period are primed (such as \(k'\)).

The value function \(V : \mathbb{R}_+ \to \mathbb{R}\) specifies the value to the planner, where the value is the discounted infinite utility from optimal planner choices in all future periods. The value function satisfies the **Bellman equation**:

\[
V(k) = \max_{c \geq 0, k' \geq 0} u(c) + \beta V(k') \\
\text{subject to} \quad c + k' = f(k, 1) + (1 - \delta) k 
\]
Notice that the aggregate resource constraint has internalized the law of motion for capital:

\[ k' = (1 - \delta) k + i. \]  

(4.50)

The key thing to recognize about the Bellman equation is its recursive nature. The value function on the left hand side is the same as the value function on the right hand side. The values \( V(k) \) and \( V(k') \) are different when \( k \neq k' \), but the function itself is the same.

In order to solve one of these growth models, we need to solve for the value function. The following section introduces a simple method to solve for the value function.

Before solving for the value function, let’s write down 3 properties of the value function that hold in all economies.

### 4.3.3 Property 1

Property 1: Assuming that \( u \) is differentiable and concave, then \( V \) is differentiable and concave.

Property 1 means that we can take derivatives of the value function. This is good, because our first step is to take the first order conditions of the Bellman equation. Denote \( \lambda \) as the Lagrange multiplier associated with the aggregate resource constraint in the Bellman equation:

\[ f(k, 1) + (1 - \delta) k - c - k' = 0. \]  

(4.51)

The first order conditions with respect to consumption \( c \) is given by:

\[ Du(c) - \lambda = 0. \]  

(4.52)

The first order condition with respect to the capital choice \( k' \) is given by:

\[ \beta DV(k') - \lambda = 0. \]  

(4.53)

Combining the first order conditions yields:

\[ Du(c) = \beta DV(k'). \]  

(4.54)
4.3.4 Property 2

Property 2: Assuming that $u$ is differentiable, strictly increasing, and strictly concave, then $V$ is strictly increasing and there exists a unique optimal solution $(c, k')$.

From the above optimality condition (4.54), we know that if $Du(c) > 0$, then $DV(k') > 0$.

The following argument shows that if $u$ is strictly concave, then there exists a unique optimal solution $(c, k')$. Suppose otherwise. That is, there exist two optimal solutions $(c_1, k_1')$ and $(c_2, k_2')$, with $(c_1, k_1') \neq (c_2, k_2')$. Since both are optimal solutions, then both satisfy the aggregate resource constraint (4.51) and maximize the objective function:

$$u(c_1) + \beta V(k_1') = u(c_2) + \beta V(k_2') \geq u(c) + \beta V(k') \quad \text{for any} \quad (c, k') \text{ satisfying (4.51)}.$$  

Consider any convex combination

$$(c_\theta, k_\theta) = \theta(c_1, k_1') + (1 - \theta)(c_2, k_2')$$  

for $\theta \in (0, 1)$. The convex combination $(c_\theta, k_\theta)$ satisfies the aggregate resource constraint (4.51) since both $(c_1, k_1')$ and $(c_2, k_2')$ do. By the definition of the strict concavity of $u$:

$$u(\theta c_1 + (1 - \theta)c_2) > \theta u(c_1) + (1 - \theta)u(c_2). \quad (4.57)$$

From Property 1, $V$ is a concave function and from the definition of the concavity of $V$:

$$V(\theta k_1' + (1 - \theta)k_2') \geq \theta V(k_1') + (1 - \theta)V(k_2'). \quad (4.58)$$

Taken together, this implies that

$$u(c_\theta) + \beta V(k_\theta) > u(c_1) + \beta V(k_1') = u(c_2) + \beta V(k_2'), \quad (4.59)$$

which contradicts that either $(c_1, k_1')$ or $(c_2, k_2')$ are optimal solutions.

Thus, there exists a unique solution $(c, k')$.

Define the policy function $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ that maps from the current period state variable
$k$ into the next period state variable $k'$:

$$k' = g(k). \quad (4.60)$$

The choice of $k'$ must be optimal, meaning that it must satisfy the first order conditions (4.54). Using the aggregate resource constraint $c = f(k, 1) + (1 - \delta) k - g(k)$, this means that the function $g$ must be such that:

$$Du (f(k, 1) + (1 - \delta) k - g(k)) = \beta DV (g(k)). \quad (4.61)$$

Let us use the Bellman equation to evaluate the derivative $DV(k')$. The derivative value $DV(k)$ for any $k \geq 0$ can be found by applying the chain rule to the Bellman equation:

$$V(k) = u(f(k, 1) + (1 - \delta) k - g(k)) + \beta V(g(k)). \quad (4.62)$$

Notice that the Bellman equation does not include the max operator. This is because $g(k)$, by definition, dictates that the optimal choice is being made. The variable $k$ appears 3 times in the expression for $V(k)$. The chain rule implies:

$$DV(k) = (D_1 f(k, 1) + 1 - \delta) Du(c) - Dg(k) Du(c) + \beta Dg(k) DV(g(k)). \quad (4.63)$$

Collect the terms with the derivative $Dg(k)$:

$$DV(k) = (D_1 f(k, 1) + 1 - \delta) Du(c) - Dg(k) \{ Du(c) - \beta DV(g(k)) \}. \quad (4.64)$$

The term

$$Du(c) - \beta DV(g(k)) = 0 \quad (4.65)$$

from the first order condition (4.61). Thus, the derivative

$$DV(k) = (D_1 f(k, 1) + 1 - \delta) Du(c). \quad (4.66)$$

The fact that $DV(k) = (D_1 f(k, 1) + 1 - \delta) Du(c)$ is an outcome of the Envelope Theorem. The preceding paragraph offered a proof of this theorem for our particular economy (see the Mathematical Preliminaries chapter for the full statement of the theorem).
When we evaluate this derivative mapping at $k^0$, the derivative value is

$$DV(k^0) = (D_1f(k^0,1) + 1 - \delta)Du(c').$$

Plugging this expression into the equation (4.54) yields the Euler equation:

$$Du(c) = \beta Du(c') (D_1f(k^0,1) + 1 - \delta).$$

(4.68)

This is identical to the Euler equation derived in the previous section, where we switch from the Bellman equation notation to the notation in which periods are indicated with subscripts:

$$Du(c_t) = \beta Du(c_{t+1}) (D_1f(k_{t+1},1) + 1 - \delta).$$

(4.69)

As a check of our work, we can verify this with our step-by-step procedure for finding the Euler equation. Here, solving for consumption yields the functions $\gamma(k,k') = f(k,1) + (1 - \delta)k - k'$ and $\gamma(k',k'') = f(k',1) + (1 - \delta)k' - k''$. Recall that the Euler equation is always of the following form:

$$Du(c) D_2 \gamma(k,k') + \beta Du(c') D_1 \gamma(k',k'') = 0.$$

(4.70)

For the case of the Bellman equation, $D_2 \gamma(k,k') = -1$ and $D_1 \gamma(k',k'') = D_1f(k',1) + 1 - \delta$. Putting these together, then the above equation is given by:

$$Du(c) = \beta Du(c') (D_1f(k',1) + 1 - \delta).$$

(4.71)

4.3.5 Property 3

We are now positioned to verify an additional property about the value function $V : \mathbb{R}_+ \rightarrow \mathbb{R}$ and the policy function $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$.

Property 3: Assuming that $u$ is $C^2$, strictly increasing, and strictly concave, then $V$ is $C^2$ and $g$ is $C^1$ and strictly increasing.

The fact that $V$ is $C^2$ is a complicated and technical result. However, if $V$ is $C^2$, we can show without too much effort that $g$ is $C^1$ and strictly increasing by using the Implicit Function Theorem. Define the following equation in terms of the two variables $(k,k')$:

$$G(k,k') = -Du(f(k,1) + (1 - \delta)k - k') + \beta DV(k').$$

(4.72)
Notice that if \( G(k, k') = 0 \), then \( k' \) is the optimal choice for the household. Here, the parameter is \( p = k \) and the variable is \( v = k' \). The Implicit Function Theorem says that if \( D_{k'}G(k, k') \neq 0 \), then the implicit function \( g : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) from the parameter \( p = k \) to the variable \( v = k' \) is \( C^1 \) and the derivative of this implicit function is given by:

\[
Dg(k) = -\frac{D_kG(k, k')}{D_{k'}G(k, k')}. \tag{4.73}
\]

What is the derivative \( D_{k'}G(k, k') \)? From the chain rule:

\[
D_{k'}G(k, k') = D^2u(c) + \beta D^2V(k'). \tag{4.74}
\]

Since \( u \) is strictly concave, then \( D^2u(c) < 0 \). From Property 1, since \( V \) is concave, then \( D^2V(k') \leq 0 \). The sum \( D_{k'}G(k, k') < 0 \).

What is the derivative \( D_kG(k, k') \)? From the chain rule:

\[
D_kG(k, k') = -D^2u(c) D_1f(k, 1). \tag{4.75}
\]

Since \( u \) is strictly concave and \( f \) is strictly increasing, then

\[
D_kG(k, k') = -D^2u(c) D_1f(k, 1) > 0. \tag{4.76}
\]

Thus the derivative of the implicit function is

\[
Dg(k) = -\frac{-D^2u(c) D_1f(k, 1)}{D^2u(c) + \beta D^2V(k')} = -\frac{(+)}{(-)} > 0. \tag{4.77}
\]

The derivative \( Dg(k) > 0 \) for any \( k \), meaning that \( g \) is strictly increasing.

### 4.4 The AK model

#### 4.4.1 Sneak peek

**Summary**

The AK model is less of a model in its own right and more a specification of economic parameter values that make solving the Bellman equation straightforward. Recall in the neoclassical growth model that the two factors of production are capital and labor. In the
AK specification, firm output only depends upon capital, and that dependence is linear in terms of the capital stock. This can equivalently be viewed as a Cobb-Douglas production function with capital and labor as inputs in which the capital share equals 1 and the labor share equals 0.

The name 'AK' refers to the fact that the production function is of the form such that output is equal to a constant term $A$ multiplied by the current capital stock $K$. An additional feature of the AK specification is that the utility function for households is natural log utility.

In this setting, we are able to solve the Bellman equation by solving for the optimal savings rate and value function for the planner. This section provides a cookbook procedure to apply the tools of dynamic programming to solve a Bellman equation. This same procedure will be applied throughout the remainder of the chapter to solve more complicated versions of the neoclassical growth model.

One of the main predictions of the AK specification is that the savings rate is a constant value, meaning that the amount saved every period is a constant fraction of the total output of the economy. As it turns out, this is not just a feature of the AK specification, but of any specification of the neoclassical growth model with Cobb-Douglas production and natural log utility. We adopt such specifications in this chapter as we want to be able to solve the models using pencil and paper. More complicated models, perhaps with parameter specifications more consistent with data, can be solved on the computer.

**Notation**

The variables to be introduced in this section are given in the following table:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>cake</td>
<td>a sweet food, baked, often served as dessert</td>
</tr>
<tr>
<td>$s : \mathbb{R}_+ \to \mathbb{R}$</td>
<td>savings rate</td>
</tr>
<tr>
<td>$m$</td>
<td>slope coefficient</td>
</tr>
<tr>
<td>$b$</td>
<td>intercept coefficient</td>
</tr>
</tbody>
</table>

**Main takeaways**

After completing this section, you will be able to answer the following questions:

- How do you solve for the value function and the policy function by guessing and checking the value function?
- How do you solve for the policy function by guessing and checking the policy function?
4.4.2 The economy specification

For the simplest economy, suppose that \( f(k, 1) = Ak \) with coefficient \( A = 1 \), \( \delta = 1 \) (complete depreciation), and \( u(c) = \ln(c) \). This problem is commonly referred to as the cake-eating problem. Suppose that \( k_0 \) is the initial size of the cake. The state variable \( k_t \) specifies the amount of cake remaining at the beginning of period \( t \). This amount \( k_t \) can be split between (i) amount consumed in period \( t \), denoted \( c_t \), and (ii) the amount not consumed. Whatever is not consumed is denoted \( k_{t+1} \) and is stored for the next period. In the next period, there is no cake spoilage (loss) nor cake regeneration (growth), so the amount of cake remaining is \( k_{t+1} \). This means that the production function is given by \( f(k_{t+1}, 1) = k_{t+1} \). Such a production function is referred to as a storage technology.

The laws of nature require that the sum of the consumed amount and the stored amount must be equal to the original amount of cake in the beginning of the period. This is simply the planner’s aggregate resource constraint:

\[
c_t + k_{t+1} = k_t. \quad (4.78)
\]

Let’s write down the Bellman equation for the planner:

\[
V(k) = \max_{c \geq 0, k' \geq 0} \ln(c) + \beta V(k') \quad \text{subject to} \quad c + k' = k. \quad (4.79)
\]

4.4.3 Guess and check the value function

There are two equivalent methods that can be used to solve these growth models. Both methods employ the time-honored approach of guess and check. The first method will have us solving for the value function. I list the following steps of this cookbook procedure and then illustrate in the following subsection with an example:

1. From the constraint, solve for the consumption.
2. Insert this expression for consumption into the utility function in the Bellman equation.
3. Guess the functional form \( V(k) = m \ln(k) + b \) and use this guess in the Bellman equation.
4. Take the first order condition with respect to the choice variable \( k' \). Use the first order condition to solve for the optimal \( k' \).
5. Insert the expressions for the optimal $k'$ back into the Bellman equation.

6. The Bellman equation must be satisfied for all values of the state variable $k$. Divide the terms in the Bellman equation into one of two types: (i) those containing $\ln(k)$ and (ii) constant terms.

7. Set up two equations. The first will require equality between the terms containing $\ln(k)$ on both sides of the equation, while the second will require equality between the constant terms on both sides of the equation.

8. Solve the two equations for the two unknown coefficients $m$ and $b$.

Congratulations. You have now solved for the equilibrium value function.

4.4.4 Example: guess and check the value function

Step 1

From the constraint, $c = k - k'$.

Step 2

The Bellman equation is updated as:

$$V(k) = \max_{c \geq 0, k' \geq 0} \ln(k - k') + \beta V(k') \tag{4.80}$$

Step 3

We do not know what the value function $V$ is, but we can take a guess about its functional form. We guess that

$$V(k) = m \ln(k) + b \tag{4.81}$$

for some unknown coefficients $m$ and $b$.

The Bellman equation is updated as:

$$m \ln(k) + b = \max_{k' \geq 0} \ln(k - k') + \beta \{m \ln(k') + b\} \tag{4.82}$$
Step 4

Let’s consider the first order conditions of the Bellman equation, which are the partial derivatives with respect to the choice variable $k'$:

$$\frac{1}{k-k'} - \frac{\beta m}{k'} = 0. \quad (4.83)$$

Let’s go ahead and solve for $k'$ as a function of $k$. First, bring the negative term to the other side of the equation and cross-multiply:

$$k' = \beta m (k-k'). \quad (4.84)$$

Next add $\beta mk'$ to both sides in order to gather the $k'$ terms together:

$$k' (1 + \beta m) = \beta mk. \quad (4.85)$$

Finally divide both sides by $(1 + \beta m)$ in order to solve for $k'$:

$$k' = \frac{\beta m}{1 + \beta m} \cdot k. \quad (4.86)$$

The consumption is $c = k - k'$, which can be evaluated as:

$$c = k - \frac{\beta m}{1 + \beta m} k = \frac{1}{1 + \beta m} k. \quad (4.87)$$

Step 5

Now that we know the optimal choices $(c, k')$ as a function of $k$, we can write down the Bellman equation without the max operator:

$$m \ln(k) + b = \ln \left( \frac{1}{1+\beta m} k \right) + \beta \left\{ m \ln \left( \frac{\beta m}{1+\beta m} k \right) + b \right\}. \quad (4.88)$$
Step 6

We can then use the properties of natural logs, namely \( \ln(ab) = \ln(a) + \ln(b) \) and \( \ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b) \), to expand out the right-hand side:

\[
m \ln(k) + b = \ln(k) - \ln(1 + \beta m) + \beta m \ln(\beta m) + \beta m \ln(k) - \beta m \ln(1 + \beta m) + \beta b.
\]

Step 7

The left-hand side and the right-hand side must be identical for all values of \( k \). Thus, it must be that the terms involving \( \ln(k) \) must cancel out on the left and right hand sides and the constant terms must cancel out on the left and right hand sides.

This implies the following 2 equations must be satisfied:

\[
m \ln(k) = \ln(k) + \beta m \ln(k).
\]

\[
b = -\ln(1 + \beta m) + \beta m \ln(\beta m) - \beta m \ln(1 + \beta m) + \beta b.
\]

Step 8

From the first equation, we obtain an equation only in terms of the unknown coefficient \( m \):

\[
m = 1 + \beta m.
\]

Solving this equation for \( m \) yields \( m = \frac{1}{1-\beta} \).

Using this value for \( m \), the second equation can be solved for the unknown coefficient \( b \). As we will see, solving for \( b \) is not important at all, so the details are omitted. The solution for the curious reader is

\[
b = \frac{\ln(1 - \beta) + \frac{\beta}{1-\beta} \ln(\beta)}{1 - \beta}.
\]

The policy function is given by

\[
g(k) = \frac{\beta m}{1 + \beta m} k.
\]
4. NEOCLASSICAL GROWTH MODEL

from above. Since $m = \frac{1}{1-\beta}$, then we can solve for the policy function:

$$g(k) = \frac{\beta \left( \frac{1}{1-\beta} \right)}{1 + \beta \left( \frac{1}{1-\beta} \right)} k.$$  \hspace{1cm} (4.94)

We can multiply all terms in the numerator and denominator by $1 - \beta$. This yields

$$g(k) = \frac{\beta}{1 - \beta + \beta} k = \beta k. \hspace{1cm} (4.95)$$

And the consumption is then given by $c = (1 - \beta) k$. This means that the constant fraction $(1 - \beta)$ of the cake is eaten in every period.

The savings rate $s$ will be defined to satisfy the following equation:

$$k' = g(k) = (1 + s) k.$$  \hspace{1cm} (4.96)

Here, the savings rate is equal to $s = \beta - 1$. This savings rate is constant, but negative. This makes perfect sense as the opposite of savings is taking place in this particular economy (the cake is slowly being eaten).

4.4.5 Guess and check the policy function

The second method will solve for the policy function directly. This method is considered easier as it has fewer steps, but there is also the potential for an economic error. One has to be extremely careful to write down the Euler equation correctly. I list the following steps of this cookbook procedure and then illustrate in the following subsection with an example:

1. Find the Euler equation.

2. Express the consumption terms in the Euler equation only in terms of the state variable $k$.

3. Guess the functional form $g(k) = mk$ and use this guess in the Euler equation. We guess that $g(k)$ is a linear function of the firm output. Here output is $f(k, 1) = k$.

4. Solve for the unknown coefficient $m$.

Congratulations. You have now solved for the equilibrium policy function.
4.4.6 Example: Guess and check the policy function

Step 1

The Bellman equation is given by:

$$V(k) = \max_{c \geq 0, k' \geq 0} \ln(c) + \beta V(k') \quad \text{subject to } \quad c + k' = k.$$  \hspace{1cm} (4.97)

The first order conditions with respect to consumption $c$ is given by:

$$\frac{1}{c} - \lambda = 0.$$  \hspace{1cm} (4.98)

The first order condition with respect to the capital choice $k'$ is given by:

$$\beta D V(k') - \lambda = 0.$$  \hspace{1cm} (4.99)

Combining the first order conditions yields:

$$\frac{1}{c} = \beta D V(k').$$  \hspace{1cm} (4.100)

From the Envelope Theorem,

$$D V(k') = Du(c') D_1 f(k', 1) = \frac{1}{c'}.$$  \hspace{1cm} (4.101)

This means that the Euler equation is given by:

$$\frac{1}{c} = \beta \frac{1}{c'}.$$  \hspace{1cm} (4.102)

As a check of our work, we can verify this with our step-by-step procedure for finding the Euler equation. Here, solving for consumption yields the functions $\gamma(k, k') = k - k'$ and $\gamma(k', k'') = k' - k''$. Recall that the Euler equation is always of the following form:

$$Du(c) D_2 \gamma(k, k') + \beta Du(c') D_1 \gamma(k', k'') = 0.$$  \hspace{1cm} (4.103)

For the case of this simple economy, $Du(c) = \frac{1}{c}$, $D_2 \gamma(k, k') = -1$, $Du(c') = \frac{1}{c'}$, and
4. NEOCLASSICAL GROWTH MODEL

$D_1 \gamma (k', k'') = 1$. Putting these together, then the above equation is given by:

$$\frac{1}{c} = \beta \frac{1}{c'}.$$  \hspace{1cm} (4.104)

**Step 2**

The current period consumption is expressed by:

$$c = f(k, 1) - g(k) = k - g(k).$$  \hspace{1cm} (4.105)

The next period consumption is expressed as:

$$c' = f(k') - g(k') = k' - g(k').$$  \hspace{1cm} (4.106)

This means that the Euler equation is updated as:

$$\frac{1}{k - g(k)} = \beta \frac{1}{k' - g(k')}.$$  \hspace{1cm} (4.107)

**Step 3**

We guess that the policy function is of the form $g(k) = mk$ for some unknown coefficient $m$. This means that $g(k') = mk'$. Plugging this functional form into the Euler equation yields:

$$\frac{1}{k - mk} = \beta \frac{1}{k' - mk'}.$$  \hspace{1cm} (4.108)

**Step 4**

This equation can be solved for $m$:

$$\frac{1}{k (1 - m)} = \beta \frac{1}{k' (1 - m)}.$$  \hspace{1cm} (4.109)

The terms that can cancel are $(1 - m)$:

$$\frac{1}{k} = \frac{\beta}{k'}.$$  \hspace{1cm} (4.110)
Use the policy function guess again, \( k' = g(k) = mk \). The Euler equation becomes:

\[
\frac{1}{k} = \frac{\beta}{mk}.
\]

(4.111)

This means that \( m = \beta \), so the policy function is given by \( g(k) = \beta k \). This is identical to what we obtained by solving for the value function.

## 4.5 Equilibrium solution with taxes

### 4.5.1 Sneak peek

**Summary**

The previous section considered one specification of the economy, namely the AK specification in which the production function only depends upon the capital stock. Previous descriptions of the model assumed that households supply labor inelastically, meaning that the labor supply is fixed at a certain value and not an actual choice of households.

This section relaxes these requirements by considering the standard specification of the neoclassical growth model in which labor is elastically supplied (meaning the labor supply is chosen by households) and firm output is determined by a Cobb-Douglas production function in terms of the two factors of production: capital and labor.

This section will keep one important specification, namely that the utility function for households is of the natural log form. This allows for a closed form solution to the Bellman equation.

The neoclassical growth model is extended in this section to include a tax. The type of tax modeled is an investment tax. It is possible to include different types of taxes in the model, but the investment tax is the easiest to work with.

With this model, we are interested in finding the equilibrium solution and determining the relationship between the level of taxation and the savings rate and labor supply. Solving the planner’s problem does us no good for the present model as the presence of taxes renders the welfare theorems invalid. This means that the solution to the planner’s problem is NOT equivalent to the equilibrium solution. We are left with no choice but to solve for the equilibrium solution directly. This is not as daunting as it might sound, but I will introduce the additional steps that we need to add to our cookbook procedure in order to accomplish this objective.
4. NEOCLASSICAL GROWTH MODEL

Notation

The variables to be introduced in this section are given in the following table:

- $R$: rate of return on capital
- $w$: wage rate
- $\tau$: investment tax rate
- $\gamma$: labor disutility parameter
- $k_{ss}$: steady state capital stock
- $Y_{ss}$: steady state output

Main takeaways

After completing this section, you will be able to answer the following questions:

- What is the equilibrium policy function and consumption function in the model with investment tax?
- What effect does investment taxation have on steady state output?

4.5.2 Adding taxes to the model

The production function $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ is assumed to be of the Cobb-Douglas form:

$$ f(K_t, N_t) = (K_t)^\theta (N_t)^{1-\theta} $$

(4.112)

for the capital share $\theta \in (0, 1)$.

The government imposes a tax on capital investment $i_t$ by the households. The tax rate is $\tau \geq 0$. Suppose that depreciation $\delta = 1$, which implies that investment equals next period capital stock: $k_{t+1} = i_t$. Adding in the investment tax changes the budget constraint of the household to

$$ c_t + (1 + \tau) k_{t+1} = R_t k_t + w_t n_t + T_t. $$

(4.113)

The term $T_t$ is a lump-sum transfer that the household receives from the government. The transfer can be either positive or negative. As with the prices $R_t$ and $w_t$, the household takes the lump-sum transfer as given. The tax enters as $(1 + \tau)$ since household expenditures are increased by $\tau k_{t+1}$, the size of the tax collected by the government.
In this section, we will introduce a labor-leisure trade-off. This means that labor is elastically supplied as households choose the fraction of time that they want to devote to labor each period. The utility function with the labor-leisure trade-off is given by:

\[
\sum_{t=0}^{\infty} \beta^t \{ u(c_t) - \gamma n_t \}.
\] (4.114)

The term \( \gamma n_t \) is the disutility from labor. The parameter \( \gamma > 0 \) captures the size of the disutility.

### 4.5.3 Equilibrium

The market clearing conditions for the capital and labor markets are the same as previously introduced: \( k_t = K_t \) and \( n_t = N_t \). In this model, the household labor supply is elastically supplied, meaning that households choose the fraction of time \( n_t \in [0, 1] \) that they choose to devote to labor.

In equilibrium, the firm is maximizing profit, meaning that the first order conditions of the firm problem must be satisfied:

\[
R_t = D_1 f(K_t, N_t) = \theta (K_t)^{\theta-1} (N_t)^{1-\theta}.
\] (4.115)
\[
w_t = D_2 f(K_t, N_t) = (1 - \theta) (K_t)^{\theta} (N_t)^{-\theta}.
\]

The government collects the tax \( \tau k_{t+1} \) from the household in period \( t \). The government distributes the lump-sum transfer \( T_t \) to the household in period \( t \). The government serves no other purpose and does not incur government expenditures. In equilibrium, the government must balance its budget:

\[
\tau k_{t+1} = T_t.
\] (4.116)

An equilibrium consists of an optimal solution to the household problem, which is the dynamic maximization problem with a budget constraint in all periods, the satisfaction of the government budget constraint, and the satisfaction of the market clearing conditions. In the neoclassical growth model, we only need to verify that the capital and labor market clearing conditions hold. This is because Walras’ Law states that the total income in the economy is equal to the total expenditures. With a single commodity and both the capital and labor markets clearing, then consumption plus investment equals output, which is a restatement of the aggregate resource constraint.
4.5.4 Solving the Bellman equation

The Bellman equation is given by

\[ V(k) = \max_{c \geq 0, k' \geq 0, n \in [0,1]} \ln(c) - \gamma n + \beta V(k') \]
subject to \( c + (1 + \tau) k' = Rk + wn + T \). \hfill (4.117)

To solve the Bellman equation, we adopt the method where we solve for the policy function. The cookbook procedure steps need to be updated a bit in order to account for the facts that (i) the constraint is now the household budget constraint instead of the aggregate resource constraint and (ii) the household is choosing its labor supply. The steps are given by:

1. Find the Euler equation.

2. Express the consumption terms in the Euler equation only in terms of the state variable \( k \).

   (a) This means that the term \( Rk + wn \) needs to be expressed as a function of \( k \) only (using the equilibrium definitions of \( R \) and \( w \)). You will find that \( Rk + wn = k^\theta n^{1-\theta} \), which equals the output of the firm.

   (b) This means that the term \( R'k' + w'n' \) needs to be expressed as a function of \( k' \) only (using the equilibrium definitions of \( R' \) and \( w' \)).

3. Guess that the policy function has the form \( g(k) = mk^\theta n^{1-\theta} \) and use this guess in the Euler equation. The term \( k^\theta n^{1-\theta} \) is the firm production, so the guess imposes that the policy function is linear in the firm production \( k^\theta n^{1-\theta} \).

4. Solve for the unknown coefficient \( m \).

5. Take the first order condition with respect to the labor supply choice \( n \) to solve for the labor supply.

The following is very important, and I cannot stress it enough. When following these steps, make sure you go in order. You should not, under any circumstances, express the term \( Rk + wn \) in terms of \( k \) (Step 2a) before finding the Euler equation (Step 1).
4.5. EQUILIBRIUM SOLUTION WITH TAXES

Step 1

The Bellman equation is given by:

\[
V (k) = \max_{c \geq 0, k' \geq 0, n \in [0,1]} \ln(c) - \gamma n + \beta V (k')
\]
subject to \( c + (1 + \tau) k' = Rk + wn + T \) \( . \) (4.118)

Denote \( \lambda \) as the Lagrange multiplier of the budget constraint.

The first order condition with respect to consumption \( c \) is given by:

\[
\frac{1}{c} - \lambda = 0.
\] (4.119)

The first order condition with respect to the capital choice \( k' \) is given by:

\[
\betaDV (k') - \lambda (1 + \tau) = 0.
\] (4.120)

Combining the first order conditions yields:

\[
\frac{1 + \tau}{c} = \beta DV (k').
\] (4.121)

From the Envelope Theorem,

\[
DV (k') = Du (c') R'
\]
\[
= \frac{R'}{c'}.
\] (4.122)

Here, \( R' \) is the rate of return on capital in the next period (recall that next period variables are primed). This means that the Euler equation is given by:

\[
\frac{1 + \tau}{c} = \beta \frac{R'}{c'}.
\] (4.123)

As a check of our work, we can verify this with our step-by-step procedure for finding the Euler equation. Here, solving for consumption yields the functions \( \gamma (k, k') = Rk + wn + T - (1 + \tau) k' \) and \( \gamma (k', k'') = R'k' + w'n' + T' - (1 + \tau) k'' \). Recall that the Euler equation is always of the following form:

\[
Du (c) D_2 \gamma (k, k') + \beta Du (c') D_1 \gamma (k', k'') = 0.
\] (4.124)
For the case of this simple economy, \( D_u(c) = \frac{1}{c}, \ D_2\gamma(k,k') = -(1+\tau), \ D_u(c') = \frac{1}{c'}, \) and \( D_1\gamma(k',k'') = R'. \) Putting these together, then the above equation is given by:

\[
\frac{1 + \tau}{c} = \beta \frac{R'}{c'}.
\] (4.125)

**Step 2(a)**

The current period consumption is given by:

\[
c = Rk + wn + T - (1 + \tau) g(k).
\] (4.126)

We need to express the income \( Rk + wn \) only in terms of \( k. \) How do we do this? Consider the equilibrium expressions for \( R \) and \( w \) from the first order conditions of the firm problem:

\[
R = \theta k^{\theta - 1} n^{1-\theta},
\]

\[
w = (1-\theta) k^\theta n^{-\theta}.
\] (4.127)

This means that the sum

\[
Rk + wn = \theta k^{\theta - 1} n^{1-\theta} k + (1-\theta) k^\theta n^{-\theta} n
\]

\[
= \theta k^\theta n^{1-\theta} + (1-\theta) k^\theta n^{1-\theta} = k^\theta n^{1-\theta}.
\] (4.128)

Here \( n \) is actually a function of \( k \) and should be written as \( n(k). \) However, we keep the notation simple and only write \( n. \)

In equilibrium, the term \( T = \tau k' \), meaning that:

\[
c = k^{\theta} n^{1-\theta} - g(k).
\] (4.129)

**Step 2(b)**

The next period consumption is given by:

\[
c' = R'k' + w'n' + T' - (1 + \tau) g(k').
\] (4.130)

We need to express the income \( R'k' + w'n' \) only in terms of \( k'. \) How do we do this? Exactly as in Step 2(a), we obtain:

\[
R'k' + w'n' = (k')^{\theta} (n')^{1-\theta}.
\] (4.131)
4.5. **EQUILIBRIUM SOLUTION WITH TAXES**

Here $n'$ is actually a function of $k'$ and should be written as $n(k')$, though we simply write $n'$. In equilibrium, the term $T' = \tau g(k')$, meaning that:

\[ c' = (k')^\theta (n')^{1-\theta} - g(k'). \] (4.132)

**Step 3**

In the Euler equation, the term

\[ R' = \theta (k')^{\theta-1} (n')^{1-\theta}. \] (4.133)

The Euler equation is now given by:

\[
\frac{1 + \tau}{k^\theta n^{1-\theta} - g(k)} = \beta \frac{\theta (k')^{\theta-1} (n')^{1-\theta}}{(k')^{\theta-1} (n')^{1-\theta} - g(k')}. \] (4.134)

We guess that the policy function is proportional to the firm output:

\[ g(k) = mk^\theta n^{1-\theta}. \] (4.135)

The policy function coefficient is the same no matter which capital stock is being considered, so $g(k') = m (k')^{\theta} (n')^{1-\theta}$.

**Step 4**

Plug these guesses into the Euler equation:

\[
\frac{1 + \tau}{k^\theta n^{1-\theta} - mk^\theta n^{1-\theta}} = \frac{\theta (k')^{\theta-1} (n')^{1-\theta}}{(k')^{\theta-1} (n')^{1-\theta} - m (k')^{\theta} (n')^{1-\theta}}.
\] (4.136)

In the above equation, we can cancel $\theta (k')^{\theta-1} (n')^{1-\theta}$ from both the numerator and the denominator in the right-hand side:

\[
\frac{1 + \tau}{k^\theta n^{1-\theta} - mk^\theta n^{1-\theta}} = \frac{\beta \theta}{k' - mk'}.
\] (4.137)

Now the term $(1 - m)$ can be canceled from both denominators:

\[
\frac{1 + \tau}{k^\theta n^{1-\theta}} = \frac{\beta \theta}{k'}. \] (4.138)
Use the policy function guess \( k' = g(k) = mk^\theta n^{1-\theta} \) again:

\[
\frac{1 + \tau}{k^\theta n^{1-\theta}} = \frac{\beta \theta}{mk^\theta n^{1-\theta}}.
\] (4.139)

Cancelling out \( k^\theta n^{1-\theta} \) from both denominators, we are left with:

\[
1 + \tau = \frac{\beta \theta}{m}.
\] (4.140)

Solving for \( m \) yields:

\[
m = \frac{\theta \beta}{1 + \tau}.
\] (4.141)

Thus, the policy function is given by:

\[
g(k) = \frac{\theta \beta}{1 + \tau} k^\theta n^{1-\theta}.
\] (4.142)

After using the government budget balance equilibrium condition, the consumption was previously found to be:

\[
c = (1 - m) k^\theta n^{1-\theta},
\] (4.143)

meaning that the consumption function is given by:

\[
c(k) = \left(1 - \frac{\theta \beta}{1 + \tau}\right) k^\theta n^{1-\theta}.
\] (4.144)

Notice that the policy function \( k' = g(k) \) is strictly decreasing in \( \tau \) and the consumption function \( c(k) \) is strictly increasing in \( \tau \). The presence of the tax distorts the household’s consumption-savings decision problem and leads to an "inefficient" low level of investment. The term "inefficient" means that the households are investing less than what the planner would prescribe.

**Step 5**

Recall the Bellman equation

\[
V(k) = \max_{c \geq 0, k' \geq 0, n \in [0,1]} \ln(c) - \gamma n + \beta V(k')
\]

subject to \( c + (1 + \tau) k' = Rk + wn + T \). (4.145)
4.5. *EQUILIBRIUM SOLUTION WITH TAXES*

The first order condition with respect to consumption $c$ is given by:

\[
\frac{1}{c} - \lambda = 0. \tag{4.146}
\]

The first order condition with respect to labor $n$ is given by:

\[-\gamma + w\lambda = 0. \tag{4.147}\]

Combining the two previous equations:

\[
\frac{w}{c} = \gamma. \tag{4.148}
\]

This equation says that the marginal benefit from labor (wages times the marginal utility) must be equal to the marginal cost of labor.

From the equilibrium expression for $w$ from the first order condition of the firm problem:

\[w = (1 - \theta) k^\theta n^{-\theta}. \tag{4.149}\]

We have just found that the consumption function is given by:

\[c(k) = \left(1 - \frac{\theta \beta}{1 + \tau}\right) k^\theta n^{1-\theta}. \tag{4.150}\]

The first order condition with respect to labor can then be solved for the labor supply:

\[
\frac{(1 - \theta) k^\theta n^{-\theta}}{(1 - \frac{\theta \beta}{1 + \tau}) k^\theta n^{1-\theta}} = \gamma. \tag{4.151}\]

The term $k^\theta n^{-\theta}$ cancels out of both the numerator and the denominator:

\[
\frac{(1 - \theta)}{(1 - \frac{\theta \beta}{1 + \tau})} n = \gamma. \tag{4.152}\]

This means that the labor supply is

\[n = \frac{(1 - \theta)}{(1 - \frac{\theta \beta}{1 + \tau}) \gamma}. \tag{4.153}\]

Notice that the labor supply is constant and does not vary with $k$. 
4.5.5 Steady state tax analysis

A steady state is defined such that consumption, output, and capital stock remain constant across periods. We ensure that all variables are constant by making sure that the capital stock (the state variable) is constant: \( k = k' \) or \( k_{ss} = g(k_{ss}) \). We can evaluate the steady state capital stock:

\[
k_{ss} = \frac{\theta \beta}{1 + \tau} (k_{ss})^\theta n_{ss}^{1-\theta}.
\] (4.154)

Multiply both sides of the equation by \((k_{ss})^{-\theta}\):

\[
(k_{ss})^{1-\theta} = \frac{\theta \beta}{1 + \tau} n_{ss}^{1-\theta}.
\] (4.155)

Raise both sides of the equation to the exponent \( \frac{1}{1-\theta} \):

\[
k_{ss} = \left( \frac{\theta \beta}{1 + \tau} \right)^{\frac{1}{1-\theta}} n_{ss}.
\] (4.156)

As a check of this, we can use the Euler equation directly in order to find the equation for steady state capital. Recall the Euler equation (evaluated at steady state values)

\[
\frac{1 + \tau}{c_{ss}} = \beta \frac{R_{ss}}{c_{ss}},
\] (4.157)

where the steady state rate of return on capital is \( R_{ss} = \theta (k_{ss})^{\theta-1} (n_{ss})^{1-\theta} \). Solving the Euler equation for \( k_{ss} \) yields:

\[
k_{ss} = \left( \frac{\theta \beta}{1 + \tau} \right)^{\frac{1}{1-\theta}} n_{ss}.
\] (4.158)

The aggregate output of the economy (or the GDP) in steady state is given by:

\[
Y_{ss} = f(k_{ss}, n_{ss}) = (k_{ss})^\theta n_{ss}^{1-\theta}.
\] (4.159)

Plugging in the expression for the steady state capital yields:

\[
Y_{ss} = \left( \frac{\theta \beta}{1 + \tau} \right)^{\frac{\theta}{1-\theta}} n_{ss}.
\] (4.160)

Notice that the steady state output is linear in the labor supply \( n_{ss} \). Labor supply is constant in every period and equal to its steady state value. We can now insert the equilibrium labor
4.5. EQUILIBRIUM SOLUTION WITH TAXES

supply from (4.153) into the steady state output equation:

\[ Y_{ss} = \left( \frac{\theta \beta}{1 + \tau} \right)^{\frac{\theta}{1-\tau}} \frac{(1 - \theta)}{(1 - \frac{\theta \beta}{1+\tau})^\gamma}. \]  \hspace{1cm} (4.161)

Consider country A with a high investment tax (\( \tau^A \)) and country B with a low investment tax (\( \tau^B \)). Suppose that \( \tau^A = 35\% \) and \( \tau^B = 0\% \). For these parameterizations, consider that country A is the US with a capital tax rate of 35\% and country B does not have any investment tax. All other economic factors (production function \( f : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+ \), capital share \( \theta \), discount factor \( \beta \), and labor disutility \( \gamma \)) are identical. Define \( Y^A_{ss} \) as the steady state GDP of country A and \( Y^B_{ss} \) as the steady state GDP of country B.

The ratio of the GDPs in the two countries is given by:

\[ \frac{Y^A_{ss}}{Y^B_{ss}} = \left( \frac{1 + \tau^B}{1 + \tau^A} \right)^{\frac{\theta}{1-\tau}} \frac{(1 - \frac{\theta \beta}{1+\tau^A})}{(1 - \frac{\theta \beta}{1+\tau^B})}. \]  \hspace{1cm} (4.162)

Empirically, the value for the capital share has been shown to be consistently in the ballpark of \( \theta = \frac{1}{3} \), so we choose this value. Dealing with quarterly GDP values, the length of a period is one quarter of a year, so the discount factor is approximately \( \beta = 0.99 \). With the higher investment tax, country A has a lower steady state GDP, and the relative GDP is:

\[ \frac{Y^A_{ss}}{Y^B_{ss}} = \left( \frac{1}{1.35} \right)^{\frac{1}{1}} \frac{(1 - 0.33)}{(1 - 0.24)} \approx 76\%. \]  \hspace{1cm} (4.163)

If we want, we can decompose the tax effect. The first term \( \left( \frac{1}{1.35} \right)^{\frac{1}{1}} \approx 86\% \) captures the disincentive to investment. The second term \( \frac{(1 - 0.33)}{(1 - 0.24)} \approx 89\% \) captures the difference in the steady state labor supplies in the two countries. It is clear that an investment tax affects how much households save, but the labor effect needs to be explained in greater detail. With a higher investment tax, households increase their consumption (investment down, consumption up). The marginal utility will decrease, meaning that households value an extra unit of income less than they used to. Households are not willing to work as long since (i) the marginal benefit from work has decreased and (ii) the marginal cost of labor has remained fixed. Thus, the labor supply decreases.
4.6 Equilibrium solution with taxes and human capital

4.6.1 Sneak peek

Summary

Up until this point, the variations of the neoclassical growth model contained only two factors of production: capital and labor. This section is interested in decomposing the labor supply decision into two components: the human capital stock and the hours worked. Economists view human capital as the amount of schooling or training that workers receive. Human capital can be valuable to a firm as it allows them to be more productive. With such added productivity, households have an incentive to invest in human capital as they will be compensated in terms of a higher salary or a higher wage.

In order to capture the fact that human capital allows the firms to be more productive, the production function has to be extended as there are now three factors of production: physical capital, human capital, and labor. The production function will still be of the Cobb-Douglas form, meaning that the sum of the exponents on the three factors must sum to 1.

As with physical capital, which is the type of capital we have exclusively considered up until this point, human capital will earn a market-based return. The return is an equilibrium variable and will be determined in equilibrium such that firms are willing to employ the equilibrium level of human capital and households are willing to supply that level.

With this structure, the total renumeration for household labor is divided into two terms: the return for human capital and the wage rate. It would be ideal if these two terms entered multiplicatively into the household income expression, meaning that more human capital allows household to earn a higher wage, and then households decided how many hours to work at that wage, but the two terms enter additively. This makes the model tractable, meaning that we can find a closed-form solution to the Bellman equation.

The model continues to contain an investment tax, where the tax is now imposed on both physical capital and human capital investment. This section adheres as closely as possible to the previous one in order to get a comparison between the predictions with and without human capital. Without human capital, there are only two factors of production, with capital receiving the capital share (roughly 1/3) and labor receiving the labor share (roughly 2/3). With human capital, the labor share is further subdivided into the components associated with human capital and the components associated with hours worked. By including a
4.6. EQUILIBRIUM SOLUTION WITH TAXES AND HUMAN CAPITAL

dynamic element to the labor supply decision (the decision is dynamic because it takes time
and foresight to acquire human capital, as students well know), the model with human capital
provides some stark predictions about how output changes over time.

Notation

The variables to be introduced in this section are given in the following table:

- \( \eta \): human capital share
- \( h \): human capital stock of households
- \( H \): human capital stock rented by firms
- \( \tilde{R} \): rate of return on human capital

Main takeaways

After completing this section, you will be able to answer the following questions:

- What effect does investment taxation have on steady state output in the model with
  human capital?

- How does this compare to the model without human capital accumulation?

4.6.2 Adding human capital to the model

The production function states that output is produced using three factors of production:
physical capital \( K_t \), human capital \( H_t \), and labor \( N_t \). The production function \( f : \mathbb{R}^3_+ \rightarrow \mathbb{R}_+ \)
is of the Cobb-Douglas form once again:

\[
f (K_t, H_t, N_t) = (K_t)\theta (H_t)\eta (N_t)^{1-\theta-\eta}
\]

for \( \theta, \eta \in (0,1) \) such that \( \theta + \eta < 1 \).

As in the previous section, the government imposes a tax \( \tau \) on investment. In this model,
that investment is both the physical capital investment and the human capital investment.
As in the previous section, the depreciation rate for physical capital equals \( \delta = 1 \). The
depreciation rate for human capital will equal 1 as well. The rate of return on physical
capital is denoted \( R_t \), the rate of return on human capital is denoted \( \tilde{R}_t \), and the wage rate
is denoted \( w_t \). Taking these factor prices as given, the household choices are the capital stocks
(both physical and human) and the labor supply. The household’s stock of physical capital brought into the current period is \( k_t \), the household’s stock of human capital is \( h_t \), and the labor supply is \( n_t \).

The household budget constraint is given by:

\[
c_t + (1 + \tau) (k_{t+1} + h_{t+1}) = R_t k_t + \tilde{R}_t h_t + w_t n_t + T_t.
\]

(4.165)

As before, \( T_t \) is a lump-sum transfer that the household receives from the government.

### 4.6.3 Equilibrium

In equilibrium, the firm choices (upper case variables) must be equal to the household choices (lower case variables): \( K_t = k_t \), \( H_t = h_t \), and \( N_t = n_t \).

Additionally, the government must balance its budget constraint:

\[
\tau (k_{t+1} + h_{t+1}) = T_t.
\]

(4.166)

The firm problem is to choose the physical capital input \( K_t \), the human capital input \( H_t \), and the labor input \( N_t \) to maximize the profit function:

\[
\pi (K_t, H_t, N_t) = f (K_t, H_t, N_t) - R_t K_t - \tilde{R}_t H_t - w_t N_t.
\]

(4.167)

The optimal choices by the firm are found from the first order conditions \( D_1 \pi (K_t, H_t, N_t) = 0 \), \( D_2 \pi (K_t, H_t, N_t) = 0 \), and \( D_3 \pi (K_t, N_t) = 0 \). In equilibrium, the first order conditions lead to the pricing equations:

\[
R_t = \theta (k_t)^{\theta-1} (h_t)^{\eta} (n_t)^{1-\theta-\eta}.
\]

\[
\tilde{R}_t = \eta (k_t)^{\theta} (h_t)^{\eta-1} (n_t)^{1-\theta-\eta}.
\]

\[
w_t = (1 - \theta - \eta) (k_t)^{\theta} (h_t)^{\eta} (n_t)^{-\theta-\eta}.
\]

(4.168)

There are two vehicles for household saving: investment in physical capital and investment in human capital. In equilibrium, the returns on these investments must be identical. Otherwise, there would be an arbitrage opportunity as households would only invest in the higher return capital. Identical returns requires that:

\[
R_t = \tilde{R}_t.
\]

(4.169)
This condition is the no arbitrage condition.

Let’s consider the argument behind no arbitrage in greater detail. Suppose that $R_t > \tilde{R}_t$ (an arbitrage opportunity exists). All households would switch investment from human capital to physical capital (the capital with the highest return). Given factor price equations (4.168), this switch would both decrease $R_t$ and increase $\tilde{R}_t$. This adjustment must continue until the rates of return equilibrate at $R_t = \tilde{R}_t$.

Using the factor price equations (4.168), the no arbitrage condition (4.169) implies that:

$$
\theta (k_t)^{\theta-1} (h_t)^\eta (n_t)^{1-\theta-\eta} = \eta (k_t)^\theta (h_t)^{\eta-1} (n_t)^{1-\theta-\eta}.
$$

(4.170)

We can solve for $h$:

$$
h_t = \frac{\eta}{\theta} k_t.
$$

(4.171)

By expressing the human capital in terms of the physical capital, the production function can be reduced to:

$$
f \left( k_t, \frac{\eta}{\theta} k_t, n_t \right) = \left( \frac{\eta}{\theta} \right)^\eta k_t^\theta + n_t^{1-\theta-\eta}.
$$

(4.172)

### 4.6.4 Solving the Bellman equation

The budget constraint for the Bellman equation is given by:

$$
c + (1 + \tau) (k + h') = Rk + \tilde{R}h + wn + T.
$$

(4.173)

Since $R = R'$ and $h = \frac{\eta}{\theta} k$, then we can express this constraint only in terms of physical capital:

$$
c + (1 +\tau) \left( 1 + \frac{\eta}{\theta} \right) k' = R \left( 1 + \frac{\eta}{\theta} \right) k + wn + T.
$$

(4.174)

The Bellman equation is then given by:

$$
V(k) = \max_{c \geq 0, k' \geq 0, n \in [0,1]} \ln (c) - \gamma n + \beta V(k')
$$

subject to $c + (1 + \tau) \left( 1 + \frac{\eta}{\theta} \right) k' = R \left( 1 + \frac{\eta}{\theta} \right) k + wn + T$.

(4.175)

To solve this Bellman equation, let’s use the same 5-step method as in the previous section.

1. Find the Euler equation.
For this, we can use our step-by-step procedure for finding the Euler equation. Here, solving for consumption yields the functions

\[ \gamma (k, k') = R \left(1 + \frac{\eta}{\theta}\right) k + wn + T - (1 + \tau) \left(1 + \frac{\eta}{\theta}\right) k' \]  

(4.176)

and

\[ \gamma (k', k'') = R' \left(1 + \frac{\eta}{\theta}\right) k' + w'n' + T' - (1 + \tau) \left(1 + \frac{\eta}{\theta}\right) k''. \]  

(4.177)

Recall that the Euler equation is always of the following form:

\[ Du(c)D_2 \gamma (k, k') + \beta Du(c')D_1 \gamma (k', k'') = 0. \]  

(4.178)

For the case of this simple economy, \( Du(c) = \frac{1}{c}, \) \( D_2 \gamma (k, k') = -(1 + \tau) \left(1 + \frac{\eta}{\theta}\right), \) \( Du(c') = \frac{1}{c'}, \) and \( D_1 \gamma (k', k'') = R' \left(1 + \frac{2}{\theta}\right). \) Putting these together, the Euler equation is given by:

\[ \frac{1 + \tau}{c} = \beta \frac{R'}{c'}. \]  

(4.179)

2. Express the consumption terms in the Euler equation only in terms of the state variable \( k. \)

(a) This means that the term \( R \left(1 + \frac{2}{\theta}\right) k + wn \) needs to be expressed as a function of \( k \) only (using the equilibrium definitions of \( R \) and \( w \)).

For Step 2(a), in order to express the household income \( R \left(1 + \frac{2}{\theta}\right) k + wn \) only in terms of the physical capital stock \( k, \) we begin by replacing \( R \) and \( w \) with the equilibrium equations from (4.168):

\[ R \left(1 + \frac{\eta}{\theta}\right) k + wn = (\theta k^\theta h^\eta n^{1-\theta-\eta}) \left(1 + \frac{\eta}{\theta}\right) + (1 - \theta - \eta) k^\theta h^\eta n^{1-\theta-\eta}. \]  

(4.180)

We can then expand the first term and reduce:

\[ R \left(1 + \frac{\eta}{\theta}\right) k + wn = k^\theta h^\eta n^{1-\theta-\eta}. \]  

(4.181)

We can then replace \( h \) with the equilibrium condition from (4.169):

\[ R \left(1 + \frac{\eta}{\theta}\right) k + wn = \left(\frac{\eta}{\theta}\right)^\eta k^{\theta+\eta} n^{1-\theta-\eta}. \]  

(4.182)

The household income is equal to the firm production.
(b) This means that the term \( R' \left( 1 + \frac{\eta}{\beta} \right) k' + w'n' \) needs to be expressed as a function of \( k' \) only (using the equilibrium definitions of \( R' \) and \( w' \)).

The derivation is similar to Step 2a:

\[
R' \left( 1 + \frac{\eta}{\beta} \right) k' + w'n' = \left( \frac{\eta}{\theta} \right)^\eta (k')^{\theta+\eta} (n')^{1-\theta-\eta}. \tag{4.183}
\]

3. Guess the functional form \( g(k) = m \left( \frac{\eta}{\beta} \right)^\eta k^{\theta+\eta} n^{1-\theta-\eta} \) and use this guess in the Euler equation. Notice that we have guessed that the policy function is a linear function of the firm output.

4. Solve for the unknown coefficient \( m \).

5. Take the first order condition with respect to the labor supply choice \( n \) and solve for the labor supply function \( n \).

One of the end-of-chapter exercises asks you to follow the first 4 steps and solve for the policy function:

\[
g(k) = \frac{\theta \beta}{1+\tau} \left( \frac{\eta}{\beta} \right)^\eta k^{\theta+\eta} n^{1-\theta-\eta}. \tag{4.184}
\]

Using the definition for the human capital \( h = \frac{\eta}{\theta} k \), this can be rewritten as:

\[
g(k) = \frac{\theta \beta}{1+\tau} k^{\theta+\eta} n^{1-\theta-\eta}. \tag{4.185}
\]

Notice that the coefficient \( m = \frac{\theta \beta}{1+\tau} \) is identical to the model without human capital. In other words, the investment-to-output fraction for each household is identical in the two models.

One of the end-of-chapter exercises asks you to solve for the consumption function:

\[
c(k) = \left( 1 - \frac{(\theta + \eta) \beta}{1+\tau} \right) \left( \frac{\eta}{\theta} \right)^\eta k^{\theta+\eta} n^{1-\theta-\eta}. \tag{4.186}
\]

One of the end-of-chapter exercises asks you to complete step 5 and solve for the labor supply:

\[
n = \frac{1 - \theta - \eta}{1 - (\theta+\eta) \beta \gamma}. \tag{4.187}
\]
4.6.5 Steady state tax analysis

The steady state capital holdings are found to satisfy $k_{ss} = g(k_{ss})$. From the policy function equation:

$$k_{ss} = \frac{\theta \beta}{1 + \tau} \left( \frac{\eta}{\beta} \right)^{\eta} (k_{ss})^{\theta+\eta} n_{ss}^{1-\theta-\eta}. \quad (4.188)$$

Multiply both sides by $(k_{ss})^{-(\theta+\eta)}$:

$$(k_{ss})^{1-(\theta+\eta)} = \frac{\theta \beta}{1 + \tau} n_{ss}^{1-\theta-\eta}. \quad (4.189)$$

To solve for the steady state capital stock, raise both sides of the equation to the exponent $\frac{1}{1-(\theta+\eta)}$:

$$k_{ss} = \left( \frac{\theta \beta}{1 + \tau} \right)^{\frac{1}{1-(\theta+\eta)}} n_{ss}. \quad (4.190)$$

As a check of this, we can use the Euler equation directly in order to find the equation for steady state capital. Recall the Euler equation (evaluated at steady state values)

$$\frac{1 + \tau}{c_{ss}} = \beta \frac{R_{ss}}{c_{ss}}, \quad (4.191)$$

where the steady state rate of return on capital is

$$R_{ss} = \theta (k_{ss})^{\theta-1} (h_{ss})^{\eta} (n_{ss})^{1-\theta-\eta}. \quad (4.192)$$

Using the no arbitrage condition, $h_{ss} = \frac{\eta}{\beta} k_{ss}$, then the steady state rate of return on capital is equal to:

$$R_{ss} = \theta \left( \frac{\eta}{\beta} \right)^{\eta} (k_{ss})^{\theta+\eta-1} (n_{ss})^{1-\theta-\eta}. \quad (4.193)$$

Solving the Euler equation for $k_{ss}$ yields:

$$k_{ss} = \left( \frac{\theta \beta}{1 + \tau} \right)^{\frac{1}{1-(\theta+\eta)}} n_{ss}. \quad (4.194)$$

The aggregate output of the economy (or the GDP) in any time period $t$ is given by:

$$Y_{ss} = f \left( k_{ss}, \frac{\eta}{\theta} k_{ss}, n_{ss} \right) = \left( \frac{\eta}{\theta} \right)^{\eta} (k_{ss})^{\theta+\eta} n_{ss}^{1-\theta-\eta}. \quad (4.195)$$
Plugging in the expression for the steady state capital yields:

\[ Y_{ss} = \left( \frac{\theta \beta}{1 + \tau} \right)^{\frac{\theta + \eta}{1 - (\theta + \eta)}} \left( \frac{\eta}{\theta} \right)^{\frac{\eta}{1 - (\theta + \eta)}} n_{ss}. \]  

(4.196)

Labor supply is constant in all periods and equal to its steady state value. We can now insert the expression for the equilibrium labor supply into the steady state output equation:

\[ Y_{ss} = \left( \frac{\theta \beta}{1 + \tau} \right)^{\frac{\theta + \eta}{1 - (\theta + \eta)}} \left( \frac{\eta}{\theta} \right)^{\frac{\eta}{1 - (\theta + \eta)}} \frac{1 - \theta - \eta}{\left( 1 - \frac{(\theta + \eta) \beta}{1 + \tau} \right) \gamma}. \]  

(4.197)

Consider country A with a high investment tax \((\tau^A)\) and country B with a low investment tax \((\tau^B)\). The investment tax is on the combined investment in both physical capital and human capital. It is not clear what value to use for a tax rate on human capital, so we will maintain the values from the previous section, namely \(\tau^A = 35\%\) and \(\tau^B = 0\%.\) All other factors (production function \(f : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+\), physical capital share \(\theta\), human capital share \(\eta\), and discount factor \(\beta\)) are identical. Define \(Y^A_{ss}\) as the steady state GDP of country A and \(Y^B_{ss}\) as the steady state GDP of country B. The ratio of the GDPs in the two countries is given by:

\[ \frac{Y^A_{ss}}{Y^B_{ss}} = \left( \frac{1 + \tau^B}{1 + \tau^A} \right)^{\frac{\theta + \eta}{1 - (\theta + \eta)}} \frac{1 - \frac{(\theta + \eta) \beta}{1 + \tau^B}}{\left( 1 - \frac{(\theta + \eta) \beta}{1 + \tau^B} \right) \gamma}. \]  

(4.198)

We maintain the value of \(\theta = \frac{1}{3}\) for the physical capital share and the value of \(\beta = 0.99\) for the discount factor. The labor share is the remaining fraction \(1 - \theta = \frac{2}{3}\). In this application, the share owed to labor is split between the share owed for the hours worked and the share owed for the human capital stock. Both aspects would contribute to the labor share in the data. Empirical work using the returns on education has been able to approximate the value for \(\eta = \frac{1}{3}\) (we trust the value for the physical capital share much more than this value for the human capital share, but it is a first approximation). With the higher investment tax, country A has a lower steady state GDP, and the relative GDP is given by:

\[ \frac{Y^A_{ss}}{Y^B_{ss}} = \left( \frac{1}{1.35} \right)^{\frac{2}{3}} \frac{(1 - 0.66)}{(1 - 0.49)} \approx 37\%. \]  

(4.199)

The model in this lesson is identical to that in the previous lesson, with the lone exception
that human capital is included alongside physical capital. With this change, the negative
effects of an investment tax on steady state production are exacerbated. Previously, the
GDP level in the high tax country was 76% of the low tax country, and now the ratio is only
37%.

If we want, we can decompose the tax effect. The first term \((\frac{1}{1.35})^2\) \approx 55% captures the
disincentive to investment. The second term \((\frac{1-0.66}{1-0.49}) \approx 67%\) captures the difference in the
steady state labor supplies in the two countries.

### 4.7 Exercises

1. **Introducing the model**

Suppose that the household consumes \(c_t = c_{hi}\) in all even periods and \(c_t = c_{lo}\) in all
odd periods. Write an expression for the infinite discounted utility \(\sum_{t=0}^{\infty} \beta^t u(c_t)\).

2. **Introducing the model**

Assume the production function is \(f(K, N) = AK^\theta N^{1-\theta}\). Prove that the firm profit is
equal to 0 in equilibrium.

3. **Euler equation**

Derive the Euler equations associated with the household problem

\[
\begin{align*}
\text{maximize} & \quad \sum_{t=0}^{\infty} \beta^t u(c_t) \\
\text{subject to} & \quad c_t + k_{t+1} \leq R_t k_t + w_t + (1 - \delta)k_t + \pi_t \text{ in all time periods } \\
& \quad k_0 \text{ given}
\end{align*}
\]

4. **Euler equation**

If the equilibrium is such that consumption is constant, i.e., \(c_t = \bar{c}\) for all time periods,
what is the relation between the rate of return on capital (which is also constant,
\(R_t = \bar{R}\) for all time periods) and the discount factor?

5. **The AK model**

Let the production function be \(f(K, N) = AK\) and let \(\delta\) take any value. In this case,
the Bellman equation is given by:

\[ V(k) = \max_{c \geq 0, k' \geq 0} c \ln(c) + \beta V(k') \]

subject to \( c + k' = Ak + (1 - \delta)k \).

Define the savings rate \( s \) such that:

\[ k' = (1 + s) k. \]

Solve for the policy function \( g(k) \) that solves the Bellman equation and find the savings rate \( s \). When is the savings rate strictly positive?

6. The AK model

Consider the model with inelastic labor supply \((n = 1)\), utility function \( u(c) = \ln(c) \), and production function \( f(K, N) = K^\theta N^{1-\theta} \). In this case, the Bellman equation is given by:

\[ V(k) = \max_{c \geq 0, k' \geq 0} \ln(c) + \beta V(k') \]

subject to \( c + k' = k^\theta \).

Solve for the policy function \( g(k) \) and the consumption function \( c(k) \).

7. The AK model

Consider the model with elastic labor supply, utility function \( u(c, n) = \ln(c) - \gamma n \), and production function \( f(K, N) = K^\theta N^{1-\theta} \). In this case, the Bellman equation is given by:

\[ V(k) = \max_{c \geq 0, k' \geq 0, n \in [0, 1]} \ln(c) - \gamma n + \beta V(k') \]

subject to \( c + k' = k^\theta n^{1-\theta} \).

Solve for the policy function \( g(k) \), the consumption function \( c(k) \), and the labor supply function \( n(k) \).

8. Equilibrium solution with taxes

For the model with the investment tax, suppose that the disutility from labor is equal to \( \gamma (n_t)^p \), for some parameter \( p > 1 \). With only this change in the model, calculate the ratio of steady state output \( \frac{Y_A^{SS}}{Y_B^{SS}} \) and determine the ratio of steady state labor supply \( \frac{n_A^{SS}}{n_B^{SS}} \).
9. *Equilibrium solution with taxes and human capital*

For the model with taxes and human capital, together with the investment tax, solve for the policy function and show that

\[ g(k) = \frac{\theta \beta}{1 + \tau} \left( \frac{\eta}{\theta} \right)^\eta k^{\theta + \eta} n^{1 - \theta - \eta}. \]

10. *Equilibrium solution with taxes and human capital*

For the model with taxes and human capital, together with the investment tax, solve for the consumption function and show that

\[ c(k) = \left( 1 - \frac{(\theta + \eta) \beta}{1 + \tau} \right) \left( \frac{\eta}{\theta} \right)^\eta k^{\theta + \eta} n^{1 - \theta - \eta}. \]

11. *Equilibrium solution with taxes and human capital*

For the model with taxes and human capital, together with the investment tax, solve for the labor supply and show that

\[ n = \frac{1 - \theta - \eta}{\left( 1 - \frac{(\theta + \eta) \beta}{1 + \tau} \right)^\gamma}. \]

12. *Equilibrium solution with taxes and human capital*

For the model with taxes and human capital, suppose that the disutility from labor is equal to \( \gamma (n_t)^p \), for some parameter \( p > 1 \). With only this change in the model, calculate the ratio of steady state output \( \frac{Y_{SS}^A}{Y_{SS}^B} \) and determine the ratio of steady state labor supply \( \frac{n_{SS}^A}{n_{SS}^B} \).
Bibliography


5

Endogenous Growth Theory

5.1 Monopolistic competition

5.1.1 Sneak peek

Summary

For growth models with technological change, firms no longer operate in a setting of perfect competition. With perfect competition and a Cobb-Douglas production function, the equilibrium firm profit equals 0. In order for firms to be willing to accept a trade-off involving R&D investments in the current period, it must be that firms are able to earn strictly positive profits in the future. For that reason, we need to model a setting in which firms earn strictly positive profits in equilibrium.

This section introduces the properties of such a model. In the model, firms interact in a setting of monopolistic competition. The setting is somewhere in between perfect competition and pure monopoly. Namely, firms have some market power, certainly moreso than the setting of perfect competition, but are not pure monopolists. Recall that a pure monopolist is able to internalize the demand function of the consumers and maximize profit accordingly.

The key parameter in this setting is the households’ elasticity of substitution. The economy contains many varieties of the same good. Each variety is produced by a different firm. The degree to which households are willing to substitute from one variety to another (from Nike to Adidas) is the elasticity of substitution. Households respond according to their preferences (as governed by the elasticity of substitution parameter) and the price charged by firms. Firms set prices and internalize the demand function of the consumers into their
profit maximization problem. Firms have a decision to make as an increase in the price has two effects: (i) firms earn more profit for each unit sold and (ii) households substitute to nearby varieties with lower prices and this reduces the number of units sold. With this trade-off, we can uniquely determine the price charged by firms (and paid by consumers).

A model of monopolistic competition is not only used in all endogenous growth models (such as the one introduced later in this chapter), but also in the New Keynesian models to be introduced in later chapters.

**Notation**

The variables/parameters to be introduced in this section are given in the following table:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>number of varieties/firms</td>
</tr>
<tr>
<td>$u : \mathbb{R}_+ \to \mathbb{R}$</td>
<td>utility of the consumption index</td>
</tr>
<tr>
<td>$c_i$</td>
<td>household consumption of variety $i$</td>
</tr>
<tr>
<td>$C = \left( \int_0^N c_i^{-\epsilon} , di \right)^{1/\epsilon}$</td>
<td>consumption index</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>elasticity of substitution</td>
</tr>
<tr>
<td>$p_i$</td>
<td>price for variety $i$ (produced by firm $i$)</td>
</tr>
<tr>
<td>$m$</td>
<td>household income</td>
</tr>
<tr>
<td>$P = \left( \int_0^N p_i^{1-\epsilon} , di \right)^{1/\epsilon}$</td>
<td>price index</td>
</tr>
<tr>
<td>$\pi_i$</td>
<td>profit for firm $i$</td>
</tr>
<tr>
<td>$\psi$</td>
<td>marginal cost of production for each firm</td>
</tr>
</tbody>
</table>

**Main takeaways**

After completing this section, you will be able to answer the following questions:

- What is the elasticity of substitution for households under CES utility?
- How is the price index defined?
- What is the constant markup charged by firms under monopolistic competition?

### 5.1.2 Dixit-Stiglitz aggregator

We consider the industry for one good. The industry contains a continuum of firms with mass equal to $N$. The number $N$ is viewed as both the number of firms and the number of
5.1. **MONOPOLISTIC COMPETITION**

varieties of the good. The consumption of variety $i$ is denoted $c_i$. Each variety is produced by a single monopolistically competitive firm. Additionally, each firm only produces one variety.

The firm (equivalently the variety) index is $i \in [0, N]$.

We want to aggregate the consumption over all varieties into one index. The consumption index is defined by

$$C = \left( \int_0^N \frac{c_i}{c_{i+1}} \, di \right)^{\frac{1}{\epsilon+1}}. \tag{5.1}$$

This aggregator is referred to either as the Dixit-Stiglitz aggregator (in honor of its founders) or the CES aggregator (for reasons that will be clear to us shortly). In the aggregator, $\epsilon > 1$ is the elasticity of substitution between the varieties. High elasticity $\epsilon$ means that varieties are more substitutable, which will lead to less market power for firms.

The household utility function is $u(C)$, where $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfies the standard assumptions of $C^2$, strictly increasing, and strictly concave. Together the consumption index and the utility function describe the household preferences.

The Dixit-Stiglitz aggregator has several important properties and we will spend the remainder of this section understanding each of them: (i) households have love-for-variety, (ii) the constant elasticity of substitution is equal to $\epsilon$, and (iii) the profit-maximizing price chosen by the firms is a constant markup over marginal cost.

### 5.1.3 Love of variety

In the model, we assume that even though each firm is producing a different variety of the good, all firms are identical in that they have the same marginal cost of production. This implies that the production of each variety, and therefore the consumption of each variety, is identical.

For the love-of-variety property, suppose that there is a fixed supply of aggregate resources, independent of the number of firms in the economy. The supply of aggregate resources equals $Y$, and suppose that the aggregate resources are evenly produced across all firms. Since the firms have mass equal to $N$, each firm would produce $\frac{Y}{N}$ units. Do households prefer to have a large number of firms and varieties (but a smaller amount for each variety) or do they prefer to have just one firm producing $Y$ units of one variety? Plug
$c_i = \frac{Y}{N}$ into the consumption index:

$$C = \left( \int_0^N c_i \, di \right)^{\frac{\epsilon - 1}{\epsilon}}$$

$$= \left( N \left( \frac{Y}{N} \right)^{\frac{\epsilon - 1}{\epsilon}} \right)^{\frac{\epsilon - 1}{\epsilon}}$$

$$= N^{\frac{1}{\epsilon - 1}} \left( \frac{Y}{N} \right).$$

Combining the terms involving $N$:

$$C = N^{\frac{1}{\epsilon - 1}} Y.$$ (5.3)

This means that the household utility is given by

$$u(C) = u \left( N^{\frac{1}{\epsilon - 1}} Y \right).$$ (5.4)

Since $\epsilon > 1$, the utility function is a strictly increasing function of $N$. That is, more variety (higher $N$) means higher utility for the households.

**5.1.4 Household problem**

Denote the price of the variety $i$ as $p_i$. We will eventually derive the price as an optimal choice of the firm problem, but for now we are only considering the household problem. In the household problem, the prices are taken as given. Denote the total income for the household as $m$. The household problem is given by:

$$\max_{\{c_i\}_{i \in [0, N]}} u \left( \left( \int_0^N c_i \, di \right)^{\frac{\epsilon - 1}{\epsilon}} \right),$$

subject to $\int_0^N p_i c_i \, di \leq m$. (5.5)

Denote $\lambda$ as the Lagrange multiplier for the budget constraint. Since $u$ is strictly increasing, we seek only to maximize the consumption index $C$. So the household problem is equivalently given by:

$$\max_{\{c_i\}_{i \in [0, N]}} \left( \int_0^N c_i \, di \right)^{\frac{\epsilon - 1}{\epsilon}},$$

subject to $\int_0^N p_i c_i \, di \leq m$. (5.6)

Let’s write down the first order conditions for variety $i$. When taking the derivative of the
5.1. MONOPOLISTIC COMPETITION

objective function, remember to use the chain rule. The first order condition for variety \( i \) is:

\[
\left[ \frac{\epsilon}{\epsilon - 1} \left( \int_0^N c_i^{\frac{-1}{\epsilon}} \, di \right)^{\frac{\epsilon - 1}{\epsilon}} \right] \left[ \frac{\epsilon - 1}{\epsilon} (c_i)^{\frac{-1}{\epsilon}} - 1 \right] - \lambda p_i = 0. \tag{5.7}
\]

The exponents can be simplified and the terms \( \left( \frac{\epsilon}{\epsilon - 1} \right) \) and \( \left( \frac{\epsilon - 1}{\epsilon} \right) \) cancelled:

\[
\left( \int_0^N c_i^{\frac{-1}{\epsilon}} \, di \right)^{\frac{1}{\epsilon}} (c_i)^{-\frac{1}{\epsilon}} - \lambda p_i = 0. \tag{5.8}
\]

The first order condition for any other variety \( j \) is going to take a very similar form:

\[
\left( \int_0^N c_j^{\frac{-1}{\epsilon}} \, dj \right)^{\frac{1}{\epsilon}} (c_j)^{-\frac{1}{\epsilon}} - \lambda p_j = 0. \tag{5.9}
\]

We can combine the first order conditions for two varieties \( i \) and \( j \) by solving both first order conditions for \( \lambda \):

\[
\lambda = \frac{\left( \int_0^N c_i^{\frac{-1}{\epsilon}} \, di \right)^{\frac{1}{\epsilon}} (c_i)^{-\frac{1}{\epsilon}}}{p_i} = \frac{\left( \int_0^N c_j^{\frac{-1}{\epsilon}} \, dj \right)^{\frac{1}{\epsilon}} (c_j)^{-\frac{1}{\epsilon}}}{p_j}. \tag{5.10}
\]

The key equality that we obtain is the relation between the demand functions for variety \( i \) and variety \( j \):

\[
\frac{\left( \int_0^N c_i^{\frac{-1}{\epsilon}} \, di \right)^{\frac{1}{\epsilon}} (c_i)^{-\frac{1}{\epsilon}}}{p_i} = \frac{\left( \int_0^N c_j^{\frac{-1}{\epsilon}} \, dj \right)^{\frac{1}{\epsilon}} (c_j)^{-\frac{1}{\epsilon}}}{p_j}. \tag{5.11}
\]

The term \( \left( \int_0^N c_i^{\frac{-1}{\epsilon}} \, di \right)^{\frac{1}{\epsilon}} \) is the same in both numerators, meaning that

\[
\frac{(c_i)^{-\frac{1}{\epsilon}}}{p_i} = \frac{(c_j)^{-\frac{1}{\epsilon}}}{p_j}. \tag{5.12}
\]

This implies that

\[
\left( \frac{c_i}{c_j} \right) = \left( \frac{p_i}{p_j} \right)^{-\epsilon}. \tag{5.13}
\]

The elasticity of substitution between any two varieties \( i \) and \( j \) is defined as follows:

\[
\text{Elasticity} = -\frac{\left( \Delta \frac{c_i}{c_j} / \left( \frac{c_i}{c_j} \right) \right)}{\left( \Delta \frac{p_i}{p_j} / \left( \frac{p_i}{p_j} \right) \right)}.
\]
From (5.13), \(\frac{c_i}{c_j} = \left(\frac{p_i}{p_j}\right)^{-\epsilon}\). We evaluate the ratio of consumption change to price change 
\[
\frac{\Delta \frac{c_i}{c_j}}{\Delta \frac{p_i}{p_j}}
\] 
with a derivative:
\[
\frac{\Delta \frac{c_i}{c_j}}{\Delta \frac{p_i}{p_j}} = \frac{\partial \left(\frac{c_i}{c_j}\right)}{\partial \left(\frac{p_i}{p_j}\right)} = -\epsilon \left(\frac{p_i}{p_j}\right)^{-\epsilon - 1}.
\] 
(5.15)

The equation for the elasticity is then expressed as:
\[
\text{Elasticity} = \epsilon \left(\frac{p_i}{p_j}\right)^{-\epsilon - 1}\left(\frac{p_i}{p_j}\right).\] 
(5.16)

When we multiply \(\left(\frac{p_i}{p_j}\right)^{-\epsilon - 1}\) and \(\left(\frac{p_i}{p_j}\right)\), we are adding exponents:
\[
\text{Elasticity} = \epsilon \left(\frac{p_i}{p_j}\right)^{-\epsilon}
\] 
\[
\frac{c_i}{c_j}
\] 
\] 
(5.17)

Since \(\frac{c_i}{c_j} = \left(\frac{p_i}{p_j}\right)^{-\epsilon}\), then the elasticity of substitution is constant and equal to \(\epsilon\):
\[
\text{Elasticity} = \epsilon \left(\frac{p_i}{p_j}\right)^{-\epsilon} = \epsilon.
\] 
(5.18)

### 5.1.5 Price index

Define the price index \(P\) such that
\[
\left(\frac{c_i}{C}\right)^{-\frac{1}{\gamma}} = \frac{p_i}{P}.
\] 
(5.19)

This definition is consistent with the household demand functions (5.13). Multiply both sides of the equation by \(c_i\):
\[
\frac{c_i^{-\frac{1}{\gamma}}}{C^{-\frac{1}{\gamma}}} = \frac{p_i c_i}{P}.
\] 
(5.20)
The term in the numerator of the left-hand side term is $c_{i}^{-\frac{1}{\epsilon}}c_{i} = c_{i}^{\frac{\epsilon - 1}{\epsilon}}$ after adding the exponents. We then want to integrate both sides of the equality over all varieties $i \in [0, N]$:

$$\int_{0}^{N} \frac{c_{i}^{\frac{\epsilon - 1}{\epsilon}}}{C^{\frac{1}{\epsilon}}} di = \int_{0}^{N} p_{i}c_{i}di.$$  \hfill (5.21)

The definition of the consumption index states that

$$C = \left( \int_{0}^{N} c_{i}^{\frac{\epsilon - 1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon - 1}}.$$  \hfill (5.22)

This means that the term in the numerator of the left-hand side of equation (5.21) is

$$\int_{0}^{N} c_{i}^{\frac{\epsilon - 1}{\epsilon}} di = C^{\frac{\epsilon - 1}{\epsilon}}.$$  

This means that equation (5.21) is simplified to:

$$\frac{C^{\frac{\epsilon - 1}{\epsilon}}}{C^{\frac{1}{\epsilon}}} = \frac{\int_{0}^{N} p_{i}c_{i}di}{P}.$$  \hfill (5.23)

The left-hand side is simplified by subtracting the exponents:

$$\frac{C^{\frac{\epsilon - 1}{\epsilon}}}{C^{\frac{1}{\epsilon}}} = C^{\frac{\epsilon - 1}{\epsilon} - \frac{1}{\epsilon}} = C^{\frac{\epsilon - 1}{\epsilon}} = C.$$  \hfill (5.24)

Cross-multiplying equation (5.23) yields

$$PC = \int_{0}^{N} p_{i}c_{i}di.$$  \hfill (5.25)

So the price index has the property that the value of expenditures on all varieties is equal to the product of the consumption index $C$ and the price index $P$.

From the definition of $P$,

$$P = \frac{C^{\frac{1}{\epsilon}}}{c_{i}},$$  \hfill (5.26)

$$P^{-\epsilon} = \frac{c_{i}^{-\epsilon}}{c_{i}}.$$  

Multiply both sides of the equation by $P$ and multiply the right-hand side of the equation
5. ENDOGENOUS GROWTH THEORY

by \( \frac{p_i}{p_c} \):

\[ P^{1-\epsilon} = \frac{p_i^{1-\epsilon} PC}{p_i c_i}. \]  

(5.27)

Bring the term \( p_i c_i \) to the left-hand side and integrate both sides of the equality over all varieties \( i \in [0, N] \):

\[ P^{1-\epsilon} \int_0^N p_i c_i di = PC \int_0^N p_i^{1-\epsilon} di. \]  

(5.28)

From the previous derivation, we know that \( PC = \int_0^N p_i c_i di \). Thus, we have the expression for the price index:

\[ P^{1-\epsilon} = \int_0^N p_i^{1-\epsilon} di, \text{ or} \]

\[ P = \left( \int_0^N p_i^{1-\epsilon} di \right)^{\frac{1}{1-\epsilon}}. \]  

(5.29)

5.1.6 Firm problem

Finally, we want to consider the problem of each firm in the monopolistically competitive market. A firm is the sole producer of a variety. The marginal cost of producing each of the varieties is assumed to be constant and equal to \( \psi \). The objective of the firm is to choose the price \( p_i \) in order to maximize profit:

\[ \max_{p_i} \pi_i = c_i (p_i - \psi). \]  

(5.30)

The profit equals revenue minus cost, where revenue is \( p_i c_i \) and cost is \( \psi c_i \).

The demand function for the households is equal to \( c_i = (\frac{p_i}{\bar{p}})^{-\epsilon} C \) from the definition of the price index \( P \). With a continuum of firms, each firm’s choice of \( p_i \) has zero impact on the price index \( P = \left( \int_0^N p_i^{1-\epsilon} di \right)^{\frac{1}{1-\epsilon}} \). The profit maximization problem is

\[ \max_{p_i} \pi_i = \left( \frac{p_i}{\bar{p}} \right)^{-\epsilon} C (p_i - \psi). \]  

(5.31)

An end-of-chapter exercise asks you to show that the profit maximizing price for the firm is:

\[ p_i = \frac{\epsilon}{\epsilon - 1} \psi. \]  

(5.32)

High \( \epsilon \) means that the markup \( \frac{\epsilon}{\epsilon - 1} \) is small, which is consistent with small market power for firms. In the limit as \( \epsilon \to \infty \), varieties are perfect substitutes, firms have 0 market power, and there is no price markup over marginal cost.
The markup is the ratio $\frac{\psi}{\epsilon - 1} > 1$. The price choices for all firms are identical, so $p_i = p = \frac{\epsilon}{\epsilon - 1} \psi$ for all varieties $i \in [0, N]$.

The profits for all firms are identical, so $\pi_i = \pi$ for all $i \in [0, N]$. An end-of-chapter exercise asks you to compute the firm profit and verify the comparative statics conclusions: $\frac{\partial \pi}{\partial N} < 0$, $\frac{\partial \pi}{\partial \epsilon} < 0$, $\frac{\partial \pi}{\partial m} > 0$, and $\frac{\partial \pi}{\partial \psi} = 0$. In words, the profit for each firm is strictly decreasing in the number of firms $N$, strictly increasing in market power (equivalently, strictly decreasing in elasticity $\epsilon$), strictly increasing in household income $m$, and independent of firm marginal cost $\psi$.

### 5.2 Technological change

#### 5.2.1 Sneak peek

**Summary**

This section looks at one type of growth model with technological change. Recall that endogenous technological change is one factor that drives long-run growth. In the model, entrepreneurs are able to make investments in research and development (R&D) in order to obtain a patent that grants them perpetual rights to sell a particular variety. With R&D activity, the number of varieties in the economy increases. As we saw in the previous section, more varieties means higher utility for households. This is the mechanism in the model that leads to technology and welfare growth.

The cost of the investment is measured in terms of the labor supply devoted to R&D activities, labor that the entrepreneur could have alternatively used for production of current output. The gain from R&D investment is that the entrepreneur receives the profit from selling the new variety in all future periods. As with all business accounting, future profit streams must be discounted back to the time at which the investment was made, a process called net present value discounting. Given the costs and benefits for R&D, there is an equilibrium amount of R&D that takes place in the economy. This amount of R&D determines how fast technology and welfare grow.

In the present model of technological change, long-run growth emerges as an equilibrium outcome. With long-run growth, no steady state is approached. Rather, the economy converges to a balanced growth path (BGP). Along a balanced growth path, the growth rates are constant. The solution and the policy analysis for this model are determined along the balanced growth path.
5. ENDOGENOUS GROWTH THEORY

Notation

The variables to be introduced in this section are given in the following table:

- $N(t)$: mass of firms in period $t$
- $L_{R}(t)$: labor supplied to research sector in period $t$
- $l_{i}(t)$: labor used in production of variety $i$ in period $t$
- $\eta$: growth factor for the number of varieties
- $c_{i}(t)$: household consumption of variety $i$ in period $t$
- $y_{i}(t)$: production of variety $i$ in period $t$
- $p_{i}(t)$: price for variety $i$ in period $t$
- $w(t)$: wage rate in period $t$
- $\pi_{i}(t)$: profit for firm $i$ in period $t$
- $V_{i}(t)$: net present discounted value for firm $i$ in period $t$
- $r(t)$: real interest rate in period $t$
- $g_{\pi}$: growth rate for profit along a balanced growth path
- $g_{C}$: growth rate for consumption along a balanced growth path
- $g_{N}$: growth rate for number of varieties along a balanced growth path

Main takeaways

After completing this section, you will be able to answer the following questions:

- What factors affect the investment in the research sector?
- Along a balanced growth path, what is the relation between the growth rates for consumption, firm profit, and the number of varieties?

5.2.2 Labor markets

We consider a model with an infinite number of discrete time periods $t \in \{0, 1, 2, \ldots\}$. Denote the number of varieties in period $t$ as $N(t)$.

Labor is inelastically supplied by households and can either be devoted toward the production sector or the research sector. The production sector consists of the firms producing the varieties of the output good. The labor hired by the firm producing variety $i$ in period $t$ is denoted $l_{i}(t)$. The labor employed in the research sector is denoted $L_{R}(t)$. The unit mass
of homogeneous households supply labor inelastically, meaning that the total labor supply in each period is 1. The labor market clearing condition is:

$$\int_0^{N(t)} l_i(t) \, di + L_R(t) = 1. \quad (5.33)$$

The production function for the growth of varieties is given by:

$$N(t + 1) = (1 + \eta L_R(t)) N(t) \quad (5.34)$$

for some growth factor $\eta > 0$. Simply put, more labor devoted to research means more varieties in the following period.

### 5.2.3 Household problem

The household consumption of variety $i$ in period $t$ is denoted $c_i(t)$. The price of variety $i$ in period $t$ is denoted $p_i(t)$. We normalize the prices in all periods such that the price index

$$P(t) = \left( \int_0^{N(t)} (p_i(t))^{1-\epsilon} \, di \right)^{\frac{1}{1-\epsilon}} = 1. \quad (5.35)$$

Notice that the price normalization imposes a zero inflation rate. Any changes in household income or firm profit will be real changes. The real asset holdings for households are $a(t)$ and the real interest rate is $r(t)$. The households maximize the following infinite discounted utility:

$$\max \left\{ \left\{ c_i(t) \right\}_{i \in [0,N(t)]}, \{a(t)\} \right\} \sum_{t=0}^{\infty} \beta^t U(C(t))$$

$$C(t) = \left( \int_0^{N(t)} (c_i(t))^{\frac{1}{1-\epsilon}} \, di \right)^{1-\frac{1}{\epsilon}}$$

subject to

$$\int_0^{N(t)} p_i(t) c_i(t) \, di + a(t) \leq w(t) + (1 + r(t)) a(t - 1) \quad \forall t$$

The utility function is $U(C(t)) = \ln(C(t))$. Recall that the price index was defined such that

$$\left( \frac{c_i(t)}{C(t)} \right)^{-\frac{1}{\epsilon}} = \frac{p_i(t)}{P(t)} . \quad (5.37)$$

Using the price normalization, the variety demand functions are:

$$c_i(t) = (p_i(t))^{-\epsilon} C(t) . \quad (5.38)$$
5. ENDOGENOUS GROWTH THEORY

5.2.4 Firm problem

For each variety $i \in [0, N(t)]$, a single firm produces the output $y_i(t)$ using the production function:

$$y_i(t) = l_i(t). \quad (5.39)$$

Assuming that households allocate their labor in a competitive market, $w(t)$ is the wage rate both for labor supplied in the production sector and labor supplied in the research sector. The profit function for firm $i$ in period $t$ is

$$\max_{p_i(t)} \pi_i(t) = c_i(t) (p_i(t) - w(t)) . \quad (5.40)$$

Here, the marginal cost of production is simply the wage rate $w(t)$. This is the cost (per unit of output) that must be paid by the firm.

5.2.5 Market clearing

The model is not an entirely closed economy. Growth requires that household real asset holdings change over time. Such changes require a government that issues debt in the form of real assets or foreign investors/borrowers holding real assets. These considerations lie outside the scope of the present discussion. For our purposes, the inclusion of the real assets provides the Euler equation that determines the real interest rate.

The household consumptions of all varieties are equal:

$$c_i(t) = c(t) \ \forall i \in [0, N(t)]. \quad (5.41)$$

Since the varieties of the good produced must be consumed by the household, market clearing dictates that the amount produced by each firm must then be identical:

$$y_i(t) = y(t) = c(t) \ \forall i \in [0, N(t)]. \quad (5.42)$$

From the firm production function, this requires that the labor supplied for each variety is constant:

$$l_i(t) = l(t) = y(t) = c(t) \ \forall i \in [0, N(t)]. \quad (5.43)$$

The total labor supplied to the production sector is $1 - L_R(t)$, so the labor supplied to
5.2. TECHNOLOGICAL CHANGE

the production of each variety is:

\[ l(t) = \frac{1 - L_R(t)}{N(t)}. \]  

(5.44)

5.2.6 Firm profit

The profit function for firm \( i \) in period \( t \) is

\[ \max_{p_i(t)} \pi_i(t) = c_i(t) (p_i(t) - w(t)) . \]  

(5.45)

Using the demand functions for varieties and the result from the previous section, the optimal price chosen by each firm is equal to:

\[ p_i(t) = \frac{\epsilon}{\epsilon - 1} w(t). \]  

(5.46)

Firm profit in period \( t \) is \( \pi_i(t) = \pi(t) \) for each firm \( i \), where

\[ \pi(t) = c(t) \left( \frac{\epsilon}{\epsilon - 1} w(t) - w(t) \right) = \frac{1}{\epsilon - 1} c(t) w(t). \]  

(5.47)

The amount produced is simply equal to the labor supply \( l(t) \). Recall \( l(t) = \frac{1 - L_R(t)}{N(t)} \). The profit is given by:

\[ \pi(t) = \frac{1}{\epsilon - 1} \frac{1 - L_R(t)}{N(t)} w(t). \]  

(5.48)

Firms care about the net present discounted value of the profits in each time period. If \( r(t) \) is the real interest rate for a bond made available to the households, then the household Euler equation can be used to obtain an expression for \( r(t) \) :

\[ DU(C(t - 1)) = \beta (1 + r(t)) DU(C(t)). \]  

(5.49)

Since \( U(C) = \ln(C) \), the Euler equation can be solved for the real interest rate:

\[ 1 + r(t) = \frac{1}{\beta C(t - 1)} \]  

(5.50)
The net present discounted value in period \( t \) is \( V_i(t) = V(t) \) for each firm \( i \), where

\[
V(t) = \pi(t) + \frac{\pi(t + 1)}{1 + r(t + 1)} + \frac{\pi(t + 2)}{(1 + r(t + 2))(1 + r(t + 1))} + \ldots \tag{5.51}
\]

\[
= \pi(t) + \sum_{\tau=t+1}^{\infty} \left( \prod_{s=t+1}^{\tau} \frac{1}{1 + r(s)} \right) \pi(\tau).
\]

### 5.2.7 Free entry condition

The research sector must satisfy the following free entry condition, which equates the marginal benefit of research (left-hand side) with the marginal cost (right-hand side):

\[
\eta N(t) \left( \frac{V(t + 1)}{1 + r(t + 1)} \right) = w(t). \tag{5.52}
\]

One extra unit of \( L_R(t) \) results in \( \eta N(t) \) additional firms in the following period \( t + 1 \). Why is this? Recall the equation for the growth of varieties:

\[
N(t + 1) = (1 + \eta L_R(t)) N(t). \tag{5.53}
\]

The derivative of this equation with respect to \( L_R(t) \) equals \( \eta N(t) \).

The net present discounted value of each of these firms in period \( t + 1 \) is \( V(t + 1) \). This is expressed in the following expanded form:

\[
V(t + 1) = \pi(t + 1) + \frac{\pi(t + 2)}{1 + r(t + 2)} + \ldots \tag{5.54}
\]

This value discounted back to period \( t \) equals:

\[
\frac{V(t + 1)}{1 + r(t + 1)} = \frac{\pi(t + 1)}{1 + r(t + 1)} + \frac{\pi(t + 2)}{(1 + r(t + 2))(1 + r(t + 1))} + \ldots \tag{5.55}
\]

We must discount back to period \( t \) as that is when the research investment decision is made.

From the definition of net present discounted value in its expanded form (5.51), we see that

\[
V(t) = \pi(t) + \frac{V(t + 1)}{1 + r(t + 1)}. \tag{5.56}
\]

This implies that \( \frac{V(t+1)}{1+r(t+1)} = V(t) - \pi(t) \) and the free entry condition (5.52) is more conve-
niently expressed as:

\[ \eta N(t) (V(t) - \pi(t)) = w(t). \] (5.57)

The left-hand side is the marginal value generated for the economy by devoting labor resources to the research sector, i.e., the total benefit from an extra unit of \( L_R(t) \). The right-hand side is the cost of that extra unit, the wage rate \( w(t) \).

In a competitive setting with free entry, the marginal benefit and the marginal cost are identical in equilibrium. Let’s consider this mechanism with an example. If \( \eta N(t) (V(t) - \pi(t)) > w(t) \), then a firm currently outside the market could hire research labor at wage \( w(t) \) and become the patent holder for the new varieties produced using this research. Such an investment scheme would make net present discounted value \( \eta N(t) (V(t) - \pi(t)) \), which is strictly bigger than the labor cost. This means that it is desirable for many firms to enter the research sector. As new firms enter, two things happen. First, the additional firms drive up the wage rate (the labor supply is fixed). Second, the additional firms create competition that drives down the net present value. Both effects continue until equilibrium is reached at \( \eta N(t) (V(t) - \pi(t)) = w(t) \). The reverse mechanism takes place if \( \eta N(t) (V(t) - \pi(t)) < w(t) \). As we will see shortly, the free entry condition determines the equilibrium number of firms to enter the market in each period.

### 5.2.8 Profit and free entry

An end-of-chapter exercise asks you to show that the profit equation (5.48) and the free entry condition (5.57) together yield the equation:

\[ \pi(t) = \frac{(1 - L_R(t)) \eta}{\epsilon - 1 + (1 - L_R(t)) \eta} V(t). \] (5.58)

### 5.2.9 Balanced growth path

In a balanced growth path, \( L_R(t) = L_R \) in all time periods. This implies that the profit and the net present discounted value must grow at the same rate. Define \( g_\pi \) as this growth rate, where \( (1 + g_\pi) = \frac{\pi(t+1)}{\pi(t)} = \frac{V(t+1)}{V(t)} \). Further, define the growth rate \( g_C \) for the consumption
index and the growth rate $g_N$ for the number of varieties:

\[
1 + g_C = \frac{C(t+1)}{C(t)}.
\]

\[
1 + g_N = \frac{N(t+1)}{N(t)}.
\]

(5.59)

I now proceed to characterize the 4 equilibrium equations under the balanced growth path. It is important that we have 4 equations, since we have 4 unknown variables that we are trying to solve for (namely, $g_\pi$, $g_C$, $g_N$, and $L_R$).

**Equation 1**

From the equation for the growth in the number of varieties, $N(t+1) = (1 + \eta L_R) N(t)$. When we divide both sides by $N(t)$ we obtain

\[
1 + g_N = 1 + \eta L_R.
\]

(5.60)

**Equation 2**

Since $c_i(t) = l(t)$ for all varieties:

\[
C(t) = \left( \int_0^{N(t)} (c_i(t))^{\frac{1}{\gamma-1}} \, di \right)^{\gamma-1} = \left( N(t) (l(t))^{\frac{\gamma-1}{\gamma}} \right)^{\gamma-1} = l(t) N(t)^{\frac{\gamma-1}{\gamma}}.
\]

We have already found from market clearing that $l(t) = \left( \frac{1 - L_R}{N(t)} \right)$, so the consumption index is given by

\[
C(t) = \left( \frac{1 - L_R}{N(t)} \right) N(t)^{\frac{\gamma-1}{\gamma}} = (1 - L_R) N(t)^{\frac{\gamma-1}{\gamma}},
\]

(5.62)

after simplifying the term $\frac{N(t)^{\frac{\gamma-1}{\gamma}}}{N(t)} = N(t)^{\frac{\gamma-1}{\gamma}-1} = N(t)^{\frac{1}{\gamma}}$.

Write the equation $C(t) = (1 - L_R) N(t)^{\frac{1}{\gamma}}$ for both time periods $t$ and $t+1$ :

\[
C(t) = (1 - L_R) N(t)^{\frac{1}{\gamma}}.
\]

\[
C(t+1) = (1 - L_R) N(t+1)^{\frac{1}{\gamma}}.
\]

(5.63)
5.2. **TECHNOLOGICAL CHANGE**

Dividing the second by the first yields

\[ (1 + g_C) = (1 + g_N)^{\frac{1}{1-\epsilon}}. \]  \[(5.64)\]

**Equation 3**

From the price normalization

\[ P(t) = \left( \int_0^{N(t)} (p_i(t))^{1-\epsilon} \, di \right)^{\frac{1}{1-\epsilon}} = 1 \]  \[(5.65)\]

and the fact that firms set the same price \( p_i(t) = p(t) = \frac{\epsilon}{\epsilon-1} w(t) \), then we arrive at the following equality:

\[ \frac{\epsilon}{\epsilon-1} w(t) N(t)^{\frac{1}{1-\epsilon}} = 1. \]  \[(5.66)\]

Solve for \( w(t) \) yields:

\[ w(t) = \frac{\epsilon-1}{\epsilon} N(t)^{\frac{1}{1-\epsilon}}. \]  \[(5.67)\]

Since both \( C(t) \) and \( w(t) \) depend on \( N(t) \) in the same manner, then the growth rate for wage equals the growth rate for consumption:

\[ \frac{w(t+1)}{w(t)} = \frac{C(t+1)}{C(t)} = 1 + g_C. \]  \[(5.68)\]

Let’s write down the free entry conditions (5.57) in time periods \( t \) and \( t + 1 \):

\[ \eta N(t) \left( V(t) - \pi(t) \right) = w(t). \]  \[(5.69)\]

\[ \eta N(t+1) \left( V(t+1) - \pi(t+1) \right) = w(t+1). \]

When we divide the second by the first, we get that:

\[ \frac{N(t+1) \left( V(t+1) - \pi(t+1) \right)}{N(t) \left( V(t) - \pi(t) \right)} = \frac{w(t+1)}{w(t)}. \]  \[(5.70)\]

Since both the profit and the net present discounted value grow at the same rate, then the difference \( V(t) - \pi(t) \) grows at that same rate, which is \( g_\pi \). Thus, the free entry condition yields that

\[ (1 + g_N) (1 + g_\pi) = 1 + g_C. \]  \[(5.71)\]
5. ENDOGENOUS GROWTH THEORY

Equation 4

Recall that \( 1 + r(t) = \frac{C(t)}{C(t-1)} \). This implies that

\[
(1 + r(t + 2)) (1 + r(t + 1)) = \frac{1}{\beta^2} \frac{C(t + 2)}{C(t)}.
\]  

(5.72)

The net present discounted value \( V(t) \) is defined by

\[
V(t) = \pi(t) + \frac{\pi(t + 1)}{1 + r(t + 1)} + \frac{\pi(t + 2)}{(1 + r(t + 2)) (1 + r(t + 1))} + \ldots.
\]  

(5.73)

The terms \( \pi(t + 1) = (1 + g_C) \pi(t) \) and \( \pi(t + 2) = (1 + g_C)^2 \pi(t) \), by definition. The terms \( 1 + r(t + 1) = \frac{1}{\beta} (1 + g_C) \) and

\[
(1 + r(t + 2)) (1 + r(t + 1)) = \frac{1}{\beta^2} (1 + g_C)^2,
\]  

(5.74)

by definition. Using these facts, then

\[
V(t) = \pi(t) + \frac{\beta (1 + g_C)}{(1 + g_C)} \pi(t) + \left( \frac{\beta (1 + g_C)}{(1 + g_C)} \right)^2 \pi(t) + \ldots
\]  

(5.75)

Using our knowledge about discounting, we know that:

\[
V(t) = \frac{\pi(t)}{1 - \frac{\beta(1 + g_C)}{(1 + g_C)}}.
\]  

(5.76)

Recall the equation from the free entry condition and the firm profit condition (5.58):

\[
\pi(t) = \frac{(1 - L_R) \eta}{\epsilon - 1 + (1 - L_R) \eta} V(t).
\]  

(5.77)

Together, equations (5.76) and (5.77) imply that:

\[
1 - \frac{\beta (1 + g_C)}{(1 + g_C)} = \frac{(1 - L_R) \eta}{\epsilon - 1 + (1 - L_R) \eta}.
\]  

(5.78)
5.2.10 Solving the balanced growth path equilibrium

The 4 equations from the previous subsection are (5.60), (5.71), (5.64), and (5.78) and can be used to solve for the four unknowns: $g_C$, $g_N$, and $L_R$. An end-of-chapter exercise asks you to show that the balanced growth path research labor supply is given by:

$$L_R = \frac{\beta \eta + (\epsilon - 1)(\beta - 1)}{\beta \eta + (\epsilon - 1) \eta}. \tag{5.79}$$

Since $\eta > 0 > \beta - 1$, then $L_R < 1$. The values of the parameters $(\beta, \eta, \epsilon)$ must be chosen such that $L_R \geq 0$. If $(\epsilon - 1)(1 - \beta) > \beta \eta$, the labor supply equation (5.79) predicts that $L_R < 0$, an impossibility. What happens here is that elasticity $\epsilon$ is too high, technological growth $\eta$ is too low, and/or patience $\beta$ is too low to incentivize any investment in research. The equilibrium research labor supply $L_R = 0$ (the lower bound) in this rare setting.

An end-of-chapter exercise asks you to verify that the comparative statics solutions are given by: (i) $\frac{\partial L_R}{\partial \eta} > 0$, (ii) $\frac{\partial L_R}{\partial \epsilon} < 0$, and (iii) $\frac{\partial L_R}{\partial \beta} > 0$.

In words, the comparative statics report that (i) if the technology growth rate increases, labor research increases, (ii) if goods are more substitutable (firm market power is lower), labor research is lower, and (iii) if households are more patient, labor research is higher. In all cases, labor research is higher whenever the new invention is more profitable. Profit is high when the goods are less substitutable. Net present discounted profit is high when the interest rates are low (household discount factor high).

With the balanced growth path research labor supply given in (5.79), the growth rates are given as follows:

- Balanced growth rate in the number of varieties (from (5.60)):

$$g_N = \eta L_R = \frac{\beta \eta - (\epsilon - 1)(1 - \beta)}{\beta + \epsilon - 1}. \tag{5.80}$$

No matter the size of the labor supply, $g_N \geq 0$ (nonnegative growth in varieties) with $g_N > 0$ whenever $L_R > 0$ (strictly positive growth in varieties). We will see that this is important for welfare.

- Balanced growth rate in the consumption and wages (from (5.64)):

$$1 + g_C = (1 + g_N) \frac{1}{1 + \epsilon}. \tag{5.81}$$
Our measure of welfare is in terms of the consumption index $C(t)$ and more specifically in terms of its growth rate. If $g_N > 0$ (strictly positive growth in varieties), then since $\frac{1}{c-1} > 0$, it must be that $g_C > 0$ (strictly positive growth in consumption). From market clearing, we use $C(t)$ as our proxy of aggregate output (GDP) in the economy.

An end-of-chapter exercise asks you to find the conditions under which $g_C > g_N$. This is important not just to determine if consumption growth is greater than variety growth, but moreso for the next result.

• Balanced growth rate in the profit and net present discounted value (from (5.71)):  

\[ 1 + g_\pi = \frac{1 + g_C}{1 + g_N}. \] 

(5.82)

An end-of-chapter exercise asks you to find the conditions under which $g_\pi > 0$. While the model will always deliver growth in varieties ($g_N > 0$) and growth in consumption ($g_C > 0$), it is not clear whether the growth in profit for each firm will increase or decrease. The total profit across all firms will increase, but growth also leads to more firms (and more competition). Does this make an individual firm better off or worse off (in terms of profit)? This determination will play a major role in any policy assessment related to incumbent firms’ willingness to accept technological growth in the economy.

5.3 Exercises

1. Monopolistic competition

Show that the profit maximizing price for the firm results in a constant markup over marginal cost: 

\[ p_i = \frac{\epsilon}{\epsilon - 1} \psi. \]

2. Monopolistic competition

Compute the profit $\pi_i$ and show that the comparative statics conclusions are $\frac{\partial \pi_i}{\partial N} < 0$, $\frac{\partial \pi_i}{\partial \epsilon} < 0$, $\frac{\partial \pi_i}{\partial m} > 0$, and $\frac{\partial \pi_i}{\partial \psi} = 0$.

3. Technological change

In the model of technological change, show that the profit equation and the free entry
condition lead to the equation

\[ \pi(t) = \frac{(1 - LR(t)) \eta}{\epsilon - 1 + (1 - LR(t)) \eta} V(t). \]

4. **Technological change**

In the model of technological change with a balanced growth path, write down expressions for the growth rates \( g_\pi, g_C, \) and \( g_N \) in terms of \( LR \) and the parameters \( (\beta, \epsilon, \eta) \). Use these expressions and the equation

\[ 1 - \frac{\beta (1 + g_\pi)}{(1 + g_C)} = \frac{(1 - LR) \eta}{\epsilon - 1 + (1 - LR) \eta} \]

from the fourth numbered item to verify that the labor devoted toward research in the balanced growth path is:

\[ LR = \frac{\beta \eta + (\epsilon - 1) (\beta - 1)}{\beta \eta + (\epsilon - 1) \eta}. \]

5. **Technological change**

Define \( g_a \) as the growth rate of household real asset holdings, where \( 1 + g_a = \frac{a(t)}{a(t-1)} \) for any period \( t \). Show that along the balanced growth path, \( g_a = g_C \).

6. **Technological change**

Suppose the initial number of varieties is \( N(0) \) and household initially have zero real asset holdings. Solve for \( a(0) \) as a function of \( N(0) \). Are households net borrowers or net lenders of the real asset along the balanced growth path?

7. **Technological change**

In the model of technological change with a balanced growth path, show that the comparative statics conclusions are \( \frac{\partial LR}{\partial \eta} > 0, \frac{\partial LR}{\partial \epsilon} < 0, \) and \( \frac{\partial LR}{\partial \beta} > 0 \).

8. **Technological change**

In the model of technological change with a balanced growth path, under what conditions (on the parameter \( \epsilon \)) is \( g_C > g_N \)?

9. **Technological change**

In the model of technological change with a balanced growth path, under what conditions (on the parameter \( \epsilon \)) is \( g_\pi > 0 \)?
10. Technological change

A government is added to the model that taxes firm profit at the rate $\tau$ and uses the tax revenue to increase the growth of varieties according to the following equation:

$$N(t + 1) = \left(1 + \eta L_R(t) + \phi \frac{g(t)}{C(t)}\right) N(t).$$

The equation has the usual term $\eta L_R(t)$ that represents the growth in varieties due to R&D in the private sector. The new term in the equation is $\phi \frac{g(t)}{C(t)}$, where $\phi > 0$ is a parameter and $\frac{g(t)}{C(t)}$ is the ratio of government spending to output (consumption index $C(t)$ is the measure of output in this model). The term $\phi \frac{g(t)}{C(t)}$ represents the growth in varieties due to R&D by the government.

The government balances its budget every period, so $g(t)$ must be equal to the tax revenue collected. Firm profit after taxation is given by:

$$\pi(t) = (p(t)y(t) - w(t)l(t))(1 - \tau).$$

There are a total of $N(t)$ firms, meaning that total tax revenue is equal to

$$\tau (p(t)y(t) - w(t)l(t)) N(t).$$

Write down the expression for firm profit and the free entry condition for firms (Hint: the free entry condition is the same as in the notes).

Consider a balanced growth path in with the research labor supply is $L_R$, the growth rate for consumption is $g_C$, the growth rate for varieties is $g_N$, and the growth rate for profit is $g_\pi$. Write down Equations 1, 2, 3, and 4 that characterize a solution for the balanced growth path (Hint: Equations 2 and 3 are the same as in the notes).

Solve for the balanced growth path research labor supply $L_R$. How big does the value $\phi$ need to be so that the growth rate for varieties $g_N$ is higher with both private sector and government compared to private sector alone? Does $\phi$ need to be bigger than $\eta$? Recall that if the growth rate for varieties $g_N$ is higher, then the growth rate for consumption $g_C$ is higher as well.


Part III

Classical Monetary Theory
Cash-in-Advance Model

6.1 The basic model

6.1.1 Sneak peek

Summary

This chapter and the next consider very simple models of monetary economies in a deterministic setting, meaning that there are no shocks and no uncertainty. To begin to understand how monetary policy affects the economy and how monetary policy can be optimally designed, we must first model the reasons that agents in the economy choose to hold an intrinsically worthless piece of fiat currency.

Understanding why agents in the economy hold money and how their money demand responds to changes in the markets have been pursuits of economists since Adam Smith. One simple formulation is to include money directly in the utility function. Then households that maximize utility would hold money simply because it gives them extra utility. But this simplification is a bit unsatisfying as the utility function should contain commodities that a household actually consumes. Money, although it has many uses, is not something that households consume. Money can be used by households to obtain things that they consume to increase utility.

The simplest monetary model is the cash-in-advance model. As the name suggests, there is a timing assumption in the model that requires households to purchase consumption (using money) before they have had the opportunity to earn income. The model only contains households in a pure-exchange economy (no firms and no production).
One of the implications of the cash-in-advance timing is the Quantity Theory of Money. In this simple setting, the velocity is fixed and equal to 1.

Notation

The variables/parameters to be introduced in this section are given in the following table:

- $e_t$: endowment in period $t$
- $c_t$: consumption in period $t$
- $m_t$: money carried over from period $t$ to $t+1$
- $p_t$: commodity price in period $t$
- $b_t$: nominal bond holding chosen in period $t$ (with payout in period $t+1$)
- $n_t$: nominal interest rate in period $t$
- $T_t$: transfers from government to household
- $\hat{m}_t$: money "shopper" uses to purchase consumption
- $M_t$: total money supply in period $t$
- $\lambda_t$: Lagrange multiplier for the budget constraint
- $\mu_t$: Lagrange multiplier for the CIA constraint
- $a_t$: real bond holding chosen in period $t$ (with payout in period $t+1$)
- $r_t$: real interest rate in period $t$
- $\pi_t$: inflation rate in period $t$

Main takeaways

After completing this section, you will be able to answer the following questions:

- Why do households hold money in this model?
- What are the effects of monetary policy (i.e., the changing of the money supply $M(t)$) on equilibrium?

6.1.2 Model setup

The model contains two economic parties: (i) a government and (ii) a unit mass of homogeneous households.

The economy consists of an infinite number of discrete time periods $t \in \{0, 1, 2, \ldots\}$. In each period, a single commodity is traded and consumed. Denote the endowment of each
household in period $t$ as $e_t$. Denote the consumption of each household in period $t$ as $c_t$. The utility function for the households is given by the discounted utility function

$$
\sum_{t=0}^{\infty} \beta^t u(c_t),
$$

(6.1)

where $\beta \in (0, 1)$ is the discount factor, $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is the one-period utility function, and $c_t$ is the consumption of the household in period $t$. The utility function $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is assumed to be $C^2$, strictly increasing, strictly concave, and satisfy the Inada condition:

$$
\lim_{c \to 0} Du(c) = +\infty.
$$

(6.2)

Each time period is divided into two subperiods:

1. In the morning, the "shopper" half of a household leaves to purchase consumption.

2. During the day, the "storekeeper" half of a household stays at home to sell the endowment.

This timing implies that consumption cannot be purchased using the income obtained from the sale of endowments.

We assume that households cannot consume their own endowment (it is taboo), so they must trade on the commodity markets.

### 6.1.3 Budget constraint

A household has two instruments that allows for transfers of income across time periods. First, a household can carry money holdings from one period to the next. Second, a household can hold a nominal bond with a nominal interest rate equal to $n_t$. A positive amount of bond refers to savings by the household, while a negative amount of bond refers to borrowing by the household.

Define $m_t$ as the money holdings at the end of period $t$ that a household carries into the next period. Define $b_t$ as the period $t$ holdings of the nominal bond (with payout in period $t+1$), and $T_t$ as the period $t$ transfers that are received by the household from the government.

Money holdings are nonnegative $m_t \geq 0$, but bond holdings and transfers can either be positive or negative. If $b_t > 0$, then households are saving; if $b_t < 0$, households are
6. CASH-IN-ADVANCE MODEL

borrowing. If \( T_t > 0 \), the households are receiving a subsidy; if \( T_t < 0 \), households are paying a tax.

The budget constraint is expressed in units of account, typically dollars. In the morning of period \( t \), the household receives the transfers \( T_t \) from the government and the bond payouts \( b_{t-1} \). The household chooses the new bond holdings \( b_t \) and the amount of money \( \hat{m}_t \) to give to the "shopper" to use on the market to purchase consumption. You can think of this as the cash in the wallet of the "shopper." This amount of cash \( \hat{m}_t \) must satisfy the budget constraint:

\[
\frac{b_t}{1 + n_t} + \hat{m}_t \leq m_{t-1} + b_{t-1} + T_t.
\] (6.3)

The "shopper" leaves with \( \hat{m}_t \) and the "storekeeper" stays in the store to sell the endowment.

Are you not used to seeing the interest rate in the denominator? I can show you that this formulation is equivalent to one you are more familiar with. Instead of the bond expenditure being equal to \( \frac{b_t}{1 + n_t} \), we define the bond holdings \( \tilde{b}_t = \frac{b_t}{1 + n_t} \). By holding 1 unit of bond \( \tilde{b}_{t-1} \), households are entitled to the payout \( (1 + n_{t-1}) \) in period \( t \). Notice that the interest rate is determined in the period in which the bond is purchased (period \( t - 1 \)). The budget constraint can then be written as:

\[
\tilde{b}_t + \hat{m}_t \leq m_{t-1} + (1 + n_{t-1})\tilde{b}_{t-1} + T_t.
\] (6.4)

6.1.4 Cash-in-advance constraint

Define \( p_t \) as the period \( t \) nominal price of the commodity. The cash-in-advance constraint requires that consumption can only be purchased using the cash \( \hat{m}_t(t) \) held by the "shopper":

\[
p_t c_t \leq \hat{m}_t.
\] (6.5)

When the "shopper" comes home at the end of the day, the money holdings of the household (that are carried into next period) are

\[
m_t = (\hat{m}_t - p_t c_t) + p_t e_t.
\] (6.6)

The first term \( \hat{m}_t - p_t c_t \) is the amount that that "shopper" has remaining in its wallet and the amount \( p_t e_t \) is the income received by the "storekeeper" from endowment sales. Using
6.1. THE BASIC MODEL

this definition, the cash-in-advance constraint (6.5) is equivalently written as:

\[ m_t \geq p_t e_t. \]  \hspace{1cm} (6.7)

Using the definition \( m_t = (\hat{m}_t - p_t c_t) + p_t e_t \) to replace \( \hat{m}_t \) from the original budget constraint, the updated budget constraint for a household in period \( t \) is given by:

\[ p_t e_t + m_t + \frac{b_t}{1 + n_t} \leq p_t e_t + m_{t-1} + b_{t-1} + T_t. \]  \hspace{1cm} (6.8)

### 6.1.5 Government budget constraint

What is the budget constraint for the government? In period \( t \), the government chooses the money supply \( M_t \). The current period money supply \( M_t \) is an asset on the government’s balance sheet. By increasing the money supply issued to the economy, the government is able to spend more. On the liabilities side of the government’s balance sheet, there are two terms: the previous period money supply \( M_{t-1} \) and the transfers \( T_t \). Any money that was issued in period \( t - 1 \) can be redeemed by households in period \( t \), so the government must be able to collect revenue to support this redemption. Positive net transfers \( T_t \) mean that transfers are made from the government to the households.

The budget must be balanced in each period, so neither a deficit nor a surplus is permitted. The budget constraint in period \( t \) is such that the assets equal the sum of the liabilities:

\[ M_t = M_{t-1} + T_t. \]  \hspace{1cm} (6.9)

### 6.1.6 Market clearing conditions

In equilibrium, there is a market clearing condition for consumption, money holdings, and nominal bonds. All households are identical, so the total consumption \( c_t \) in period \( t \) must be equal to the aggregate resources \( e_t \) in period \( t \):

\[ c_t = e_t. \]  \hspace{1cm} (6.10)

Second, the money holdings of the household equal the money supply chosen by the government:

\[ m_t = M_t. \]  \hspace{1cm} (6.11)
Third, the bond is in zero net supply, meaning that the total net savings must equal 0 (total lending equals total borrowing). As all households are identical and the government does not issue debt:

\[ b_t = 0. \]  

(6.12)

With a strictly increasing utility function, the household budget constraints (6.8) and cash-in-advance constraints (6.7) will both bind.

Out of the several possible roles for money, this model explicitly considers two of them: as a medium of exchange and as a store of value. Households have access to 2 financial assets with risk-free payout. The risk-free return of holding money is 1. The risk-free return of the nominal bond is \( 1 + n_t \). Consider what happens if \( n_t < 0 \), meaning that the return on money dominates the return on the nominal bond. Households would opt for the following portfolio: (i) hold an arbitrarily large negative nominal bond position, and (ii) support such a negative bond position with an even larger positive money holding position. This allows both for (i) an infinitely large portfolio payout and (ii) for households to have sufficient money to satisfy the cash-in-advance constraint (6.7). Such a portfolio is not in the set of real numbers (the positions are unbounded).

Thus, an equilibrium requires that \( n_t \geq 0 \). It is certainly possible that \( n_t > 0 \), and the return on the nominal bond dominates the return on money. Households still hold money, since the cash-in-advance constraint (6.7) requires money in order to purchase consumption (this is the medium of exchange role of money).

Combining the cash-in-advance constraint (6.7), holding with equality, and the money supply market clearing condition (6.11), we obtain the following:

\[ M_t = p_t e_t. \]  

(6.13)

This is a statement of the Quantity Theory of Money, which states that the nominal price level \( p_t \) is proportional to the money supply \( M_t \).
6.1. THE BASIC MODEL

6.1.7 Kuhn-Tucker conditions

The household utility maximization problem is given by:

\[
\max_{\{c_t, m_t\}_{t \in \mathbb{N}}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
\text{subj. to} \quad \text{budget constraint (6.8) for all } t \quad \text{cash-in-advance constraint (6.7) for all } t
\]  

(6.14)

The constraints are always written as nonnegative inequalities, meaning that the budget constraint is expressed as

\[
p_t e_t + m_{t-1} + b_{t-1} + T_t - p_t c_t - m_t - \frac{b_t}{1 + n_t} \geq 0
\]  

(6.15)

and the cash-in-advance constraint as

\[
m_t - p_t e_t \geq 0.
\]  

(6.16)

Consider the Kuhn-Tucker conditions associated with this constrained maximization problem. Denote the Lagrange multiplier for the period \( t \) budget constraint (6.15) as \( \lambda_t \). Denote the Lagrange multiplier for the period \( t \) cash-in-advance constraint (6.16) as \( \mu_t \). The Kuhn-Tucker conditions are given by:

- **First order conditions:**
  - With respect to consumption \( c_t \)
    \[
    \beta^t Du(c_t) - \lambda_t p_t = 0.
    \]  
    (6.17)
  - With respect to money holdings \( m_t \)
    \[
    -\lambda_t + \lambda_{t+1} + \mu_t = 0.
    \]  
    (6.18)
  - With respect to bond holding \( b_t \)
    \[
    -\frac{\lambda_t}{1 + n_t} + \lambda_{t+1} = 0.
    \]  
    (6.19)
6. CASH-IN-ADVANCE MODEL

- Complimentary slackness conditions:
  
  - For the budget constraint
    \[ \lambda_t \left( p_t c_t + m_{t-1} + b_{t-1} + T_t - p_t c_t - m_t - \frac{b_t}{1 + n_t} \right) = 0. \tag{6.20} \]
  
  - For the cash-in-advance constraint
    \[ \mu_t \{ m_t - p_t e_t \} = 0. \tag{6.21} \]

  The Euler equation is found by combining the first order conditions with respect to consumption and money holdings:

  \[ \frac{Du(c_t)}{pt} = \beta \frac{Du(c_{t+1})}{pt+1} + \frac{\mu_t}{\beta}. \tag{6.22} \]

  If \( \mu_t = 0 \), the Euler equation is given by:

  \[ \frac{Du(c_t)}{pt} = \beta \frac{Du(c_{t+1})}{pt+1}. \tag{6.23} \]

  If \( \mu_t > 0 \), the Euler equation is given by:

  \[ \frac{Du(c_t)}{pt} > \beta \frac{Du(c_{t+1})}{pt+1}. \tag{6.24} \]

  If \( \mu_t > 0 \), a distortion is introduced in the household’s optimal choice problem. Namely, the requirement to hold money forces households to optimize at \( \frac{Du(c_t)}{pt} > \beta \frac{Du(c_{t+1})}{pt+1} \) instead of the desired \( \frac{Du(c_t)}{pt} = \beta \frac{Du(c_{t+1})}{pt+1} \). Households would prefer not to hold money, as this would allow them to increase their current consumption \( c_t \). As \( c_t \) increases, then \( Du(c_t) \) decreases and the household moves closer to the ideal \( \frac{Du(c_t)}{pt} = \beta \frac{Du(c_{t+1})}{pt+1} \).

6.1.8 Interest rates

Using the first order condition with respect to the bond holding (6.19), we can derive the Euler equation for the nominal bond:

\[ \beta \frac{Du(c_{t+1})}{pt+1} (1 + n_t) = \frac{Du(c_t)}{pt}. \tag{6.25} \]
Solving for the nominal interest rate yields:

\[ 1 + n_t = \frac{1}{\beta} \frac{p_{t+1}}{p_t} \frac{Du(c_t)}{Du(c_{t+1})}. \]  

(6.26)

Define the real interest rate in period \( t \) as \( r_t \). To find the value for the real interest rate, we introduce a real bond \( a_t \). The real asset payout is in units of the commodity (real units). When it enters the budget constraint (measured in units of account, typically dollars), we must multiply the payout by the commodity price. The budget constraint with both the nominal and the real bond is given by:

\[ p_t c_t + m_t + \frac{b_t}{1 + n_t} + \frac{p_t a_t}{1 + r_t} \leq p_t e_t + m_{t-1} + b_{t-1} + p_t a_{t-1} + T_t. \]  

(6.27)

The cash-in-advance constraint (6.7) remains unchanged.

Market clearing dictates that \( a_t = 0 \). The equilibrium allocation remains \( c_t = e_t \). The first order condition with respect to \( a_t \) is given by:

\[ -\frac{\lambda_t p_t}{1 + r_t} + \lambda_{t+1} p_{t+1} = 0. \]  

(6.28)

Using this first order condition (6.28), we can find the Euler equation for the real bond:

\[ \beta Du(c_{t+1}) (1 + r_t) = Du(c_t). \]  

(6.29)

Solving for the real interest rate yields:

\[ 1 + r_t = \frac{1}{\beta} \frac{Du(c_t)}{Du(c_{t+1})}. \]  

(6.30)

Define \( 1 + \pi_{t+1} = \frac{p_{t+1}}{p_t} \), where \( \pi_{t+1} \) is the inflation rate from period \( t \) to period \( t + 1 \). The two equations (6.26) and (6.30) provide the relation between the inflation rate, the nominal interest rate, and the real interest rate:

\[ (1 + n_t) = (1 + \pi_{t+1}) (1 + r_t). \]  

(6.31)

For values \( \pi_{t+1} \approx 0 \) and \( r_t \approx 0 \), the equality relation between the nominal and real interest
rates (6.31) can be approximated by what is commonly referred to as the Fisher equation:

\[ n_t \approx \pi_{t+1} + r_t. \] (6.32)

Actually, the mathematics reveal that

\[ n_t = \pi_{t+1} + r_t + \pi_{t+1} r_t, \] (6.33)

but the final product is small and ignored for values \( \pi_{t+1} \approx 0 \) and \( r_t \approx 0 \).

### 6.1.9 Constant endowment economy

Consider an economy with constant endowment: \( e_t = e \) for all time periods. Commodity market clearing implies that the economy has constant consumption: \( c_t = e \) for all time periods.

A constant monetary growth rate implies the existence of \( \alpha \) such that \( \frac{M_{t+1}}{M_t} = \alpha \) in all time periods. From the Quantity Theory of Money, then it must be that the inflation rate is equal to the money supply growth rate: \( \frac{p_{t+1}}{p_t} = \frac{M_{t+1}}{M_t} \) and \( 1 + \pi_{t+1} = \alpha \). The real interest rate in all time periods equals

\[ 1 + r_t = \frac{1}{\beta} \frac{Du(c_t)}{Du(c_{t+1})} = \frac{1}{\beta} > 1. \] (6.34)

The nominal interest rate in all time periods equals

\[ 1 + n_t = \frac{1}{\beta} \frac{p_{t+1}}{p_t} \frac{Du(c_t)}{Du(c_{t+1})} = \frac{\alpha}{\beta}. \] (6.35)

As discussed above, an equilibrium requires that \( n_t \geq 0 \), meaning that \( 1 + n_t = \frac{\alpha}{\beta} \geq 1 \). Therefore, the money supply growth rate must satisfy \( \alpha \geq \beta \).

### 6.1.10 Friedman rule

The Friedman rule is defined such that the money supply growth rate and inflation rate are both equal to \( \beta - 1 \). Since \( \beta < 1 \), this means that the growth rate for money is negative and the growth rate for prices is negative (deflation).
In the economy with constant endowments, the Euler equation when $\mu_t = 0$ is given by:

$$\frac{Du(c_t)}{p_t} = \beta \frac{Du(c_{t+1})}{p_{t+1}}.$$  \hspace{1cm} (6.36)

Since $c_t = c_{t+1}$, the Euler equation is satisfied when

$$\beta = \frac{p_{t+1}}{p_t},$$

meaning that the inflation rate $\pi_{t+1} = \beta - 1$. As we found in the previous subsection, the monetary growth rate is equal to the inflation rate under constant endowment.

Since $\mu_t = 0$, there are no distortionary effects from the cash-in-advance constraint. We say that the Friedman rule is optimal as it equates the opportunity cost from holding currency with the social cost of currency creation. Since the marginal cost of creating fiat currency is zero, the opportunity cost from holding currency must be zero to satisfy this notion of optimality.

### 6.2 Open market operations

#### 6.2.1 Sneak peek

**Summary**

Using the cash-in-advance setup, we can analyze the traditional monetary policy called open market operations. This monetary policy involves changes in the nominal interest rate (supported by changes in the amount of nominal bonds purchased or sold by the central bank) in response to business cycle movements. The policy of open market operations is characterized by the setting of the Federal Funds Rate by the US Federal Reserve (the central bank of the US).

A country contains both a fiscal authority and a monetary authority. The fiscal authority is responsible for collecting taxes, balancing the budget, and issuing debt in order to finance a deficit (a budget shortfall). The monetary authority, or central bank, is tasked with printing money, lending to financial institutions, holding the reserves of financial institutions, and implementing monetary policy.

For our purposes in this section, we consider that the fiscal authority and the monetary authority have perfectly aligned objectives and act in concert. Together, the fiscal authority
and the monetary authority form what we call the government. This entity called the government will choose the debt level, the money supply, and the nominal interest rate.

**Notation**

The variables to be introduced in this section are given in the following table:

- $B_t$: government debt in period $t$
- $n^*$: interest rate target under interest rate targeting
- $\alpha$: monetary growth rate target under money supply targeting

**Main takeaways**

After completing this section, you will be able to answer the following questions:

- What are the mechanical steps of open market operations?
- What are the macroeconomic consequences of interest rate targeting and money supply targeting?

**6.2.2 The mechanism for open market operations**

Open market operations works as follows. Suppose that the government observes that the economy has entered a recession and wants to lower the interest rate (to encourage household borrowing). A government is composed of a fiscal authority (that levies taxes and issues debt) and a monetary authority or central bank (that chooses the monetary policy).

1. To lower the interest rate, the monetary authority must buy bonds. If the number of bond buyers increases, then this will drive down the interest rate. The price of a bond is the inverse of the interest rate, so as the number of bond buyers increases, the bond price must increase as well. This is simply the law of demand (demand goes up means the price goes up).

2. The fiscal authority issues debt in the form of these bonds. If the monetary authority is buying bonds, then the overall government debt position is decreased.

3. Where does the monetary authority get the funds to purchase these bonds? The monetary authority has the ability to print money and increase the money supply.
6.2. OPEN MARKET OPERATIONS

The fiscal authority has the ability to increase the taxes and transfer these funds to the monetary authority. Either route is an appropriate means to raise the funds for the monetary authority to purchase the bonds.

The reverse mechanism occurs during periods of growth. This class will consider the government as a single entity consisting of both the fiscal authority and the monetary authority. We consider that the government always operates as one cohesive unit.

6.2.3 Government debt

The government issues debt as a risk-free nominal bond, the same one that is traded by the households. This nominal bond can be viewed as a short-term treasury bond (or T-bill). Denote the government debt in period $t$ as the nonnegative variable $B_t \geq 0$.

There are now two participants in the bond market: households and the government. We have assumed that there is a unit mass of homogeneous households and one government, so the government has the same weight as all households in the model. This assumption can be modified by assuming a different mass for the households, but that subtle change is irrelevant for the qualitative predictions of the model. In the bond market, the government debt must exactly equal to the household savings (the government issues debt and the only other agent in the model that can hold the debt is the households):

$$b_t = B_t. \quad (6.37)$$

Equation (6.26) in the previous section was derived from the household Euler equation and will be referred to as the interest rate equation:

$$1 + n_t = \frac{1}{\beta} \frac{p_{t+1}}{p_t} \frac{Du(c_t)}{Du(c_{t+1})}. \quad (6.38)$$

The assets for the government are (i) the current money supply $M_t$ which is issued to the households and (ii) the new government debt level $\frac{B_t}{1+n_t}$ (scaled so that interest payments are already included in the issuance price). The liabilities for the government are (i) the money supply $M_{t-1}$ issued in the previous period to households (and redeemed in the current period), (ii) the previous period debt obligations $B_{t-1}$, and (iii) the transfers $T_t$ made by the government to the households. The government budget constraint requires that the total
value of the assets equals the total value of the liabilities:

\[ M_t + \frac{B_t}{1 + n_t} = M_{t-1} + B_{t-1} + T_t. \] (6.39)

In this simple model with homogeneous households, there is no trading amongst households and the equilibrium consumption is always \( c_t = e_t \). In this sense, monetary policy cannot have real effects in this model. What we want to focus on is the financial effects (price changes) caused by monetary policy.

Using the Quantity Theory of Money \( (M_t = p_t e_t) \) and the commodity market clearing condition \( (c_t = e_t) \), the expression for the nominal interest rate (6.38) becomes:

\[ 1 + n_t = \frac{1}{\beta} \frac{M_{t+1}}{M_t} \frac{e_t}{e_{t+1}} \frac{Du(e_t)}{Du(e_{t+1})}. \] (6.40)

This lesson considers two common open market operations: (i) interest rate targeting and (ii) money supply targeting.

### 6.2.4 Interest rate targeting

Interest rate targeting is a monetary policy such that \( n_t = n^* \geq 0 \) in all time periods \( t \). Recall that the nominal interest rate must be nonnegative in equilibrium. Given the previous period decisions \( M_{t-1} \) and \( B_{t-1} \), the government must implement policy that satisfies (6.40):

\[ M_t = \beta M_{t-1} (1 + n^*) \frac{e_t}{e_{t-1}} \frac{Du(e_t)}{Du(e_{t-1})}, \] (6.41)

The implementation requires the government to choose nominal bonds and transfers such that its budget constraint (6.39) is satisfied:

\[ M_{t-1} \left( \beta (1 + n^*) \frac{e_t}{e_{t-1}} \frac{Du(e_t)}{Du(e_{t-1})} - 1 \right) + B_t = (1 + n^*) B_{t-1} + T_t. \] (6.42)

It is irrelevant, both for the government and the households, whether the government chooses to support the nominal interest rate \( n^* \) with bonds or transfers. This is a result due to the early 19th century economist David Ricardo called Ricardian Equivalence.

**Theorem 6.1 Ricardian Equivalence**
It does not matter whether a government finances its spending with debt or a tax increase, because the effect on the total level of household demand in the economy is the same.

To see this theorem in action, consider one possible pair \((B_t, T_t)\) satisfying (6.42). Now suppose that the government decides (for unknown political reasons) to increase \(T_t\) by $1. To continue to satisfy (6.42), the government must increase \(B_t\) by $1. This involves borrowing 1 additional unit of the nominal bond from the households. This means that the households are saving one additional unit of the nominal bond. Each household has thus received $1 extra as transfer, but is also spending $1 more on the bond \(b_t\). The budget constraint of the household indicates that this trade-off still allows for the same amount of consumption to be chosen.

To better understand the dynamics with the government bonds, suppose that all transfers are required to be the same and set equal to zero: \(T_t = 0\) for all time periods. The government debt position from (6.42) is equal to:

\[
B_t = (1 + n^*) B_{t-1} - M_{t-1} \left( \beta (1 + n^*) \frac{e_t}{e_{t-1}} \frac{Du(e_t)}{Du(e_{t-1})} - 1 \right).
\]

(6.43)

We analyze the effects of a reduction in endowments \(e_t\). That is, suppose that a recession hits in period \(t\) and the aggregate resources \(e_t\) are reduced. What is the required policy response to maintain the interest rate target?

Let’s consider a CRRA (constant relative risk aversion) utility function \(u(c) = \frac{c^{1-\rho}}{1-\rho}\) for values \(\rho > 1\). The parameter \(\rho\) is the risk aversion parameter. Under this utility function,

\[
\frac{e_t}{e_{t-1}} \frac{Du(e_t)}{Du(e_{t-1})} = \left( \frac{e_t}{e_{t-1}} \right)^{1-\rho}.
\]

(6.44)

If \(e_t\) decreases, the product \(e_tDu(e_t) = e_t^{1-\rho}\) increases as \(1 - \rho < 0\). This implies from (6.41) and (6.42) that the money supply \(M_t\) must increase and the government debt \(B_t\) must decrease. For this second result, the term multiplying \(M_{t-1}\) has increased and \(M_{t-1} > 0\). The policy effect is that the government increases the money supply and decreases the debt, while households increase borrowing.
6.2.5 Monetary growth rate targeting

Monetary growth rate targeting is a monetary policy such that $\frac{M_t}{M_{t-1}} = \alpha$ for a constant $\alpha$. From the interest rate equation (6.26):

$$1 + n_{t-1} = \frac{\alpha}{\beta} e_{t-1} \frac{Du(e_{t-1})}{Du(e_t)}.$$ (6.45)

Since equilibrium requires $n_{t-1} \geq 0$, the constant $\alpha$ must satisfy the lower bound condition:

$$\alpha \geq \beta \frac{e_t}{e_{t-1}} \frac{Du(e_t)}{Du(e_{t-1})}.$$ (6.46)

Given the equation for the nominal interest rate, the government budget constraint (6.39) with $T_t = 0$ yields the debt equation:

$$B_t = \frac{\alpha}{\beta} e_{t-1} \frac{Du(e_{t-1})}{Du(e_t)} B_{t-1} - M_{t-1} (\alpha - 1)$$ (6.47)

Again, we analyze the effects of a reduction in endowments $e_t$. The money supply is fixed according to the target, so we focus on the effect on government debt and household savings. If $e_t$ decreases, the product $e_t Du(e_t)$ increases, as discussed above. This implies that the nominal interest rate $n_{t-1}$ decreases. This term is always positive. Since $B_{t-1} > 0$, the decrease in the term $\frac{\alpha}{\beta} e_{t-1} \frac{Du(e_{t-1})}{Du(e_t)}$ means that the government decreases debt. Households would increase borrowing.

The rate of debt reduction may vary for the two policies, but under both interest rate targeting and money supply targeting, a recession causes the government to reduce debt $B_t$, which allows households to borrow more. The size of debt reduction depends on which is bigger: $(1 + n^*)$ or $\frac{\alpha}{\beta} e_{t-1} \frac{Du(e_{t-1})}{Du(e_t)}$.

6.3 Endogenous money demand

6.3.1 Sneak peek

Summary

In the standard cash-in-advance model, the velocity is assumed to be equal to one and remain constant in all periods. This is obviously a ridiculous assumption, but it allows for a simple
solution to the model.

This section extends the cash-in-advance model by incorporating real transportation costs and introducing a trade-off into the household problem between taking more money into the market or taking less money and making more trips (incurring larger transportation costs).

In this framework, the velocity (or number of trips into the market) is the number of times that each unit of currency circulates in the economy. This variable is endogenously chosen by the household. The predictions with a changing velocity value can be compared to the standard cash-in-advance model to see if policy changes are amplified or dampened when the velocity mechanism is included.

Empirically, the velocity of money is procyclical. During periods of growth, money circulates faster in the economy (money demand is higher), where the opposite occurs during recessions. The same findings are found in this model. The outcome of the model is that velocity is directly related to the nominal interest rate. The nominal interest rate is high during periods of growth (meaning velocity is high as well) and low during recessions.

The reason that velocity and the nominal interest rate are directly related in the model derives from the trade-off between money and nominal bonds for saving. Households are forced to hold money to satisfy the cash-in-advance constraint. They would prefer to save using nominal bonds. When the nominal interest rate is high, this difference between money and nominal bonds (in terms of returns from saving) is high. Households can offset this large difference by holding less money. Less money means they have to take more trips to the market and incur a higher transportation cost. They are willing to bear this cost to offset the difference between money and nominal bond returns. Trips to the market is velocity in this model, so velocity is higher when the nominal interest rate is higher.

Notation

The variables to be introduced in this section are given in the following table:

\[ v_t \] velocity of money

Main takeaways

After completing this section, you will be able to answer the following questions:

- How is velocity determined in equilibrium?
6. CASH-IN-ADVANCE MODEL

- Under interest rate targeting, how are household consumption and prices determined? Does Ricardian equivalence hold?

6.3.2 Households

Time is discrete and infinite with time periods \( t \in \{0, 1, \ldots\} \).

The model contains a unit mass of homogeneous households. In each time period, a physical commodity is traded and consumed. Denote \( c(t) \) as the consumption in period \( t \). Household preferences are given by:

\[
U(c) = \sum_{t=0}^{\infty} \beta^t \ln(c_t).
\]  (6.48)

The parameter \( \beta \in (0, 1) \) is the discount factor.

The household receives endowments \( e_t > 0 \) in period \( t \).

The nominal price level of the commodity is \( p_t \) in period \( t \). The money holdings of a household in period \( t \) are denoted \( m_t \). Money holdings must be non-negative.

Households can also hold bonds. The bond holdings chosen in period \( t \) are denoted \( b_t \). The nominal interest rate is \( n_t \).

The transfers, in units of currency, received by the household from the government are \( T_t \).

6.3.3 Monetary transactions

In the morning of period \( t \), the household receives the income consisting of transfers \( T_t \) from the government, money carried over from the previous period, and bond payouts. The household income (in units of currency) at the beginning of period \( t \) is given by:

\[
m_{t-1} + b_{t-1} + T_t.
\]

The household consists of two halves: a "shopper" half that travels to other households to purchase consumption and a "storekeeper" half that stays in the household in order to sell the endowment. Given the available income, the household decides on the amount of period \( t \) bond holdings \( b_t \) and the amount of money \( m_t \) to give to the shopper for the first shopping
trip. These choices must satisfy the budget constraint:

\[ \hat{m}_t + \frac{b_t}{1 + n_t} \leq m_{t-1} + b_{t-1} + T_t. \]  

(6.49)

The shopper leaves with \( \hat{m}_t \) and the storekeeper stays in the store to sell the endowment.

When the shopper has spent all the money \( \hat{m}_t \), the shopper returns to the store. This completes 1 shopping trip. During this period, the storekeeper has sold endowment with value identical to the amount that the shopper just spent (due to the symmetry of the model). The storekeeper replenishes the shopper’s supply of money such that the shopper leaves with \( \hat{m}_t \) units of money for the second shopping trip.

The shopper incurs a real transaction cost for each shopping trip, which can be viewed as a cost of transporting purchased goods back to the store. Velocity \( v_t \) is the number of shopping trips and is an endogenous choice of the household. As a function of the parameter \( A \), the real transaction cost is equal to the fraction \( \frac{v_t}{A + v_t} \) of the amount purchased on each shopping trip. This means that if \( c_t \) units are to be consumed, the amount purchased must be such that \( c_t = (\text{purchased}) \left(1 - \frac{v_t}{A + v_t}\right) \). This implies that \( \text{purchased} = \left(\frac{A + v_t}{A}\right) c_t \) units.

Since there are \( v(t) \) shopping trips, the cash-in-advance constraint for the shopper is:

\[ \left(\frac{A + v_t}{A}\right) p_t c_t \leq v_t \hat{m}_t. \]  

(6.50)

This means that the shopper half of the household must take at least \( \left(\frac{A + v_t}{A}\right) \left(\frac{p_t c_t}{v_t}\right) \) units of currency every time it leaves to go shopping in order to spend a total of \( \left(\frac{A + v_t}{A}\right) p_t c_t \) units of currency over \( v_t \) shopping trips. Every time the shopper goes off and spends \( \left(\frac{A + v_t}{A}\right) \left(\frac{p_t c_t}{v_t}\right) \) units of currency, the storekeeper is able to earn that exact amount by selling endowment.

After the last shopping trip, the shopper returns to the store.

Define \( m_t \) as the amount of money that the household (both the storekeeper and the shopper) has available at the end of the period to carry forward into the next period. By definition,

\[ m_t = \hat{m}_t - \left(\frac{A + v_t}{A}\right) \left(\frac{p_t c_t}{v_t}\right) + \frac{p_t e_t}{v_t}. \]  

(6.51)

On the final shopping trip, the shopper left with \( \hat{m}_t \) units of currency and spent \( \left(\frac{A + v_t}{A}\right) \left(\frac{p_t c_t}{v_t}\right) \) units of currency. During that final shopping trip, the storekeeper sold the remaining endowment, which earned \( \frac{p_t e_t}{v_t} \) units of currency. By symmetry, each trip by the shopper is
the exact length of time needed for the storekeeper to sell \( \frac{a}{v_t} \) units of endowment. The cash-in-advance constraint (6.50) can then be updated, using the definition (6.51):

\[
m_t \geq \frac{p_t e_t}{v_t}.
\]  

(6.52)

Inserting the definitions for \( \hat{m} (t) \) from (6.51) into the original budget constraint (6.49) yields the updated household budget constraint:

\[
\left( \frac{A + v_t}{A} \right) \left( \frac{p_t c_t}{v_t} \right) + m_t + \frac{b_t}{1 + n_t} \leq \frac{p_t e_t}{v_t} + m_{t-1} + b_{t-1} + T_t.
\]  

(6.53)

The household optimization problem is given by:

\[
\max_{\{c_t, m_t, b_t, v_t\}} \sum_{t=0}^{\infty} \beta^t \ln (c_t)
\]

subj. to

budget constraint (6.53) in all time periods

cash-in-advance constraint (6.52) in all time periods

(6.54)

6.3.4 Government

The government issues the money supply \( M_t \) in period \( t \). Using the same bond markets as are available to households, the government issues debt. Denote \( B_t \geq 0 \) as the debt issued in period \( t \).

The government constraints are given by:

\[
M_t + \frac{B_t}{(1 + n_t)} = M_{t-1} + B_{t-1} + T_t.
\]  

(6.55)

6.3.5 Equilibrium

An equilibrium is the household variables \( \{c_t, m_t, b_t, v_t\} \), the government variables \( \{B_t, M_t, T_t\} \), and the price variables \( \{p_t, n_t\} \) such that:

1. Given \( \{p_t, n_t, T_t\} \), the household chooses \( \{c_t, m_t, b_t, v_t\} \) to solve the household problem (6.54).

2. The government variables \( \{B_t, M_t, T_t\} \) satisfy the government constraints (6.55).

3. Markets clear:
6.3. ENDOGENOUS MONEY DEMAND

(a) \( c_t \left( \frac{A + v_t}{A} \right) = e_t \) for every \( t \).
(b) \( m_t = M_t \) for every \( t, s' \).
(c) \( b_t = B_t \) for every \( t \).

Recall that the velocity cost \( \frac{v_t}{A + v_t} \) is a real cost, meaning that resources are destroyed and household consumption will be strictly less than the endowment whenever this real cost is incurred. If \( e_t = c_t \left( \frac{A + v_t}{A} \right) \) units are purchased, only \( c_t \left( \frac{A + v_t}{A} \right) \left( 1 - \frac{v_t}{A + v_t} \right) = c_t \) units survive the transportation back to the store and are available for consumption.

In equilibrium, the cash-in-advance constraint binds:

\[
m_t = \frac{p_t e_t}{v_t}. \quad (6.56)
\]

This is the Quantity Theory of Money. Using the market clearing condition,

\[
m_t = \left( \frac{A + v_t}{A} \right) \left( \frac{p_t c_t}{v_t} \right). \quad (6.57)
\]

6.3.6 Solving for an equilibrium

The household utility maximization problem is given by:

\[
\max_{\{c_t, m_t, b_t, v_t\}} \sum_{t=0}^{\infty} \beta^t \ln (c_t)
\]
\[
\text{subj. to } \begin{array}{l}
p_t c_t \frac{e_t}{v_t} + m_{t-1} + b_{t-1} + T_t - \left( \frac{A + v_t}{A} \right) \left( \frac{p_t c_t}{v_t} \right) - m_t - \frac{b_t}{1 + n_t} \geq 0 \\
- m_t v_t - p_t e_t \geq 0
\end{array} \quad . \quad (6.58)
\]

The constraints are always written as nonnegative inequalities.

Consider the Kuhn-Tucker conditions associated with this constrained maximization problem. Denote the Lagrange multiplier for the period \( t \) budget constraint (6.15) as \( \lambda_t \). Denote the Lagrange multiplier for the cash-in-advance constraint in period \( t \) as \( \mu_t \).

- First order conditions:

  - With respect to consumption \( c_t \)

    \[
    \frac{\beta^t}{c_t} - \lambda_t \left( \frac{A + v_t}{A} \right) \frac{p_t}{v_t} = 0. \quad (6.59)
    \]
6. CASH-IN-ADVANCE MODEL

- With respect to velocity $v_t$
  \[
  \lambda_t \left\{ -\frac{p_t c_t}{v_t^2} + \frac{p_t c_t}{v_t^2} \right\} + \mu_t m_t = 0. \tag{6.60}
  \]

- With respect to money holdings $m_t$
  \[-\lambda_t + \lambda_{t+1} + \mu_t v_t = 0. \tag{6.61}\]

- With respect to bond holding $b_t$
  \[-\frac{\lambda_t}{1 + n_t} + \lambda_{t+1} = 0. \tag{6.62}\]

- Complimentary slackness conditions:

  - For the budget constraint
    \[
    \lambda_t \left\{ \frac{p_t c_t}{v_t} + m_{t-1} + b_{t-1} + T_t - \left( \frac{A + v_t}{A} \right) \left( \frac{p_t c_t}{v_t} \right) - m_t - \frac{b_t}{1 + n_t} \right\} = 0. \tag{6.63}
    \]

  - For the cash-in-advance constraint
    \[
    \mu_t \{m_tv_t - p_t c_t\} = 0. \tag{6.64}
    \]

In equilibrium, $\lambda_t > 0$, so the budget constraint is always binding. From the Quantity Theory of Money (6.57), the first order condition (6.59) is equivalently given by:

\[
\beta^t = \lambda_t m_t. \tag{6.65}
\]

Combining first order condition for consumption (6.65) and first order condition for velocity (6.60), together with the fact that the cash-in-advance constraint is binding, yields:

\[
\frac{\beta^t}{m_t} \frac{1}{A + v_t} = \mu_t. \tag{6.66}
\]

The first order condition for money (6.61), after citing (6.65) and (6.66), is given by:

\[
\frac{\beta^t}{m_t} \left( \frac{A + v_t}{A} \right) = \frac{\beta^{t+1}}{m_{t+1}}. 
\]
6.3. ENDOGENOUS MONEY DEMAND

The Euler equation for money is then given by:

\[
\left( \frac{A + v_t}{A} \right) = \beta \frac{m_t}{m_{t+1}}.
\]

(6.67)

The first order condition for bonds (6.62) is given by:

\[
\frac{\lambda_t}{1 + n_t} = \lambda_{t+1}.
\]

(6.68)

After citing (6.65), the Euler equation for the bond is given by:

\[
\frac{1}{1 + n_t} = \beta \frac{m_t}{m_{t+1}}.
\]

(6.69)

Comparing (6.67) and (6.69), we obtain:

\[
\frac{A}{A + v_t} = \frac{1}{1 + n_t};
\]

which can be simplified to:

\[
v_t = An_t.
\]

(6.71)

6.3.7 Interest rate targeting

Interest rate targeting is a monetary policy such that \( n_t = n \) for a constant \( n \). Under interest rate targeting, the money supply growth rates are constant and given by:

\[
\frac{m_{t+1}}{m_t} = \beta (1 + n).
\]

(6.72)

The velocity is given by \( v_t = An \). The inflation rate is given by

\[
\frac{p_{t+1}}{p_t} = \frac{m_{t+1}v_{t+1}}{m_tv_t} \frac{e_t}{e_{t+1}},
\]

(6.73)

which given the relations \( v_{t+1} = An \) and \( v_t = An \) is equal to:

\[
\frac{p_{t+1}}{p_t} = \beta (1 + n) \frac{e_t}{e_{t+1}};
\]

(6.74)
Defining \( 1 + \pi_{t+1} = \frac{p_{t+1}}{p_t} \), we can obtain the Fisher equation:

\[
1 + n = \frac{1}{\beta} \frac{\epsilon_{t+1}}{\epsilon_t} (1 + \pi_{t+1}),
\]

where the real interest rate is given by: \( 1 + r_t = \frac{1}{\beta} \frac{\epsilon_{t+1}}{\epsilon_t} \).

### 6.3.8 Ricardian equivalence

Under interest rate targeting, the equilibrium velocity, monetary growth rate, and inflation rate have already been determined. From the commodity market clearing condition:

\[
c_t = \frac{\epsilon_t}{1 + n},
\]

The level of government debt and transfers has no impact on the household consumption \( c_t \) and has no impact on the nominal prices \( p_t \). This is an example of Ricardian equivalence. The government debt and transfers must be jointly determined to satisfy the government budget constraint:

\[
M_t + \frac{B_t}{(1 + n_t)} = M_{t-1} + B_{t-1} + T_t.
\]

Changes in transfer policy must be balanced out by changes in the debt level, but these will not impact household consumption and prices. In this model, monetary policy alone determines the household consumption and prices.
Bibliography


Money in the Overlapping Generations Model

7.1 The model without money

7.1.1 Sneak peek

Summary

A powerful model that can be used to analyze money is the overlapping generations (OLG) model, owing to an idea originally from Paul Samuelson. The model presents a demographic structure that is akin to what we experience in the real world in which households live for a finite number of periods and then are replaced by newly born households. This is in contrast to the models introduced thus far in which households are infinite-lived.

In the model introduced in this chapter, households live for two periods. All households born in the same period are homogeneous. Households in the economy differ only by their age in the current period. Since households only live for two periods, there are only two ages of households alive in any period: households that are just born (called young) and households in the final period of life (called old).

A financial asset such as a bond is a contract between two parties that allows each of the parties to transfer resources from one period to the next. In the overlapping generations model, there is a natural limited participation constraint. When considering the financial markets in the current period, there are only two ages of households that can participate: young and old households. Households that are dead or have not yet been born cannot
participate in these markets.

But old households are not going to enter in a financial contract where they receive payouts in the following period. In the following period, they are dead. Likewise, others will not enter into a financial contract with an old household requiring repayment from an old household in the following period. The old household is dead and will not repay anything.

The only households for which the financial markets can be utilized for saving and borrowing are the young households. But in the simple model all young households are identical, meaning that they make identical decisions. This means that financial contracts cannot be traded in the overlapping generations model with households living for two periods and homogeneous cohorts.

This section focuses on the allocational efficiency of competitive markets according to the Pareto criterion. The following section introduces money that may be able to fix an inefficient allocation of resources.

**Notation**

The variables/parameters to be introduced in this section are given in the following table:

- $e_t^\tau$: endowment in $\tau$ for household born in $t$
- $e_y$: endowment for the young
- $e_o$: endowment for the old
- $c_t^\tau$: consumption in $\tau$ for household born in $t$
- $n$: population growth rate
- $N_t$: number of households born in $t$
- $T_t^\tau$: transfer in $\tau$ for household born in $t$
- $a_t$: real bond chosen in period $t$ (with payout in $t + 1$)
- $r_t$: real interest rate in period $t$
- $\lambda_t^\tau$: Lagrange multiplier for BC in $\tau$ (for household born in $t$)

**Main takeaways**

After completing this section, you will be able to answer the following questions:

- Under what conditions is an equilibrium Pareto efficient?
- Under what conditions is an equilibrium Pareto inefficient?
7.1. THE MODEL WITHOUT MONEY

7.1.2 Model setup

The model contains two groups of agents: (i) a government and (ii) households that live for 2 periods.

The economy consists of an infinite number of discrete time periods \( t \in \{0, 1, 2, \ldots \} \). In each period, a single commodity is traded and consumed. In each period \( t \), a cohort of homogeneous households is born. This set of households is referred to as the period \( t \) cohort. These households trade and consume in periods \( t \) and \( t + 1 \). The households no longer exist in period \( t + 2 \). In period \( t \), the cohort \( t \) households are referred to as "young", while in period \( t + 1 \), the cohort \( t \) households are referred to as "old".

The model allows for population growth at rate \( n \geq 0 \). This means that the mass of cohort \( t \) households \( N_t \) is related to the mass of cohort \( t - 1 \) households \( N_{t-1} \) as follows: 
\[
N_t = (1 + n)N_{t-1}.
\]

The model is an endowment economy. Denote \( e^t_\tau \) as the endowment of a cohort \( t \) household in period \( \tau \) and denote \( c^t_\tau \) as the consumption of a cohort \( t \) household in period \( \tau \). The endowments are given by:
\[
\begin{align*}
e^t_0 &= e_y \\
e^t_1 &= e_o
\end{align*}
\] (7.1)
and all other endowments are 0.

As the model begins in period 0, there must be old households in period 0 without a corresponding period of youth. These can be denoted as cohort \(-1\) households. The initial old households have utility \( u(c^{-1}(0)) : \mathbb{R}_+ \to \mathbb{R} \), meaning that their lone objective is to consume as much as they can.

7.1.3 Budget constraints

The cohort \( t \) households (for \( t \geq 0 \)) have the utility function \( u : \mathbb{R}_+^2 \to \mathbb{R} \). Denote \( a^t_\tau \) as the real bonds chosen by the cohort \( t \) households in period \( \tau \), and with payouts in period \( t + 1 \). The real interest rate in period \( t \) is given by \( r_t \). Since the cohort \( t \) households are no longer alive in period \( t + 2 \), they do not choose real bonds in period \( t + 1 \) (since these payouts occur after their death). Denote \( T^t_\tau \) as the transfers that a cohort \( t \) household receives from the government in period \( \tau \). The transfers are specified in the unit of account.

Denote \( p_t \) as the commodity price in period \( t \).
The budget constraints for cohort \( t \) households are as follows:

\[
\begin{align*}
 p_t c_t^t + \frac{p_t a_t}{1 + r_t} & \leq p_t e_y + T_t^t, \\
p_{t+1} c_{t+1}^{t+1} & \leq p_{t+1} e_o + p_{t+1} a_t + T_{t+1}^{t+1}.
\end{align*}
\]  

\( (7.2) \)

### 7.1.4 Market clearing conditions

For the market clearing conditions on the commodity markets, consider that the only households alive in period \( t \) are cohort \( t - 1 \) households and cohort \( t \) households. The market clearing condition, which sets aggregate consumption equal to aggregate endowments, is given by:

\[
N_{t-1} c_t^{t-1} + N_t c_t^t = N_{t-1} e_o + N_t e_y.  
\]  

\( (7.3) \)

We can divide by \( N_{t-1} \) to yield:

\[
c_t^{t-1} + (1 + n) c_t^t = e_o + (1 + n) e_y.  
\]  

\( (7.4) \)

The equilibrium market clearing condition for the real bond is given by:

\[
a_t = 0.  
\]  

\( (7.5) \)

The only households in the real bond market in period \( t \) are the cohort \( t \) households. Older households don’t use the markets as they won’t be alive to collect the payouts. Younger households do not have access to the markets as they have not yet been born. This restricted access to markets is completely natural in this OLG setting, and may lead to a Pareto inefficient equilibrium allocation.

Given the bond market clearing condition, the equilibrium consumption can only be autarchic:

\[
(c_t^t, c_{t+1}^{t+1}) = (e_y, e_o)  
\]  

\( (7.6) \)

for all cohorts \( t \).
7.1. THE MODEL WITHOUT MONEY

7.1.5 Euler equations

The household utility maximization problem is given by:

\[
\begin{align*}
\max_{c_t, c_{t+1}, a_t} & \quad u(c_t, c_{t+1}) \\
\text{subj. to} & \quad e_y + T_t - c_t - \frac{a_t}{1 + r_t} \geq 0 \\
& \quad e_o + a_t + T_{t+1} - c_{t+1} \geq 0
\end{align*}
\]

(7.7)

The budget constraints are always written as nonnegative inequalities.

Consider the Kuhn-Tucker conditions associated with this constrained maximization problem. Denote the Lagrange multiplier for the period \( \tau \) budget constraint for a cohort \( t \) household as \( \lambda_t^f \). The Kuhn-Tucker conditions are given by:

- **First order conditions**:

  - With respect to consumption \( c_t^f \):
    \[
    D_1 u(c_t^f, c_{t+1}^f) - p_t \lambda_t^f = 0.
    \]
    (7.8)

  - With respect to consumption \( c_{t+1}^f \):
    \[
    D_2 u(c_t^f, c_{t+1}^f) - p_{t+1} \lambda_{t+1}^f = 0.
    \]
    (7.9)

  - With respect to real bond holding \( a_t \):
    \[
    - \frac{p_t \lambda_t^f}{1 + r_t} + p_{t+1} \lambda_{t+1}^f = 0.
    \]
    (7.10)

- **Complimentary slackness conditions**:

  - For the first budget constraint
    \[
    \lambda_t^f \left\{ p_t e_y + T_t - p_t c_t^f - \frac{p_t a_t}{1 + r_t} \right\} = 0.
    \]
    (7.11)

  - For the second budget constraint
    \[
    \lambda_{t+1}^f \left\{ p_{t+1} e_o + p_{t+1} a_t + T_{t+1}^d - p_{t+1} c_{t+1}^d \right\} = 0.
    \]
    (7.12)
The convention is that $D_1 u (c_t^t, c_{t+1}^t)$ refers to the marginal utility with respect to the first consumption term $c_t^t$, and $D_2 u (c_t^t, c_{t+1}^t)$ refers to the marginal utility with respect to the second consumption term $c_{t+1}^t$. The Euler equation is obtained by combining the three first order conditions:

$$D_1 u (c_t^t, c_{t+1}^t) = D_2 u (c_t^t, c_{t+1}^t) .$$

(7.13)

Solving the Euler equation (7.13) for the real interest rate yields the real interest rate equation:

$$1 + r_t = \frac{D_1 u (c_t^t, c_{t+1}^t)}{D_2 u (c_t^t, c_{t+1}^t)} .$$

(7.14)

### 7.1.6 Pareto efficiency?

In equilibrium, equation (7.6) dictates that the consumption is autarchic: $(c_t^t, c_{t+1}^t) = (e_y, e_o)$.

The resulting real interest rate from equation (7.14) is then given by:

$$1 + r_t = \frac{D_1 u (e_y, e_o)}{D_2 u (e_y, e_o)} .$$

(7.15)

Is the equilibrium allocation Pareto efficient?

**Theorem 7.1** The allocation is Pareto efficient if $\frac{D_1 u (e_y, e_o)}{D_2 u (e_y, e_o)} \geq 1 + n$. The allocation is Pareto inefficient if $\frac{D_1 u (e_y, e_o)}{D_2 u (e_y, e_o)} < 1 + n$.

To understand this result, consider the parameters such that $\frac{D_1 u (e_y, e_o)}{D_2 u (e_y, e_o)} < 1 + n$. Let’s determine if an alternative allocation actually makes all households better off. Consider a small $\epsilon > 0$ and define $c_{t-1}^t = e_o + \epsilon (1 + n)$ and $c_t^t = e_y - \epsilon$ in all time periods. This new allocation gives a little bit more to the old households, and takes a little bit away from the young households. The initial old households (the -1 cohort) are strictly better off since the new consumption is strictly higher than the old consumption.

Figure 7.1.1 depicts the indifference curves for the cohort $t$ households. As the graph shows, the move from endowment $(e_y, e_o)$ to consumption

$$(c_t^t, c_{t+1}^t) = (e_y - \epsilon, e_o + \epsilon (1 + n))$$

(7.16)

leads to a higher indifference curve. Thus, cohort $t$ households are strictly better off with the new allocation.
A couple of comments about Figure 7.1.1 are in order. The slope of the aggregate resources line is equal to $-(1 + n)$. The line considers how the aggregate resources are split between young and old. The intersection of the aggregate resources line and the x-axis indicates the situation in which the young households consume all of the aggregate resources. The total resources have size $N_t e_y + N_{t-1} e_o$, meaning that each of the $N_t$ young households will consume $e_y + \frac{e_o}{1+n}$. Similarly, the intersection of the aggregate resources line and the y-axis occurs when the old consume all of the resources. The total resources have size $N_t e_y + N_{t-1} e_o$, meaning that each of the $N_{t-1}$ old households will consume $e_y (1+n) + e_o$.

With young consumption on the x-axis and old consumption on the y-axis, the derivative of the indifference curves equals $-\frac{D_1 u(e_y, e_o)}{D_2 u(e_y, e_o)}$ at the equilibrium allocation of autarchy. Thus, for parameters $\frac{D_1 u(e_y, e_o)}{D_2 u(e_y, e_o)} < 1 + n$, the slope of the indifference curve is smaller (in terms of absolute value) than the slope of the aggregate resources line. This means that the indifference curve intersects the aggregate resources line from the left.

### 7.2 The model with money

#### 7.2.1 Sneak peek

**Summary**

With the limited participation constraint implicit in the overlapping generations model, financial contracts are unable to be traded across generations. This prevents intergenerational risk-sharing and (under some conditions) leads to an inefficient allocation of resources.

Into that void steps money in the form of fiat currency, an intrinsically worthless piece of paper. Money is not a contract between two parties, but rather can play the role of a store of value.

The natural role for money in an overlapping generations setting is as a transfer from the young to the old households (think of social security). Young households may wish to hold units of money as savings for the next period in which they are old. But how do they know that money will be worth anything in the following period? If they wish for their money savings to lead to utility gain, then it must be that the households are able to exchange their units of money for units of consumption. The households that accept money in the following period belong to the new generation of young households. Money can have value if the self-fulfilling cycle is satisfied in which all generations believe that money will continue to have value in the future.
We say that money has value when money has purchasing power and is able to be exchanged for units of consumption. We say that money does not have value when no households are willing to accept units of money in exchange for units of consumption. When money does not have value, it is equivalent to an infinite price for commodity. If you want to buy 1 unit of commodity, it will cost you an infinite number of units of money.

**Notation**

The variables to be introduced in this section are given in the following table:

- \( m_t \): amount of money carried from \( t \) to \( t+1 \)
- \( p_t \): commodity price
- \( q_t \): inverse price level, defined as \( q_t = \frac{1}{p_t} \)
- \( M_t \): total money supply in period \( t \)
- \( \tilde{q}_t \): per-capita inverse price level, defined as \( \tilde{q}_t = \frac{q_t}{\bar{N}_t} \)

**Main takeaways**

After completing this section, you will be able to answer the following questions:

- Under what conditions does a monetary equilibrium exist?
- How many monetary equilibria are possible?
- How is the existence of a monetary equilibrium related to the Pareto inefficiency of an equilibrium without money?

### 7.2.2 Introducing money into the model

In situations in which the allocation is Pareto inefficient, can government intervention, specifically the introduction of fiat money, lead to a Pareto improvement? To answer this question, we must introduce money into the model.

Give \( M_0 \) units of currency to the initial old in period \( t = 0 \) (the cohort \(-1\) households). Consumers are allowed to carry money from one period (young age) to the next (old age). Money is a nominal asset, so is measured in the unit of account. Denote the money holdings of the cohort \( t \) households (when young) as \( m_t \). The new updated budget constraints are...
given by:

\[ p_t c_t + \frac{p_t a_t}{1 + r_t} + m_t \leq p_t e_y + T_t^t. \]  
(7.17)

\[ p_{t+1} c_{t+1}^t \leq p_{t+1} e_o + p_{t+1} a_t + m_t + T_{t+1}^t. \]

Money \( m_t \) and the real bond \( a_t \) serve identical purposes; they are redundant financial assets. Thus, the no arbitrage condition states that the returns on both money and the real bond must be identical. The nominal return on the real bond is

\[ \frac{p_{t+1}}{p_t} (1 + r_t). \]

The nominal return on money is 1. The no arbitrage condition is then given by:

\[ \frac{p_{t+1}}{p_t} (1 + r_t) = 1. \]  
(7.18)

### 7.2.3 Inverse price level

Define the inverse price level as \( q_t = \frac{1}{p_t} \). Money has value if \( q_t > 0 \ \forall t \). The following argument shows that if money has value in period \( t \) (i.e., \( q_t > 0 \)), then money will have value in all periods (i.e., \( q_{\tau} > 0 \) in all periods \( \tau \)).

Suppose that \( q_t = 0 \). Then the return on money in period \( t \) is 0, so the cohort \( t - 1 \) consumers are only willing to hold money \( m_{t-1} \) if the price of money in period \( t - 1 \) is 0, as well. This requires \( q_{t-1} = 0 \). Proceeding by induction, then \( q_{\tau} = 0 \) for all \( \tau < t \).

If \( q_t = 0 \), then the price of money in period \( t \) is 0. If the return on money in period \( t + 1 \), \( q_{t+1} \), is anything strictly positive, then the cohort \( t \) consumers are willing to hold an infinite amount of money \( m_t \). Such money holdings would violate the market clearing condition in the money market. Thus \( q_{t+1} = 0 \). Proceeding by induction, \( q_{\tau} = 0 \) for all \( \tau > t \).

### 7.2.4 Government

The government in this model is not permitted to hold debt. The government changes the money supply and funds these changes through taxes/transfers to the old in each period. Given these money supply changes, the commodity price level is determined endogenously.

The government is assumed to only make transfers to the old in each period. That is, the transfers to the young are \( T_t^t = 0 \). Denote the money supply in period \( t \) as \( M_t \). In period \( t \), the assets for the government are the money supply \( M_t \). The liabilities for the government are the previous money supply \( M_{t-1} \) (which is being redeemed by households) plus the transfers to the old households \( T_t^{t-1} N_{t-1} \), where \( N_{t-1} \) is the total mass of cohort \( t - 1 \) households (the
old households in period $t$). The government budget constraint sets the assets equal to the liabilities:

$$M_t = M_{t-1} + T_t^{t-1}N_{t-1}. \quad (7.19)$$

In equilibrium, the money holdings for each cohort $t$ household $m_t$ must be equal to the total money supply $M_t$ divided by the total mass of cohort $t$ households $N_t$:

$$m_t = \frac{M_t}{N_t}. \quad (7.20)$$

### 7.2.5 Equilibrium consumption

Given the equilibrium money equation (7.20) and the definition of the inverse price level, the equilibrium consumption for the young is given by:

$$c_t^t = e_y - \frac{q_t M_t}{N_t}. \quad (7.21)$$

Why is this? The budget constraint from (7.17) is given by:

$$p_t c_t^t = p_t e_y + T_t^t(t) - \frac{p_t a_t}{1 + r_t} - m_t. \quad (7.22)$$

We have already set the transfers $T_t^t = 0$. In equilibrium, the real bond holdings $a_t = 0$. Using the equilibrium money equation (7.20) and the definition $q_t = \frac{1}{p_t}$, then the term $\frac{m_t}{p_t} = \frac{q_t M_t}{N_t}$. This completes the argument used to derive the equilibrium consumption of the young (7.21).

The equilibrium consumption of the old is given by:

$$c_{t+1}^t = e_0 + \frac{q_{t+1} M_{t+1}}{N_t}. \quad (7.23)$$

Why is this? The budget constraint from (7.17) is given by:

$$p_{t+1} c_{t+1}^t = p_{t+1} e_o + p_{t+1} a_t + m_t + T_{t+1}^t. \quad (7.24)$$

In equilibrium, the real bond holdings $a_t = 0$. Using the equilibrium money equation (7.20) and the definition $q_{t+1} = \frac{1}{p_{t+1}}$, then the term $\frac{m_t}{p_{t+1}} = \frac{q_{t+1} M_t}{N_t}$. In equilibrium, the transfers $T_{t+1}^t$ from the government budget constraint (7.19) must satisfy:

$$\frac{T_{t+1}^t}{p_{t+1}} = \frac{M_{t+1}}{p_{t+1} N_t} - \frac{M_t}{p_{t+1} N_t} = \frac{q_{t+1} (M_{t+1} - M_t)}{N_t}. \quad (7.25)$$
7.2. THE MODEL WITH MONEY

Adding together the terms $\frac{m_t}{p_{t+1}}$ and $\frac{T_{t+1}^t}{p_{t+1}}$:

$$m_t \cdot \frac{1}{p_{t+1}} + T_{t+1}^t \cdot \frac{1}{p_{t+1}} = \frac{q_{t+1} M_{t+1}}{N_t}.$$  \hspace{1cm} (7.26)

This completes the argument used to derive the equilibrium consumption of the old (7.23).

7.2.6 Equilibrium equations

Define the per-capita inverse price level as $\tilde{q}_t = \frac{q_t}{N_t}$. We will be finding a stationary equilibria for which $\tilde{q}_{t+1} = \tilde{q}_t$. In words, the value of money grows at the same rate as the population. The definition of stationarity implies

$$\frac{q_{t+1}}{N_{t+1}} = \frac{q_t}{N_t},$$  \hspace{1cm} (7.27)

so $q_{t+1} = (1+n)q_t$ and $p_{t+1} = \frac{p_t}{1+n}$. Using the household budget constraints (7.17), the first order condition with respect to money $m_t$ is:

$$-\lambda_t^t + \lambda_{t+1}^t = 0.$$  \hspace{1cm} (7.28)

Recalling the first order conditions with respect to consumption, we can derive the Euler equation associated with money holdings from equation (7.28):

$$-\frac{D_1 u (c_t^t, c_{t+1}^t)}{p_t} + \frac{D_2 u (c_t^t, c_{t+1}^t)}{p_{t+1}} = 0.$$  \hspace{1cm} (7.29)

Using the definition of $q_t$ and $q_{t+1}$, this can be expressed as:

$$-q_t D_1 u (c_t^t, c_{t+1}^t) + q_{t+1} D_2 u (c_t^t, c_{t+1}^t) = 0.$$  \hspace{1cm} (7.30)

Recall from (7.21) and (7.23) that the equilibrium consumptions are given by:

$$c_t^t = e_y - \frac{q_t M_t}{N_t},$$  \hspace{1cm} (7.31)

$$c_{t+1}^t = e_0 + \frac{q_{t+1} M_{t+1}}{N_t}.$$
Equation (7.30) is the key equation to find stationary equilibria. Let’s express all of the terms \( q_t \) and \( q_{t+1} \) in terms of the adjusted inverse price levels \( \tilde{q}_t \) and \( \tilde{q}_{t+1} \):

\[
-\tilde{q}_t D_1 u (c^t_t, c^t_{t+1}) + (1 + n)\tilde{q}_{t+1} D_2 u (c^t_t, c^t_{t+1}) = 0,
\]

where the consumptions are given by:

\[
c^t_t = e_y - \tilde{q}_t M_t, \\
c^t_{t+1} = e_0 + (1 + n)\tilde{q}_{t+1} M_{t+1}.
\]

Taken together, the key equation to find stationary equilibria, as derived from the household Euler equation, is given by:

\[
F (\tilde{q}_t, \tilde{q}_{t+1}) = -\tilde{q}_t D_1 u (e_y - \tilde{q}_t M_t, e_0 + (1 + n)\tilde{q}_{t+1} M_{t+1}) \\
+(1 + n)\tilde{q}_{t+1} D_2 u (e_y - \tilde{q}_t M_t, e_0 + (1 + n)\tilde{q}_{t+1} M_{t+1}) = 0.
\]

The Euler equation is a function of the two equilibrium prices \((\tilde{q}_t, \tilde{q}_{t+1})\). Define \( f (\tilde{q}_t) = \tilde{q}_{t+1} \) as the implicit function such that \( F (\tilde{q}_t, \tilde{q}_{t+1}) = 0 \).

7.2.7 Properties of a stationary equilibrium

There are two facts that need to be deduced:

Fact 1: \( f(0) = 0 \).

Fact 2: \( Df(0) = \frac{1}{1+n} \frac{D_1 u(e_y,e_o)}{D_2 u(e_y,e_o)} \).

Verification of Fact 1

To see Fact 1, let \( \tilde{q}_t = 0 \). Then the value for \( \tilde{q}_{t+1} \) that solves (7.34) can only be \( \tilde{q}_{t+1} = 0 \).

Verification of Fact 2

Using the Implicit Function Theorem, where we write the consumption vector \((c^t_t, c^t_{t+1})\) simply as \((e_y, e_o)\):

\[
Df(\tilde{q}_t) = \frac{\partial \tilde{q}_{t+1}}{\partial \tilde{q}_t} = -\frac{D_1 F (\tilde{q}_t, \tilde{q}_{t+1})}{D_2 F (\tilde{q}_t, \tilde{q}_{t+1})},
\]

(7.35)
where

\[ D_1 F (\tilde{q}_t, \tilde{q}_{t+1}) = -D_1 u (c_y, c_o) + M_t (\tilde{q}_t D_{11} u (c_y, c_o) - (1 + n) \tilde{q}_{t+1} D_{21} u (c_y, c_o)) \]  \hspace{1cm} (7.36)

and

\[ D_2 F (\tilde{q}_t, \tilde{q}_{t+1}) = (1 + n) \{ D_2 u (c_y, c_o) - M_{t+1} (\tilde{q}_t D_{12} u (c_y, c_o) - (1 + n) \tilde{q}_{t+1} D_{22} u (c_y, c_o)) \}. \]  \hspace{1cm} (7.37)

When we evaluate the derivative at \( \tilde{q}_t = 0 \), we know from Fact 1 that \( \tilde{q}_{t+1} = 0 \). Thus, the second derivative terms are multiplied by 0 and disappear. Thus, we obtain:

\[ Df(0) = \frac{1}{1 + n} \]  \hspace{1cm} (7.38)

where

\[ (c_y, c_o) = (c_y - \tilde{q}_t M_t, e_0 + (1 + n) \tilde{q}_{t+1} M_{t+1}) = (e_y, e_0) \]  \hspace{1cm} (7.39)

when evaluated at \( (\tilde{q}_t, \tilde{q}_{t+1}) = (0, 0) \).

### 7.2.8 How many stationary equilibria?

In the model without money, Pareto efficiency occurs if \( \frac{D_1 u (c_y, c_o)}{D_2 u (c_y, c_o)} \geq 1 + n \) and Pareto inefficiency occurs if \( \frac{D_1 u (c_y, c_o)}{D_2 u (c_y, c_o)} < 1 + n \). Since \( Df(0) = \frac{1}{1 + n} \frac{D_1 u (c_y, c_o)}{D_2 u (c_y, c_o)} \), then Pareto efficiency occurs if \( Df(0) \geq 1 \) and Pareto inefficiency occurs if \( Df(0) < 1 \).

The stationary equilibrium definition requires \( \tilde{q}_{t+1} = \tilde{q}_t \).

Figure 7.2.1 contains two graphs. Both contain the function \( \tilde{q}_{t+1} = f (\tilde{q}_t) \) and the 45-degree line. Along the 45-degree line, the prices must be stationary: \( \tilde{q}_{t+1} = \tilde{q}_t \). The implicit function \( f \) is strictly increasing and strictly convex.

When the function \( f (\tilde{q}_t) \) intersects the 45-degree line, then (i) \( (\tilde{q}_t, \tilde{q}_{t+1}) \) solves the Euler equation (7.34) and (ii) the per-capita values of money are stationary, \( \tilde{q}_{t+1} = \tilde{q}_t \).

The graph on the left shows that if \( Df(0) \geq 1 \), then there is only one fixed point with \( \tilde{q}_{t+1} = \tilde{q}_t \). At this fixed point, \( \tilde{q}_{t+1} = \tilde{q}_t = 0 \), so this is a nonmonetary equilibrium and money is not valued.

The graph on the right shows that if \( Df(0) < 1 \), then there are two fixed points with \( \tilde{q}_{t+1} = \tilde{q}_t \). The first fixed point at \( \tilde{q}_{t+1} = \tilde{q}_t = 0 \) is a nonmonetary equilibrium. The second fixed point at \( \tilde{q}_{t+1} = \tilde{q}_t > 0 \) is a monetary equilibrium, since money is valued. The non-
monetary equilibrium is stable and the monetary equilibrium is not. How do we see this? Suppose that the inverse price level $\tilde{q}_t$ is slightly to the left of the monetary equilibrium (specifically, the corresponding intersection of $f(\tilde{q}_t)$ and the 45-degree line). The function $f(\tilde{q}_t)$ dictates that the next period inverse price level $\tilde{q}_{t+1} < \tilde{q}_t$. Using the function $f$ again, the inverse price level two periods from now is such that $\tilde{q}_{t+2} = f(\tilde{q}_{t+1}) < \tilde{q}_{t+1}$. Continuing by induction, the inverse price levels approach the nonmonetary equilibrium $\tilde{q}_{t+1} = \tilde{q}_t = 0$. 
Bibliography

Part IV

Real Business Cycle Theory
8

Real Business Cycle Model

8.1 Stochastic growth model

8.1.1 Sneak peek

Summary

Recall in the chapter on the neoclassical growth model that the model was deterministic. This means that the economy did not experience any shocks and the households and firms knew with certainty what prices would arise in future periods. In a deterministic setting, the equilibrium sequence of capital stocks and the equilibrium sequence of outputs were monotonic and converged to a steady state. At the steady state, the economic variables are constant.

The task of the real business cycle model is to capture business cycle dynamics, namely fluctuations in output. To capture such fluctuations, there must be some shocks to the economic system. A stochastic model in economics is one with uncertainty (or risk, the two terms will be used interchangeably in this text) in which the economy is hit by a shock each period. The shock can take many economic forms, where the most common include shocks to the productivity of firms, shocks from fiscal policy (tax rate changes), shocks from monetary policy (interest rate changes), and shocks to the preferences of households. The main feature of the real business cycle theory is that only real shocks, and not nominal shocks, can lead to business cycle fluctuations. Real shocks include productivity shocks, fiscal policy shocks, and preference shocks. Nominal shocks include monetary policy shocks. In real business cycle models, agents interact in perfectly competitive markets in which the
prices instantaneously adjust (to a shock) to satisfy the market clearing conditions. This section considers a productivity shock and a later section in this chapter considers a tax shock.

Mathematically, a shock is a random variable whose value changes according to an exogenous random process. Households and firms know which process the random variable follows, but they do not know the value of the shock (the random variable draw) that will occur in the future.

Since the households and firms know the random process for the shocks, they can form expectations about the market prices in the future. Namely, they assign probabilities to each possible shock that can occur in the future. In this manner, households and firms are perfectly rational in that they consider all possible future outcomes when making their current period economic decisions, and the model can still generate business cycle fluctuations.

The analysis in this chapter will consider extensions of the neoclassical growth model to a stochastic setting. To solve such problems, the tools of dynamic programming are applied. The first extension is the simplest one as it assumes that households supply labor inelastically (the labor supply is fixed).

Notation

The variables/parameters to be introduced in this section are given in the following table:

\[
\begin{align*}
 k & \quad \text{capital stock in the current period} \\
 n & \quad \text{labor supply of household} \\
 z & \quad \text{productivity shock in the production function} \\
 Z & \quad \text{set of all possible shocks} \\
 \pi(z, z') & \quad \text{probability shock } z' \text{ occurs in } t + 1, \text{ given shock } z \text{ occurs in } t \\
 V : \mathbb{R}_+ \times Z & \rightarrow \mathbb{R} \quad \text{value function } V(k, z) \\
 g : \mathbb{R}_+ \times Z & \rightarrow \mathbb{R}_+ \quad \text{policy function } k' = g(k, z) \\
 c : \mathbb{R}_+ \times Z & \rightarrow \mathbb{R}_+ \quad \text{consumption function } c(k, z) \\
 \theta & \quad \text{capital share (from the production function)} \\
 m & \quad \text{coefficient in the guess of the policy function}
\end{align*}
\]

Main takeaways

After completing this section, you will be able to answer the following questions:
8.1. STOCHASTIC GROWTH MODEL

- Concerning the role of uncertainty, what role does the size of shocks have on the equilibrium policy and consumption functions?

- Concerning the role of uncertainty, what role does the persistence, or probability that the same shock is realized two periods in a row, have on the equilibrium policy and consumption functions?

8.1.2 Model basics

The model contains an infinite number of discrete time periods \( t \in \{0, 1, 2, \ldots\} \). In each period, one of a possible \( S \) shocks can occur in each time period. Different histories of shocks lead to different household choices and equilibrium prices.

A single commodity is produced and consumed in the economy. Households and firms are homogeneous.

There are two factors of production, capital and labor, and a random variable also affects the output of the firm. The production function is given by

\[
f(K_t, N_t, z_t),
\]

(8.1)

where \( K_t \) is the capital input, \( N_t \) is the labor input, and \( z_t \) is the random variable that represents the productivity shock. As a random variable, \( z_t \) is not chosen by any agent in the economy.

The shocks are \( z_t \in Z = \{z(1), \ldots, z(S)\} \). I assume that \( z(1) < z(2) < \ldots < z(S) \) and that the production function \( f \) is strictly increasing in \( z_t \).

The parameter \( \pi(z, z') \) is the probability that shock \( z' \) occurs in period \( t \) given that shock \( z \) occurred in period \( t - 1 \). It is the probability of moving from shock \( z \) to shock \( z' \). Probabilities of this form are called Markov transition probabilities. The probabilities for the shock realization in period \( t \) only depend upon the shock in period \( t - 1 \).

There are a total of \( S^2 \) parameters \( \pi(z, z') \), one for each combination of consecutive shocks. The definition of probability requires that

\[
\sum_{z' \in Z} \pi(z, z') = 1 \quad \text{for all } z.
\]

(8.2)
8. REAL BUSINESS CYCLE MODEL

8.1.3 Planner’s problem

For the initial model the welfare theorems will hold. Tasked with finding the equilibrium solution, I take the efficient route by solving for the solution to the planner’s problem. The solution to the planner’s problem, a Pareto efficient allocation, is identical to the equilibrium solution when the welfare theorems hold.

The household choice of the capital stock in period \( t \) is \( k_t \) and the labor supply in period \( t \) is \( n_t \). The rate of depreciation for capital is \( \delta = 1 \). For now, labor is inelastically supplied by the household, meaning that the labor supply is fixed at \( n_t = 1 \).

The planner’s problem is given by:

\[
\begin{align*}
\text{maximize} & \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \\
\text{subject to} & \quad c_t + k_{t+1} = f(k_t, 1, z_t) \text{ in all periods} \quad , \\
& \quad k_0 \text{ given}
\end{align*}
\]  

where the expectation \( \mathbb{E}_0 \) is over the random process determining the shocks \( z_t \). Given the shock \( z_0 \) in the initial period, the planner knows the probabilities for all future shock realizations and derives the expected utility value using these expectations.

8.1.4 Bellman equation

We can capture the idea of the shocks and probabilities without the clumsy expectation notation \( \mathbb{E}_0 \) by using our skills in dynamic programming to write the planner’s problem in recursive form. The Bellman equation is given by:

\[
\begin{align*}
V(k, z) = \max_{c \geq 0, k' \geq 0} & \quad u(c) + \beta \sum_{z' \in Z} \pi(z, z') V(k', z') \\
\text{subject to} & \quad c + k' = f(k, 1, z)
\end{align*}
\]  

State variables must be sufficient to allow the planner to solve its problem. In this setting, there are two state variables: \( k \) and \( z \). Both state variables enter into the aggregate resource constraint and determines how much is consumed and how much is invested.

The objective function of the Bellman equation includes the expected next-period value, summed over all possible shocks that can occur next period. The shock \( z' \) is realized with probability \( \pi(z, z') \), given that the current shock is \( z \).
8.1. **STOCHASTIC GROWTH MODEL**

Define the policy rule as a function of the two state variables:

\[ k' = g(k, z). \]  
(8.5)

Define the consumption function in terms of those two state variables:

\[ c = c(k, z). \]  
(8.6)

The policy function \( g \) can be shown to be a strictly increasing function in both variables \( k \) and \( z \).

**8.1.5 Equilibrium conditions**

In equilibrium, market clearing conditions hold for both the capital market and the labor market: \( K = k \) and \( N = n = 1 \).

In the Bellman equation, insert the aggregate resource constraint into the objective function in order to obtain a problem only in terms of \( k' \):

\[ V(k, z) = \max_{k' \geq 0} \ u(f(k, 1, z) - k') + \beta \sum_{z' \in Z} \pi(z, z') V(k', z'). \]  
(8.7)

The first order condition with respect to \( k' \) is:

\[ -Du(f(k, 1, z) - k') + \beta \sum_{z' \in Z} \pi(z, z') D_1 V(k', z') = 0. \]  
(8.8)

Recall the convention that \( D_1 V(k', z') \) is the partial derivative of the function \( V : \mathbb{R}^2_+ \to \mathbb{R} \) with respect to the first element \( k' \).

The Envelope Theorem yields

\[ D_1 V(k, z) = Du(f(k, 1, z) - k') \cdot D_1 f(k, 1, z). \]  
(8.9)

Combining these derivatives yields the Euler equation:

\[ Du(f(k, 1, z) - k') = \beta \sum_{z' \in Z} \pi(z, z') Du(f(k', 1, z') - k'') \cdot D_1 f(k', 1, z'). \]  
(8.10)

Using the notation for the consumption functions \( c(k, z) = f(k, 1, z) - k' \) and \( c(k', z') = \)
8. REAL BUSINESS CYCLE MODEL

\[ f(k', 1, z') - k'' \], the Euler equation is equivalently expressed as:

\[
Du (c(k, z)) = \beta \sum_{z' \in Z} \pi (z, z') Du (c(k', z')) \cdot D_1 f(k', 1, z').
\]  

(8.11)

### 8.1.6 Solving the Bellman equation for a simple economy

The economy contains Cobb-Douglas production function and natural log utility. Assume that \( u(c) = \ln(c) \) and \( f(K, N, z) = z K^\theta N^{1-\theta} \) for some \( \theta \in (0, 1) \).

The Euler equation (8.11) for this economy is given by:

\[
\frac{1}{c(k, z)} = \beta \sum_{z' \in Z} \pi (z, z') \frac{\theta z'(k')^{\theta-1}}{c(k', z')}. \]  

(8.12)

The Bellman equation is given by (8.7). To solve the Bellman equation, we apply the guess-and-check the policy function algorithm. Recall from the 'Neoclassical Growth Model’ chapter that the steps are given by:

1. Find the Euler equation.

2. Express the consumption terms in the Euler equation only in terms of the state variables \( k \) and \( z \).

3. Guess the functional form \( g(k) = mz k^\theta n^{1-\theta} \) and use this guess in the Euler equation.

   Notice that the guess for the functional form expresses the policy function as a linear function of firm output.

4. Solve for the unknown coefficient \( m \).

**Step 1**

The Euler equation was previously found in (8.12).

**Step 2**

From the aggregate resource constraint (with inelastic labor supply \( n = n' = 1 \))

\[
c(k, z) = f(k, 1, z) - k' = z k^\theta - g(k, z)
\]  

(8.13)
and
\[ c(k', z') = z'(k')^\theta - g(k', z'). \] (8.14)

Inserting this into the Euler equation (8.12) yields:
\[ \frac{1}{zk^\theta - g(k, z)} = \beta \sum_{z' \in Z} \pi(z, z') \frac{\theta z'(k')^{\theta-1}}{z'(k')^\theta - g(k', z')}. \] (8.15)

**Step 3**

Guess the functional form \( g(k, z) = mz k^\theta \). Since this holds for all state variables, then \( g(k', z') = mz'(k')^\theta \). Substitute this functional form into the Euler equation (8.15):
\[ \frac{1}{(1 - m) zk^\theta} = \beta \sum_{z' \in Z} \pi(z, z') \frac{\theta z'(k')^{\theta-1}}{z'(k')^\theta - mz'(k')^\theta}. \] (8.16)

**Step 4**

In each term in the summation in the right-hand side of (8.16), we can factor out the common term \( z'(k')^{\theta-1} \). This yields:
\[ \frac{1}{(1 - m) zk^\theta} = \beta \sum_{z' \in Z} \pi(z, z') \frac{\theta}{(1 - m) k'}. \] (8.17)

The right-hand side term \( \frac{\theta}{(1 - m) k'} \) is independent of the shock \( z' \) and can be brought outside of the summation. This means that the Euler equation is expressed as:
\[ \frac{1}{(1 - m) zk^\theta} = \frac{\theta \beta}{(1 - m) k' \sum_{z' \in Z} \pi(z, z')} \sum_{z' \in Z} \pi(z, z'). \] (8.18)

By the definition of probabilities \( \sum_{z' \in Z} \pi(z, z') = 1 \). We are left with the equality:
\[ \frac{1}{(1 - m) zk^\theta} = \frac{\theta \beta}{(1 - m) k'}. \] (8.19)

Use the policy function guess again, namely \( k' = mz k^\theta \):
\[ \frac{1}{(1 - m) zk^\theta} = \frac{\theta \beta}{(1 - m) mz k^\theta}. \] (8.20)
The term \((1 - m) z k^\theta\) cancels. Solving for \(m\) yields:

\[
m = \theta \beta. \tag{8.21}
\]

This means that the policy function is given by:

\[
g(k, z) = \theta \beta z k^\theta. \tag{8.22}
\]

The consumption function is

\[
c(k, z) = z k^\theta - \theta \beta z k^\theta = (1 - \theta \beta) z k^\theta. \tag{8.23}
\]

The effects of the shocks propagate. Not only does a high shock value \(z_t\) increase the production in period \(t\), but it increases the capital stock \(g(k, z) = \theta \beta z k^\theta\) carried into period \(t + 1\). This generates a persistence of business cycles that is stronger than the underlying probabilities.

8.2 Stochastic growth with endogenous labor

8.2.1 Sneak peek

Summary

To capture business cycle fluctuations, we require more than just a shock to firm productivity and its effects on the capital stock and output. There is another important dimension in an economic model that should be able to respond to a shock. That margin is the labor supply decision by the households. This section extends the model from the previous section by considering labor supplied elastically by the households. To determine how much labor households choose to supply, there must be a trade-off between the costs and benefits of labor. This is referred to as the labor-leisure trade-off. The benefit of labor is an increase in labor income and the cost of labor must be a reduction in utility. Household preferences are such that households experience a disutility from labor.

The same stochastic process as in the previous section is used for the shocks to firm productivity. The canonical real business cycle model consists of these elements: shock, household capital investment choice, and household labor supply choice. Additional features can be added to the model in order to capture economic mechanisms of interest. With the
features mentioned above, the welfare theorems hold, meaning that the equilibrium solution can be found by solving the planner’s problem and citing the Second Basic Welfare Theorem.

Notation

The variables to be introduced in this section are given in the following table:

\[ n : \mathbb{R}_+ \times Z \rightarrow [0, 1] \quad \text{labour supply function, } n(k, z) \]
\[ \gamma \quad \text{disutility from labor} \]

Main takeaways

After completing this section, you will be able to answer the following questions:

- Does the endogenous labor supply choice change the equilibrium policy and consumption functions?
- How does the endogenous labor supply choice change the size or frequency of business cycles?

8.2.2 Model setup

The textbook example of a real business cycle model incorporates an endogeneous labor choice in the stochastic growth model. Given the endogenous labor supply choice of the household, we must introduce a labor-leisure trade-off. The utility function is given by \( u(c_t, n_t) \), where the utility function is strictly increasing in consumption and strictly decreasing in labor (households do not enjoy labor).

Since the welfare theorems are applicable, the solution to the planner’s problem is equivalent to the equilibrium solution. The planner’s problem is given by:

\[
\begin{align*}
\underset{\{c_t, k_{t+1}, n_t\}_{t \in \mathbb{N}}}{\text{maximize}} & \quad E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, n_t) \\
\text{subject to} & \quad c_t + k_{t+1} = f(k_t, n_t, z_t) \quad \text{in all periods} \\
& \quad k_0 \text{ given}
\end{align*}
\]

where the expectation \( E_0 \) is over the random process determining the shocks \( z_t \).
8. REAL BUSINESS CYCLE MODEL

8.2.3 Bellman equation

Using dynamic programming, this can be written in a recursive form. The Bellman equation is given by:

\[ V(k, z) = \max_{c \geq 0, k' \geq 0, n \geq 0} \left[ u(c, n) + \beta \sum_{z' \in Z} \pi(z, z') V(k', z') \right] \]

subject to \( c + k' = f(k, n, z) \)

Define the policy rule as:

\[ k' = g(k, z). \]

The consumption function is \( c(k, z) \) and the labor supply function is \( n(k, z) \).

We explicitly allow the labor supply to change with the state variables \( k \) and \( z \). For simplicity, denote \( n = n(k, z) \) and \( n' = n(k', z') \).

8.2.4 Equilibrium conditions

In equilibrium, market clearing conditions hold for both the capital market and the labor market: \( K = k \) and \( N = n \).

The Euler equation is given by (using the exact same methods as above):

\[ D_1 u(c(k, z), n) = \beta \sum_{z' \in Z} \pi(z, z') D_1 u(c(k', z'), n') \cdot D_1 f(k', n', z'). \]

Recall the convention that \( D_1 u(c, n) \) is the partial derivative of the utility function with respect to the first element \( c \). An additional convention is that the current period variables do not have primes, while the variables in the next period are primed.

The additional first order condition is the first order condition with respect to the labor choice:

\[ D_1 u(c, n) \cdot D_2 f(k, n, z) + D_2 u(c, n) = 0. \]

This first-order condition captures the labor-leisure trade-off. The first term is the benefit from labor (an increase in utility due to an increase in production) and the second term is the cost from labor (\( D_2 u(c, n) \) is negative as labor decreases utility).

8.2.5 Solving the Bellman equation for a simple economy

We want to consider a simple economy, so we assume \( u(c, n) = \ln(c) - \gamma n \) for some parameter \( \gamma > 0 \) and \( f(K, N, z) = z K^\theta N^{1-\theta} \) for some \( \theta \in (0, 1) \).
8.2. STOCHASTIC GROWTH WITH ENDOGENOUS LABOR

The Euler equation (8.27) for this economy is given by:

\[
\frac{1}{c(k, z)} = \beta \sum_{z' \in Z} \pi(z, z') \frac{\theta z'(k')^{\theta-1} (n')^{1-\theta}}{c(k', z')}.
\] (8.29)

The Bellman equation is given by (8.25). To solve the Bellman equation, we apply the guess-and-check the policy function algorithm. The steps are given by:

1. Find the Euler equation.
2. Express the consumption terms in the Euler equation only in terms of the state variables \( k \) and \( z \).
3. Guess the functional form \( g(k, z) = mz k^\theta n^{1-\theta} \) and use this guess in the Euler equation. The guess states that investment is a linear function of firm output.
4. Solve for the unknown coefficient \( m \).
5. Take the first order condition with respect to the labor choice \( n \) and solve for the labor supply.

**Step 1**

The Euler equation was previously found in (8.29).

**Step 2**

From the aggregate resource constraint

\[
c(k, z) = zk^\theta n^{1-\theta} - g(k, z)
\] (8.30)

and

\[
c(k', z') = z'(k')^{\theta} (n')^{1-\theta} - g(k', z').
\] (8.31)

Inserting the aggregate resource constraints into the Euler equation (8.29) yields:

\[
\frac{1}{zk^\theta n^{1-\theta} - g(k, z)} = \beta \sum_{z' \in Z} \pi(z, z') \frac{\theta z'(k')^{\theta-1} (n')^{1-\theta}}{z'(k')^{\theta} (n')^{1-\theta} - g(k', z')}.
\] (8.32)
8. REAL BUSINESS CYCLE MODEL

Step 3

Guess the functional form \( g(k, z) = mz \theta n^{1-\theta} \). Since the guess holds for all state variables, then \( g(k', z') = mz'(k')^{\theta} (n')^{1-\theta} \). Substitute this functional form into the Euler equation (8.32):

\[
\frac{1}{(1 - m) z k \theta n^{1-\theta}} = \beta \sum_{z' \in Z} \pi(z, z') \frac{\theta z'(k')^{\theta-1} (n')^{1-\theta}}{z'(k')^{\theta} (n')^{1-\theta} - mz'(k')^{\theta} (n')^{1-\theta}}. \tag{8.33}
\]

Step 4

In each term in the summation in the right-hand side of (8.33), we can factor out the common term \( z'(k')^{\theta-1} (n')^{1-\theta} \). This yields:

\[
\frac{1}{(1 - m) z k \theta n^{1-\theta}} = \beta \sum_{z' \in Z} \pi(z, z') \frac{\theta}{(1 - m) k'}. \tag{8.34}
\]

The right-hand side term \( \frac{\theta}{(1 - m) k'} \) is independent of the shock \( z' \) and can be brought outside of the summation. This means that the Euler equation is expressed as:

\[
\frac{1}{(1 - m) z k \theta n^{1-\theta}} = \frac{\theta \beta}{(1 - m) k'}. \tag{8.35}
\]

By the definition of probabilities \( \sum_{z' \in Z} \pi(z, z') = 1 \). We are left with the equality:

\[
\frac{1}{(1 - m) z k \theta n^{1-\theta}} = \frac{\theta \beta}{(1 - m) k'}. \tag{8.36}
\]

Use the guess for the policy function \( k' = mz \theta n^{1-\theta} \) again:

\[
\frac{1}{(1 - m) z k \theta n^{1-\theta}} = \frac{\theta \beta}{(1 - m) mz \theta n^{1-\theta}}. \tag{8.37}
\]

Solving for \( m \) yields:

\[
m = \theta \beta. \tag{8.38}
\]

This is the same value for \( m \) as found in the model with inelastic labor supply. The investment-to-output ratio is equal in both models \( (m = \theta \beta) \).
This means that the policy function is given by:

\[ g(k, z) = \theta \beta z k^\theta n^{1-\theta}. \]  

The consumption function is

\[ c(k, z) = z k^\theta n^{1-\theta} - \theta \beta z k^\theta n^{1-\theta} = (1 - \theta \beta) z k^\theta n^{1-\theta}. \]  

**Step 5**

What about the labor supply? The first order condition for labor (8.28) is given by:

\[ \frac{(1 - \theta) z k^\theta n^{-\theta}}{c(k, z)} - \gamma = 0. \]  

Given that the consumption function is

\[ c(k, z) = (1 - \theta \beta) z k^\theta n^{1-\theta}, \]  

then the first order condition for labor is simplified to:

\[ \frac{(1 - \theta) z k^\theta n^{-\theta}}{(1 - \theta \beta) z k^\theta n^{1-\theta}} = \gamma. \]  

Solving for the labor supply:

\[ n = \frac{(1 - \theta)}{\gamma (1 - \theta \beta)}. \]  

The labor supply is independent of the shocks. In this model, the addition of an elastic labor supply does not add to the persistence or size of the business cycles.

There are two effects to analyze when considering the labor decision of a household: the income effect and the substitution effect. Following a good productivity shock to the economy, households have higher income and this leads them to value leisure more and labor less. They have an incentive to reduce their labor supply. This is the income effect. For the same good productivity shock, firms are more productive and the wage rate (marginal product of labor) is higher. This encourages households to work more. This is the substitution effect. For the given economy, the income effect and the substitution effect have the same magnitude, but in opposite directions. The two effects cancel out and the labor supply is independent of firm productivity.
8.3 Stochastic growth with taxes

8.3.1 Sneak peek

Summary

A productivity shock is the essence of a real shock, as it is both real (affecting the output of the firm) and a shock (in that it is not chosen by the firms or the government). The previous sections analyzed how households respond to a productivity shock. This section extends the real business cycle model by including an investment tax shock. I use the term 'shock' because the value of the investment tax is not chosen by the households. The households may have some expectation about how the government will set the investment tax, but a change in the tax rate is a shock from the household’s perspective.

The government, another agent in the model, chooses the investment tax rate. This is a policy decision for the government (not to be confused with the optimal investment rule chosen by the households, labeled $g(k, z)$, which is called the policy function). The government may choose the tax rate in response to the productivity shock. With changes in both the productivity shock and the tax rate, households must respond to two shocks.

By adding the investment tax shock, the model can capture (i) how households respond to tax shocks and (ii) how fiscal policy affects the business cycle.

With taxation of any form, the welfare theorems are no longer applicable. Solving for the solution to the planner’s problem gets us no closer to our goal of finding the equilibrium solution as the planner’s solution and the equilibrium solution are NOT equivalent under taxation.
8.3. STOCHASTIC GROWTH WITH TAXES

Notation

The variables/parameters to be introduced in this section are given in the following table:

- \( y \) a combined productivity and tax shock \( y = (z, \tau) \)
- \( Y \) the set of all possible combinations of shocks
- \( R \) rate of return on capital
- \( w \) wage rate
- \( V : \mathbb{R}_+ \times Y \rightarrow \mathbb{R} \) value function, \( V(k, y) \)
- \( g : \mathbb{R}_+ \times Y \rightarrow \mathbb{R}_+ \) policy function, \( k' = g(k, y) \)
- \( c : \mathbb{R}_+ \times Y \rightarrow \mathbb{R}_+ \) consumption function, \( c(k, y) \)
- \( n : \mathbb{R}_+ \times Y \rightarrow [0, 1] \) labor supply function, \( n(k, y) \)
- \( \tau \) investment tax

Main takeaways

After completing this section, you will be able to answer the following questions:

- Does the addition of a stochastic investment tax change the equilibrium policy and consumption functions?
- Can changes in the tax rate contribute to the business cycle fluctuations?

8.3.2 Model setup

This section considers the addition of a stochastic investment tax to the model previously considered. With the investment tax, the welfare theorems will no longer hold. To solve for an equilibrium, we cannot solve for the planner’s problem, but rather need to solve for an equilibrium. The main problem in an equilibrium is the household problem:

\[
\begin{align*}
\text{maximize} & \quad E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, n_t) \\
\text{subject to} & \quad c_t + (1 + \tau_t) k_{t+1} = R_t k_t + w_t n_t + T_t \quad \text{in all time periods} \\
& \quad k_0 \text{ given}
\end{align*}
\]

(8.45)

where the expectation \( E_0 \) is over the random process determining the aggregate shocks \( z_t \). The tax rate can change over time and is denoted \( \tau_t \) in period \( t \). The transfers \( T_t \) are lump-
sum subsidies that households receive from the government. The rate of return on capital in period \( t \) is \( R_t \) and the wage rate in period \( t \) is \( w_t \).

The model contains both fiscal policy shocks (in the form of investment tax shocks) and firm productivity shocks. The shocks are \( y_t = (z_t, \tau_t) \in Y = \{y(1), ..., y(S)\} \), where \( Y \) is the set of all possible combinations of shocks.

### 8.3.3 Bellman equation

Using dynamic programming, this can be written in a recursive form. The Bellman equation is given by:

\[
V(k, y) = \max_{c \geq 0, k' \geq 0, n \geq 0} u(c, n) + \beta \sum_{y' \in Y} \pi(y, y') V(k', y').
\]  

subject to \( c + (1 + \tau) k' = Rk + wn + T \)  

(8.46)

Define the policy rule as:

\[
k' = g(k, y).
\]  

(8.47)

The consumption function is \( c(k, y) \) and the labor supply function is \( n(k, y) \). We simply denote \( n = n(k, y) \) and \( n' = n(k', y') \).

### 8.3.4 Equilibrium conditions

In equilibrium, market clearing conditions hold for both the capital market and the labor market: \( K = k \) and \( N = n \). In addition, the government must balance its budget: \( T = \tau k' \).

The Euler equation is given by:

\[
(1 + \tau) D_1 u(c(k, y), n) = \beta \sum_{y' \in Y} \pi(y, y') D_1 u(c(k', y'), n') \cdot R'.
\]  

(8.48)

The additional first order condition is the first order condition with respect to the labor choice:

\[
D_1 u(c, n) w + D_2 u(c, n) = 0.
\]  

(8.49)

### 8.3.5 Solving the Bellman equation for a simple economy

We want to consider a simple economy, so we assume \( u(c, n) = \ln(c) - \gamma n \) for some parameter \( \gamma > 0 \) and \( f(K, N, z) = z K^\theta N^{1-\theta} \) for some \( \theta \in (0, 1) \).
The Euler equation (8.48) for this economy is given by:

\[
\frac{1 + \tau}{c(k, y)} = \beta \sum_{y' \in Y} \pi(y, y') \frac{R'(y', y)}{c(k', y'}).
\tag{8.50}
\]

The Bellman equation is given by (8.46). To solve the Bellman equation, we apply the guess-and-check the policy function algorithm. The steps are given by:

1. Find the Euler equation.

2. Express the consumption terms in the Euler equation only in terms of the state variables \(k\) and \(y\).

3. Guess the functional form \(g(k, y) = mz k^\theta n^{1-\theta}\) and use this guess in the Euler equation. This guess states that capital investment is a constant fraction of output.

4. Solve for the unknown coefficient \(m\).

5. Take the first order condition with respect to the labor choice \(n\) and solve for the labor supply.

**Step 1**

The Euler equation was previously found in (8.50).

**Step 2**

From the first order conditions of the firm profit maximization problem:

\[
R = \theta z k^\theta n^{1-\theta}.
\tag{8.51}
\]
\[
w = (1 - \theta) z k^\theta n^{-\theta}.
\]

From the household budget constraint

\[
c(k, y) = Rk + wn + T - (1 + \tau) g(k, y).
\tag{8.52}
\]

In equilibrium, \(T = \tau k'\) and \(Rk + wn = z k^\theta n^{1-\theta}\), meaning that the consumption is given by:

\[
c(k, y) = z k^\theta n^{1-\theta} - g(k, y).
\tag{8.53}
\]
Similarly,
\[ c(k', y') = z'(k')^\theta (n')^{1-\theta} - g(k, y'). \]
Inserting the expression for \( R^* = \theta z'(k')^{\theta-1} (n')^{1-\theta} \) and the aggregate resource constraints into the Euler equation (8.50) yields:
\[
\frac{1 + \tau}{zk^\theta n^{1-\theta} - g(k, y)} = \beta \sum_{y' \in Y} \pi(y, y') \frac{\theta z'(k')^{\theta-1} (n')^{1-\theta}}{z'(k')^\theta (n')^{1-\theta} - m z'(k')^\theta (n')^{1-\theta}}.
\] (8.55)

**Step 3**

Guess the functional form \( g(k, y) = mz k^\theta n^{1-\theta} \). Substitute this functional form into the Euler equation (8.55):
\[
\frac{1 + \tau}{zk^\theta n^{1-\theta} - g(k, y)} = \beta \sum_{y' \in Y} \pi(y, y') \frac{\theta z'(k')^{\theta-1} (n')^{1-\theta}}{z'(k')^\theta (n')^{1-\theta} - mz'(k')^\theta (n')^{1-\theta}}.
\] (8.56)

**Step 4**

In each term in the summation in the right-hand side of (8.56), we can factor out the common term \( z'(k')^{\theta-1} (n')^{1-\theta} \). This yields:
\[
\frac{1 + \tau}{(1 - m)zk^\theta n^{1-\theta}} = \beta \sum_{y' \in Y} \pi(y, y') \frac{\theta}{(1 - m)k'}.
\] (8.57)

The right-hand side term \( \frac{\theta}{(1 - m)k'} \) is independent of the shock \( y' \) and can be brought outside of the summation. This means that the Euler equation is expressed as:
\[
\frac{1 + \tau}{(1 - m)zk^\theta n^{1-\theta}} = \frac{\theta \beta}{(1 - m)k'} \sum_{y' \in Y} \pi(y, y').
\] (8.58)

By the definition of probabilities \( \sum_{y' \in Y} \pi(y, y') = 1 \). We are left with the equality:
\[
\frac{1 + \tau}{(1 - m)zk^\theta n^{1-\theta}} = \frac{\theta \beta}{(1 - m)k'}.
\] (8.59)
Use the guess for the policy function $k' = mz^\theta n^{1-\theta}$ again:

$$\frac{1 + \tau}{(1 - m)zk^\theta n^{1-\theta}} = \frac{\theta \beta}{(1 - m)zk^\theta n^{1-\theta}}.$$  \hspace{1cm} (8.60)

Solving for $m$ yields:

$$m = \frac{\theta \beta}{1 + \tau}.$$  \hspace{1cm} (8.61)

This means that the policy function is given by:

$$g(k, y) = \frac{\theta \beta}{1 + \tau}zk^\theta n^{1-\theta}.$$  \hspace{1cm} (8.62)

The consumption function is

$$c(k, y) = zk^\theta n^{1-\theta} - \frac{\theta \beta}{1 + \tau}zk^\theta n^{1-\theta} = \left(1 - \frac{\theta \beta}{1 + \tau}\right)zk^\theta n^{1-\theta}.$$  \hspace{1cm} (8.63)

Recall that $y = (z, \tau)$, which is the vector of shocks.

**Step 5**

What about the labor supply? The first order condition for labor (8.49) is given by:

$$\frac{w}{c(k, y)} - \gamma = 0.$$  \hspace{1cm} (8.64)

Given that the consumption function is

$$c(k, y) = \left(1 - \frac{\theta \beta}{1 + \tau}\right)zk^\theta n^{1-\theta},$$  \hspace{1cm} (8.65)

and the wage rate is

$$w = (1 - \theta)zk^\theta n^{-\theta},$$

then the first order condition for labor is simplified to:

$$\frac{(1 - \theta)zk^\theta n^{-\theta}}{(1 - \frac{\theta \beta}{1 + \tau})zk^\theta n^{1-\theta}} = \gamma.$$  \hspace{1cm} (8.66)

Solving for the labor supply:

$$n(\tau) = \frac{(1 - \theta)}{\gamma (1 - \frac{\theta \beta}{1 + \tau})}.$$  \hspace{1cm} (8.67)
The labor supply is independent of the capital stock $k$ and the productivity shock $z$, but is not independent of the tax shock $\tau$. If the tax rate is high, the consumption $c(k, y)$ is high and labor supply $n(\tau)$ is low. A high investment tax means households work less in equilibrium.

Let’s consider the first order condition for labor (8.49) more closely. The marginal benefit of labor is equal to $D_1 u(c, n)w$, which is the size of the wage (the reward for labor) times the marginal increase in utility that extra wages provide. The marginal cost of labor is constant and equal to $\gamma$. Under the investment tax, household consumption increases and household savings decreases. This means that $D_1 u(c, n)$ decreases (higher consumption means lower marginal utility). Since the marginal benefit from labor has decreased, the optimal response by households is to supply fewer units of labor.

### 8.4 Exercises

1. **Stochastic growth model**

   Consider the stochastic growth model in which the labor supply is inelastically supplied (i.e., $n = 1$), $u(c) = \ln(c)$, and $f(K, N, z) = zK^\theta N^{1-\theta}$ for some $\theta \in (0, 1)$. Conditional on the shock $z$ and the capital stock $k$ in the current period, what are the mean and variance for the output one period from now?

2. **Stochastic growth with endogenous labor**

   Consider the stochastic growth model with endogenous labor where the parameters are $u(c, n) = \ln(c) - \gamma n^2$, and $f(K, N, z) = zK^\theta N^{1-\theta}$ for some $\theta \in (0, 1)$. What are the equilibrium policy function, the consumption function, and the labor supply?

3. **Stochastic growth with endogenous labor**

   Consider the stochastic growth model with endogenous labor where the parameters are $u(c, n) = \ln(c) - \gamma n$, and $f(K, N, z) = zK^\theta N^{1-\theta}$ for some $\theta \in (0, 1)$. Conditional on the shock $z$ and the capital stock $k$ in the current period, what are the mean and variance for the output one period from now?

4. **Stochastic growth with endogenous labor**

   Consider the real business cycle model with parameters $u(c, n) = \ln(c) - \gamma n$, and $f(K, N, z) = zK^\theta N^{1-\theta}$ for some $\theta \in (0, 1)$. Assume that the shocks are iid, meaning that $\pi(z, z') = \pi(\tilde{z}, z')$ for all shocks $z, \tilde{z}, z' \in Z$. Since the shocks are iid, then
\( \pi (z, z') = \pi (1, z') \) for all shocks \( z \) and we can simply define \( \pi (z') = \pi (1, z') \). Conditional on the shock \( z \) and the capital stock \( k \) in the current period, what are the mean and variance for the output \( t \) periods from now?

5. **Stochastic growth with taxes**

Consider the stochastic growth model with taxation and endogenous labor where the parameters are \( u(c, n) = \ln(c) - \gamma n \), and \( f(K, N, z) = zK^\theta N^{1-\theta} \) for some \( \theta \in (0, 1) \). Take as given the capital stock \( k \). Suppose that the government observes the productivity shock \( z \) and then implements a tax policy \( \tau \) such that the investment \( k' = g(k, y) \) is constant across for all values of \( z \). This policy, by construction, will smooth investment, but describe what happens to the time series for consumption and labor.

6. **Stochastic growth with taxes**

Consider a variation of the stochastic growth model with a labor income tax. There is only one tax, which is the labor income tax and the rate of the labor income tax is \( \tau \). The Bellman equation for this model is given as follows:

\[
V(k, y) = \max_{c \geq 0, k' \geq 0, n \geq 0} \ u(c, n) + \beta \sum_{y' \in Y} \pi(y, y') V(k', y') \quad \text{subject to } c + k' = Rk + wn(1 - \tau) + T
\]

The government market clearing condition (budget balance) requires that \( T = \tau wn \). Assume \( u(c, n) = \ln(c) - \gamma n \) for some parameter \( \gamma > 0 \) and \( f(K, N, z) = zK^\theta N^{1-\theta} \) for some \( \theta \in (0, 1) \). Solve for the equilibrium policy function, consumption function, and labor supply function.

7. **Stochastic growth with taxes**

Consider a variation of the stochastic growth model with a capital income tax. There is only one tax, which is the capital income tax and the rate of the capital income tax is \( \tau \). Assume that the shocks are independent and identically distribution (iid), meaning that \( \pi(z, z') = \pi(\tilde{z}, z') \) for any shocks \( z, \tilde{z}, z' \in Z \). Since the shocks are iid, then \( \pi(z, z') = \pi(1, z') \) for all shocks \( z \) and we can simply define \( \pi(z') = \pi(1, z') \). The Bellman equation for this model is given as follows:

\[
V(k, y) = \max_{c \geq 0, k' \geq 0, n \geq 0} \ u(c, n) + \beta \sum_{y' \in Y} \pi(y') V(k', y') \quad \text{subject to } c + k' = Rk(1 - \tau) + wn + T
\]
The government market clearing condition (budget balance) requires that $T = \tau R k$. Assume $u(c, n) = \ln(c) - \gamma n$ for some parameter $\gamma > 0$ and $f(K, N, z) = z K^\theta N^{1-\theta}$ for some $\theta \in (0, 1)$. Solve for the equilibrium policy function, consumption function, and labor supply function.

8. **Stochastic growth with taxes**

Consider a variation of the stochastic growth model with a consumption tax. There is only one tax, which is the consumption tax and the rate of the consumption tax is $\tau$. Assume that the shocks are independent and identically distribution (iid), meaning that $\pi(z, z') = \pi(\tilde{z}, z')$ for any shocks $z, \tilde{z}, z' \in Z$. Since the shocks are iid, then $\pi(z, z') = \pi(1, z')$ for all shocks $z$ and we can simply define $\pi(z') = \pi(1, z')$. The Bellman equation for this model is given as follows:

$$
V(k, y) = \max_{c \geq 0, k' \geq 0, n \geq 0} u(c, n) + \beta \sum_{y' \in Y} \pi(y') V(k', y')
$$

subject to

$$(1 + \tau)c + k' = R k + \gamma n + T$$

The government market clearing condition (budget balance) requires that $T = \tau c$. Assume $u(c, n) = \ln(c) - \gamma n$ for some parameter $\gamma > 0$ and $f(K, N, z) = z K^\theta N^{1-\theta}$ for some $\theta \in (0, 1)$. Solve for the equilibrium policy function, consumption function, and labor supply function.
Bibliography


9

Fiscal Policy

9.1 Tax revenue

9.1.1 Sneak peek

Summary

Up until this point, we have introduced taxes only to analyze their effects on household
decisions. While the focus has been on one type of tax in particular (an investment tax),
exercises in the previous chapter show that the same methods can allow us to study labor
income tax, capital income tax, and consumption tax. In the previous analysis, we solved
for the policy function, consumption function, and labor supply function in terms of the tax
rate. These functions capture how taxes affect the dynamics in the economy.

In the deterministic versions of the neoclassical growth model considered thus far, a
unique steady state exists. Further, for any initial conditions in the economy, the equilibrium
variables will converge to the steady state. This section focuses exclusively on the steady
states of a neoclassical growth model. This can be viewed as the long-run outcome of a
deterministic economy (no shocks). The effects of the various taxes on household decisions
can be captured by analyzing how the taxes affect the steady state values.

This section considers two popular forms of taxation: investment tax and income tax.
Given that such taxes are imposed by the government, we will now take seriously the ques-
tion about what influences the government’s decision about how to set the tax rate. The
first analysis will focus on how the tax rate affects the total tax revenue collected by the
government. In this setting, the only purpose of governments is to collect the tax revenue
and then rebate it back to the households as lump-sum subsidies.

The relation between tax rate and tax revenue is referred to as the Laffer curve, which is a plot of the tax revenue on the y-axis against the marginal tax rate on the x-axis. The Laffer curve has a unique maximizing value and this is referred to as the revenue-maximizing tax rate. Abstracting away from all other concerns of the government, the revenue-maximizing tax rate is optimal for a government that only seeks to maximize the tax revenue collected.

The next step will provide the government with a more active role in welfare. The government will still collect the tax revenue, but instead of simply returning the proceeds back to the households lump-sum, the government will now use the revenue to provide public goods to the households. The government is the only entity that can provide public goods. Households cannot buy public goods and firms cannot produce public goods. Households have preferences over both private good and public good consumption.

With public good provision, we can analyze the tax rate chosen by a benevolent government to maximize the utility of households. This utility-maximizing tax rate will always be strictly smaller than the revenue-maximizing tax rate as the government is concerned not only with tax revenue, but also with output growth (allowing for more private good consumption).

Economies with an investment tax have a very nice property concerning the relation between the revenue-maximizing tax rate and the utility-maximizing tax rate. The tax rates can be solved as simple algebraic functions of a few parameters. The results with an income tax do not have a closed-form solution, and must be solved on the computer. Comparing the two types of taxes, an income tax allows for higher utility (at the utility-maximizing tax rate) compared to the utility for an investment tax (at the utility-maximizing tax rate). While the economies considered in this subsection are quite stylized, a similar approach applied to a larger model would allow policymakers to determine which types of taxes are optimal for achieving certain fiscal objectives.
9.1. TAX REVENUE

Notation

The variables to be introduced in this section are given in the following table:

- $c_t$: household consumption
- $i_t$: household investment
- $k_t$: household capital holdings
- $K_t$: firm capital input
- $N_t$: firm labor demand
- $\tau$: tax rate
- $T_t$: transfers from government
- $R_t$: rate of return on capital
- $w_t$: wage rate

The parameters to be introduced in this section are given in the following table:

- $\beta$: discount factor
- $\theta$: Cobb-Douglas production parameter
- $\delta$: depreciation rate
- $\alpha$: weight for public good consumption

Main takeaways

After completing this section, you will be able to answer the following questions:

- What is the optimal investment tax rate if the government is seeking to maximize revenue? What is the optimal income tax rate in this setting?

- What is the optimal investment tax rate if the government uses tax revenue to provide a public good? What is the optimal income tax rate in this setting?

9.1.2 Steady state

Consider the following variation of the neoclassical growth model. The economy consists of households and firms. For simplicity, the households will supply labor inelastically, meaning that the labor supply is fixed: $n_t = 1$. Households choose consumption $c_t$, capital stock $k_t$, ...
and investment \( i_t \) to maximize the following utility function

\[
\sum_{t=0}^{\infty} \beta^t \ln (c_t).
\] (9.1)

There are two factors of production: capital and labor. Firms choose capital input \( K_t \) and labor input \( N_t \) given their available production function

\[
f (K_t, N_t) = K_t^\theta N_t^{1-\theta}.
\] (9.2)

In this tax analysis, we focus on deterministic models without any shocks.

The market prices are the rate of return for capital \( R_t \) and the wage rate \( w_t \). Firms solve the following static maximization problem

\[
\max_{K_t, N_t} f (K_t, N_t) - R_t K_t - w_t N_t.
\] (9.3)

Households face an investment tax. The budget constraint is given by:

\[
c_t + i_t (1 + \tau) \leq R_t k_t + w_t + T_t.
\] (9.4)

The variable \( T_t \) denotes the lump-sum transfers from the government. The investment tax rate \( \tau \) is fixed in all periods. The law of motion for capital provides the relation between current capital stock and next-period capital stock:

\[
k_{t+1} = (1 - \delta) k_t + i_t.
\] (9.5)

The parameter \( \delta \) is the rate of depreciation. The household budget constraint can be written to incorporate the law of motion as follows:

\[
c_t + k_{t+1} (1 + \tau) \leq R_t k_t + w_t + (1 - \delta) (1 + \tau) k_t + T_t.
\] (9.6)

In equilibrium, the market clearing conditions are \( K_t = k_t \) and \( N_t = 1 \). The government budget balance requirement is:

\[
T_t = \tau i_t = \tau (k_{t+1} - (1 - \delta) k_t).
\] (9.7)
9.1. TAX REVENUE

The first order conditions of the firm profit maximization problem are:

\[ R_t = \theta k_t^{\theta - 1}. \]  \hspace{1cm} (9.8)
\[ w_t = (1 - \theta) k_t^\theta. \]

The household Euler equation is given by:

\[ \frac{1 + \tau}{c_t} = \beta \frac{R_{t+1} + (1 - \delta)(1 + \tau)}{c_{t+1}}. \]  \hspace{1cm} (9.9)

Using the government budget balance equation and the fact that \( R_t k_t + w_t = k_t^\theta \), the household budget constraint can be rewritten as the aggregate resource constraint:

\[ c_t + i_t = k_t^\theta. \]  \hspace{1cm} (9.10)

A steady state, by definition, is \((c_{ss}, k_{ss}, i_{ss}, y_{ss})\) such that \((c_t, k_t, i_t, y_t) = (c_{ss}, k_{ss}, i_{ss}, y_{ss})\) for all time periods. The variables are constant and do not change once a steady state is reached. To solve for the steady state in this economy, we proceed using the following 4 steps:

1. From the household Euler equation, a steady state satisfies:

\[ \frac{1 + \tau}{c_{ss}} = \beta \frac{\theta k_{ss}^{\theta - 1} + (1 - \delta)(1 + \tau)}{c_{ss}}. \]  \hspace{1cm} (9.11)

This equation can be solved for \( k_{ss} \) directly:

\[ k_{ss} = \left( \frac{\theta}{(1 + \tau)(1/\beta - (1 - \delta))} \right)^{\frac{1}{\theta - 1}}. \]  \hspace{1cm} (9.12)

2. From the law of motion for capital, \( k_{t+1} = (1 - \delta) k_t + i_t \), steady state satisfies:

\[ i_{ss} = \delta k_{ss}. \]  \hspace{1cm} (9.13)

3. From the production function, \( y_t = k_t^\theta \), steady state satisfies:

\[ y_{ss} = k_{ss}^\theta. \]  \hspace{1cm} (9.14)
4. From the aggregate resource constraint, \( c_t + i_t = y_t \), steady state satisfies:

\[
c_{ss} = y_{ss} - i_{ss} = k_{ss}^\theta - \delta k_{ss}.
\]  
(9.15)

9.1.3 Laffer curves

Investment tax

The Laffer curve is a plot of the steady state tax revenue, which equals \( \tau i_{ss} \) for the case of the investment tax, against the tax rate \( \tau \). Naturally, tax revenue equals 0 if \( \tau = 0 \). The Laffer curve will be strictly increasing for low levels of the tax rate. However, the Laffer curve is strictly concave. Eventually, the marginal effect will level off and there will exist a unique revenue-maximizing tax rate \( \tau^* \). Absent any other considerations, the revenue-maximizing tax rate \( \tau^* \) is optimal for the government as it brings in the most revenue.

The tax revenue is expressed as \( T_{ss} = \tau i_{ss} \) and can be written as:

\[
T_{ss} = \tau \delta k_{ss}.
\]  
(9.16)

Taking the derivative:

\[
\frac{\partial T_{ss}}{\partial \tau} = \delta k_{ss} + \tau \delta \frac{\partial k_{ss}}{\partial \tau}.
\]  
(9.17)

The first term \( \delta k_{ss} > 0 \) represents the benefit to the government from an increase in the tax rate (higher tax revenue). The second term \( \tau \delta \frac{\partial k_{ss}}{\partial \tau} < 0 \) represents the cost to the government from an increase in the tax rate (lower tax revenue). By the chain rule,

\[
\frac{\partial k_{ss}}{\partial \tau} = -\frac{k_{ss}}{(1 - \theta)(1 + \tau)} < 0.
\]  
(9.18)

The revenue-maximizing tax rate \( \tau^* \) is such that \( \frac{\partial T_{ss}(\tau^*)}{\partial \tau} = 0 \). Algebraically,

\[
\frac{\partial T_{ss}(\tau^*)}{\partial \tau} = \delta k_{ss} - \frac{\tau^* \delta k_{ss}}{(1 - \theta)(1 + \tau^*)} = 0.
\]  
(9.19)

The variable \( k_{ss} \) cancels out and the equation can be solved for \( \tau^* \):

\[
\tau^* = \frac{1 - \theta}{\theta}.
\]  
(9.20)

Consider the following parameter values: \( \theta = \frac{1}{3} \), \( \beta = 0.96 \), and \( \delta = 0.09 \). For this
9.1. TAX REVENUE

...
To solve for the steady state in this economy, we proceed using the following 4 steps:

1. From the household Euler equation, a steady state satisfies:

   \[
   \frac{1}{c_{ss}} = \beta \left( 1 + \left( \theta k_{ss}^{\theta-1} - \delta \right) \frac{1}{c_{ss}} (1 - \tau) \right). \tag{9.27}
   \]

   This equation can be solved for \( k_{ss} \) directly:

   \[
   k_{ss} = \left( \frac{\theta (1 - \tau)}{1/\beta - 1 + \delta (1 - \tau)} \right)^{\frac{1}{1-\theta}}. \tag{9.28}
   \]

2. From the law of motion for capital, \( k_{t+1} = (1 - \delta) k_t + i_t \), steady state satisfies:

   \[
   i_{ss} = \delta k_{ss}. \tag{9.29}
   \]

3. From the production function, \( y_t = k_t^{\theta} \), steady state satisfies:

   \[
   y_{ss} = k_{ss}^{\theta}. \tag{9.30}
   \]

4. From the aggregate resource constraint, \( c_t + i_t = y_t \), steady state satisfies:

   \[
   c_{ss} = y_{ss} - i_{ss} = k_{ss}^{\theta} - \delta k_{ss}. \tag{9.31}
   \]

The tax revenue is expressed as \( T_{ss} = \tau \left( (R_{ss} - \delta) k_{ss} + w_{ss} \right) \) and can be written as:

\[
T_{ss} = \tau \left( (R_{ss} - \delta) k_{ss} + w_{ss} \right) = \tau \left( y_{ss} - \delta k_{ss} \right) \tag{9.32}
= \tau \left( k_{ss}^{\theta} - \delta k_{ss} \right). \]

Taking the derivative:

\[
\frac{\partial T_{ss}}{\partial \tau} = (k_{ss}^{\theta} - \delta k_{ss}) + \tau \left( \theta k_{ss}^{\theta-1} - \delta \right) \frac{\partial k_{ss}}{\partial \tau}. \tag{9.33}
\]

By the chain rule,

\[
\frac{\partial k_{ss}}{\partial \tau} = -\frac{(1/\beta - 1) k_{ss}}{(1 - \tau) (1 - \theta) (1/\beta - 1 + \delta (1 - \tau))}. \tag{9.34}
\]
The revenue-maximizing tax rate \( \tau^* \) is such that \( \frac{\partial T_{tt}(\tau^*)}{\partial \tau} = 0 \). An end-of-chapter exercise asks you to find the revenue-maximizing tax rate \( \tau^* \) under income taxation.

### 9.1.4 Public goods

The revenue-maximizing tax rates are quite high in the previous analysis. In the model, the governments have nothing important (or interesting) to do other than collect taxes and then rebate the proceeds lump-sum back to the households. While the high tax rate maximizes revenue, the high tax rate also decreases the steady state output (GDP) and steady state consumption. We wish to quantify the trade-off between tax revenue and growth. To do that, we allow governments in this subsection to use the tax revenue to provide public goods to households. Households will have preferences over both private goods (bought on the market) and public goods (provided by the government).

Suppose the government operates with perfect efficiency. If the tax revenue \( T_t \) is collected, then the public goods provided are \( g_t = T_t \). Households have preferences over private goods (consumption \( c_t \)) and public goods (\( g_t \)). Households maximize the following utility function

\[
\sum_{t=0}^{\infty} \beta^t \{ \ln (c_t) + \alpha \ln (g_t) \},
\]

where \( \alpha > 0 \) is the weight that households assign to the public good. This is just one possible form for the household preferences. It is chosen for simplicity.

We have already found the revenue-maximizing tax rate for the government. What is the tax rate that a benevolent government would impose, namely the tax rate that provides the optimal balance between private and public goods for households?

### Investment tax

Consider first the case with investment tax. With government expenditure, the households do not receive any lump-sum subsidy from the government (as these funds are used to provide public goods). The budget constraint, specified in terms of investment, prior to inserting the law of motion for capital, is given by:

\[
c_t + i_t (1 + \tau) \leq R_t k_t + w_t.
\]
In equilibrium, \( R_t k_t + w_t = y_t \). The aggregate resource constraint is given by \( c_t + i_t (1 + \tau) = y_t \). Since the government is perfectly efficient, the public good \( g_t = \tau i_t \), meaning that we have the following equilibrium relation between output (GDP) and the sum of all expenditures in the economy:

\[
  c_t + g_t + i_t = y_t. 
\] (9.37)

In steady state:

\[
  c_{ss} + g_{ss} + i_{ss} = y_{ss}. 
\] (9.38)

Given the new aggregate resource constraint, the steady state variables are determined as a function of \( k_{ss} \) (whose equation remains unchanged):

\[
  k_{ss} = \left( \frac{\theta}{(1 + \tau) (1/\beta - (1 - \delta))} \right)^{\frac{1}{1-\beta}}. 
\] (9.39)

\[
  i_{ss} = \delta k_{ss}. 
\]

\[
  y_{ss} = k_{ss}^\theta. 
\]

\[
  c_{ss} = k_{ss}^\theta - \delta (1 + \tau) k_{ss}. 
\]

\[
  g_{ss} = \tau \delta k_{ss}. 
\]

Taking the derivative of \( \ln (c_{ss}) + \alpha \ln (g_{ss}) \) with respect to \( \tau \) and setting it equal to 0 (first order condition), the utility-maximizing tax rate is such that

\[
  \frac{1}{c_{ss}} \frac{\partial c_{ss}}{\partial \tau} + \frac{\alpha}{g_{ss}} \frac{\partial g_{ss}}{\partial \tau} = 0. 
\] (9.40)

Recall that

\[
  \frac{\partial k_{ss}}{\partial \tau} = -\frac{k_{ss}}{(1 - \theta) (1 + \tau)}. 
\] (9.41)

\[
  \frac{\partial g_{ss}}{\partial \tau} = \delta k_{ss} + \tau \delta \frac{\partial k_{ss}}{\partial \tau}. 
\]

Using the chain rule,

\[
  \frac{\partial c_{ss}}{\partial \tau} = -\delta k_{ss} + (\theta k_{ss}^\theta - \delta (1 + \tau)) \frac{\partial k_{ss}}{\partial \tau}. 
\] (9.42)

The utility-maximizing tax rate \( \tau^{**} \) is such that

\[
  \frac{1}{c_{ss} (\tau^{**})} \frac{\partial c_{ss} (\tau^{**})}{\partial \tau} + \frac{\alpha}{g_{ss} (\tau^{**})} \frac{\partial g_{ss} (\tau^{**})}{\partial \tau} = 0. 
\] (9.43)
9.1. TAX REVENUE

Algebraically, the above equation is satisfied when

\[ \tau^{**} = \frac{\alpha}{\alpha + 1} \left( \frac{1}{\theta} \right) \]  

(9.44)

For any economy with an investment tax, the utility-maximizing tax rate \( \tau^{**} = \frac{\alpha}{\alpha + 1} \tau^{*} \), where \( \tau^{*} \) is the revenue-maximizing tax rate. Consider the following parameter values: \( \theta = \frac{1}{3}, \beta = 0.96, \delta = 0.09, \) and \( \alpha = 1 \). For this economy, the utility-maximizing tax rate \( \tau^{**} = \frac{1}{2} (200\%) = 100\% \). This means that for each $1 invested, households pay $1 in taxes to the government. Recall that the revenue-maximizing tax rate \( \tau^{*} = 200\% \) for the same economy. When the government is not solely concerned about maximizing revenue, but is instead concerned about the optimal provision of public goods (according to household preferences), the tax rate will be lower.

**Income tax**

Consider second the case with income tax. With government expenditure, the households do not receive any lump-sum subsidy from the government (as these funds are used to provide public goods). The budget constraint, specified in terms of investment, prior to inserting the law of motion for capital, is given by:

\[ c_t + i_t \leq \delta k_t + ((R_t - \delta) k_t + w_t) (1 - \tau) \]  

(9.45)

In equilibrium, \( R_t k_t + w_t = y_t \). The aggregate resource constraint is given by \( c_t + i_t = \delta k_t + (y_t - \delta k_t) (1 - \tau) \), which reduces to

\[ c_t + i_t = y_t (1 - \tau) + \tau \delta k_t \]  

(9.46)

Since the government is perfectly efficient, then the public good \( g_t = \tau (y_t - \delta k_t) \), meaning that we have the following equilibrium relation between output (GDP) and the sum of all expenditures in the economy:

\[ c_t + g_t + i_t = y_t \]  

(9.47)

In steady state:

\[ c_{ss} + g_{ss} + i_{ss} = y_{ss} \]  

(9.48)
Given the new aggregate resource constraint, the steady state variables are determined as a function of $k_{ss}$ (whose equation remains unchanged):

$$k_{ss} = \left( \frac{\theta (1 - \tau)}{1/\beta - 1 + \delta (1 - \tau)} \right)^{1/\alpha}. \tag{9.49}$$

$$i_{ss} = \delta k_{ss}.$$  
$$y_{ss} = k_{ss}^\theta.$$  
$$c_{ss} = (k_{ss}^\theta - \delta k_{ss}) (1 - \tau).$$  
$$g_{ss} = \tau (k_{ss}^\theta - \delta k_{ss}).$$

Taking the derivative of $\ln (c_{ss}) + \alpha \ln (g_{ss})$ with respect to $\tau$ and setting it equal to 0 (first order condition), the utility-maximizing tax rate is such that

$$\frac{1}{c_{ss}} \frac{\partial c_{ss}}{\partial \tau} + \frac{\alpha}{g_{ss}} \frac{\partial g_{ss}}{\partial \tau} = 0. \tag{9.50}$$

Recall that

$$\frac{\partial k_{ss}}{\partial \tau} = - \frac{1}{(1 - \tau) (1 - \theta) (1/\beta - 1 + \delta (1 - \tau))} \frac{1/\beta - 1}{k_{ss}}. \tag{9.51}$$

$$\frac{\partial g_{ss}}{\partial \tau} = (k_{ss}^\theta - \delta k_{ss}) + \tau (\theta k_{ss}^{\theta-1} - \delta) \frac{\partial k_{ss}}{\partial \tau}.$$  

Using the chain rule,

$$\frac{\partial c_{ss}}{\partial \tau} = - (k_{ss}^\theta - \delta k_{ss}) + (1 - \tau) (\theta k_{ss}^{\theta-1} - \delta) \frac{\partial k_{ss}}{\partial \tau}. \tag{9.52}$$

The utility-maximizing tax rate $\tau^{**}$ is such that

$$\frac{1}{c_{ss}(\tau^{**})} \frac{\partial c_{ss}(\tau^{**})}{\partial \tau} + \frac{\alpha}{g_{ss}(\tau^{**})} \frac{\partial g_{ss}(\tau^{**})}{\partial \tau} = 0. \tag{9.53}$$

Suppose the economy has the parameter values $\theta = \frac{1}{3}$, $\beta = 0.96$, $\delta = 0.09$, and $\alpha = 1$. For these parameters, the utility-maximizing income tax rate can be found. Using the result from an end-of-chapter exercise, the utility value for the utility-maximizing income tax rate is higher than the utility value for the utility-maximizing investment tax rate.\footnote{Under the parameters $\theta = \frac{1}{3}$, $\beta = 0.96$, $\delta = 0.09$, and $\alpha = 1$, we find (i) the utility-maximizing investment tax rate is $\tau^{**} = 100\%$ with utility $\ln (c_{ss}) + \alpha \ln (g_{ss}) = -2.195$ and (ii) utility-maximizing income tax rate} If we believe
that these parameter values are an appropriate representation of the real-world economy, the analysis suggests that the income tax leads to the highest welfare for the economy.

With the investment tax, at the utility-maximizing tax rate, the private good consumption is 6.8 times as large as the public good consumption. With the income tax, at the utility-maximizing tax rate, the private good consumption is only 1.25 times larger than the public good consumption. The income tax provides a larger stream of revenue and thus a more optimal balance between the private good and the public good. This is preferred by the household as has been made evident by the higher utility value.

9.2 Time-consistent fiscal policy

9.2.1 Sneak peek

Summary

When discussing broad questions about how economic policy is set, you will typically hear debates between policy rules and discretion. A policy rule is an algorithm that tells current and future policymakers exactly how to set policy as a function of the current economic conditions. Once the rule is in place, it must be followed. A benefit of policy rules is that it allows households and firms to accurately forecast future policy changes, and ultimately future market conditions. Discretion means that the policymakers respond each instant that new data is received and there is not a consistent relation between the economic conditions and the policy choices. Though discretion seemingly provides more flexibility, economists all agree that policy rules are a more effective means of governance.

This is closely related to the concept of commitment. If a government can commit to its policies, then it adopts a policy rule that prescribes how policy will be chosen in the current period and in all future periods. Given this information, households and firms make optimal decisions concerning consumption, investment, and labor supply. The commitment by governments must be credible. This means that households and firms must believe (correctly) that if a government adopts a policy rule, then it will carry out the mandates of that policy rule in all future periods.

It is difficult to imagine that governments have a credible commitment device, especially over a long time horizon (10 years and longer). For this reason, it is important to analyze the fiscal policies that will be chosen in a setting without commitment. A setting without

\[ \tau^{**} = 44.4\% \text{ with utility } \ln(c_{ss}) + \alpha \ln(g_{ss}) = -1.097. \]
commitment can be interpreted as a setting in which the current government is not able to force the future government to make a specific fiscal policy choice. Consider that in many countries the government administration may switch out every 4-8 years. With this kind of turnover, commitment is not possible.

If a government cannot commit to future policies, they must choose policies that are time consistent. A time consistent policy is a policy rule that prescribes how policy decisions are made in the future, and must be such that when the time comes to implement the announced decisions, governments find it in their best interest to honor the announcements. Time consistent policies are important as these are the only ones that will be believed by households and firms. Households and firms will not believe any policy announcements that they know future governments will renege on.

**Notation**

The variables to be introduced in this section are given in the following table:

- $c_t$  household consumption
- $k_t$  household capital holdings
- $K_t$  firm capital input
- $N_t$  firm labor demand
- $\tau_t$  tax rate
- $g_t$  public good consumption
- $R_t$  rate of return on capital
- $w_t$  wage rate

**Main takeaways**

After completing this section, you will be able to answer the following questions:

- What is the optimal fiscal policy if the government can commit to the policy?
- What is meant by a time-consistent policy?
- What is the optimal fiscal policy if the government cannot commit?
9.2. TIME-CONSISTENT FISCAL POLICY

9.2.2 Policy under commitment

Thus far, we have assumed an important property about fiscal policy, namely that the government can commit itself to policy decisions. The government is typically viewed as the first-mover and would choose the fiscal policy for the entire length of the economy. This could be represented by a constant tax rate or a tax rate that changes over time. Households and firms observe the fiscal policy (the tax rate) and optimally choose consumption, investment, and labor. Households and firms believe (correctly) that the government will honor its past commitments.

The model

Let’s consider a simple economy and derive the optimal fiscal policy under commitment. Unlike the previous section, we will not restrict our analysis to the steady state as the equilibrium dynamics are important.

The economy consists of households and firms. For simplicity, the households will supply labor inelastically, meaning that the labor supply is fixed: $n_t = 1$. Households choose consumption $c_t$, capital stock $k_t$, and investment $i_t$ to maximize the following utility function:

$$
\sum_{t=0}^{\infty} \beta^t \{ \ln (c_t) + \alpha \ln (g_t) \}.
$$

The amount of public good provided by the government equals $g_t$ and $\alpha$ is the utility weight assigned to the public good. There are two factors of production: capital and labor. Firms choose capital input $K_t$ and labor input $N_t$ given their available production function

$$
f (K_t, N_t) = K_t^\theta N_t^{1-\theta}.
$$

The market prices are the rate of return for capital $R_t$ and the wage rate $w_t$. Firms solve the following static profit maximization problem

$$
\max_{K_t, N_t} f (K_t, N_t) - R_t K_t - w_t N_t.
$$

Households face a capital income tax that includes taxing both the principal and the interest. The budget constraint is given by:

$$
c_t + i_t \leq R_t k_t (1 - \tau_t) + w_t.
$$
The capital income tax rate $\tau_t$ can take different values across time. The law of motion for capital provides the relation between current capital stock and next-period capital stock:

$$k_{t+1} = (1 - \delta) k_t + i_t. \quad (9.58)$$

For simplicity, assume $\delta = 1$, so $k_{t+1} = i_t$.

In equilibrium, the market clearing conditions are $K_t = k_t$ and $N_t = 1$. The amount of public good provided equals:

$$g_t = \tau_t R_t k_t. \quad (9.59)$$

The first order conditions of the firm profit maximization problem are:

$$R_t = \theta k_t^{g-1}. \quad (9.60)$$
$$w_t = (1 - \theta) k_t^g.$$

The household Euler equation is given by:

$$1 \frac{1}{c_t} = \beta \frac{R_{t+1} (1 - \tau_{t+1})}{c_{t+1}}. \quad (9.61)$$

Using the public good equation and the fact that $R_t k_t + w_t = k_t^g$, the household budget constraint can be rewritten as the aggregate resource constraint:

$$c_t + g_t + k_{t+1} = k_t^g. \quad (9.62)$$

**Optimal policy under commitment**

The optimal policy under commitment will involve a constant tax rate denoted by $\tau^C$, where $'C'$ stands for commitment. By definition, $g_t = \tau^C R_t k_t = \tau^C \theta k_t^g$. The policy function $k_{t+1}$ can be solved as a function of $k_t^g$ using the same guess-and-check the policy function approach that we utilized in the "Neoclassical Growth Model" chapter. You are asked in an end-of-chapter exercise to verify that $k_{t+1} = \theta \beta (1 - \tau^C) k_t^g$ in any period $t$. Since $g_t$ and $k_{t+1}$ are known, the aggregate resource constraint is used to determine $c_t$:

$$c_t = (1 - \theta \beta - \tau^C \theta (1 - \beta)) k_t^g$$
The Bellman equation for the government is then given by:

$$ W(k) = \max_{c, g, k'} \ln (c) + \alpha \ln (g) + \beta W(k') $$

subject to

$$ c = (1 - \theta \beta - \tau^C \theta (1 - \beta)) k^\theta $$

$$ g = \tau^C \theta k^\theta $$

$$ k' = \theta \beta (1 - \tau^C) k^\theta $$

(9.63)

The value function $W(k)$ is of the form $W(k) = A + B \ln (k)$. It is easy to verify this using the guess-and-check the value function from the "Neoclassical Growth Model" chapter. With this form for the value function, the Bellman equation, including the expressions for $c$, $g$, and $k'$, is given by:

$$ A + B \ln (k) = \ln \left( (1 - \theta \beta - \tau^C \theta (1 - \beta)) k^\theta \right) + \alpha \ln \left( \tau^C \theta k^\theta \right) + \beta A + \beta B \ln \left( \theta \beta (1 - \tau^C) k^\theta \right). $$

(9.64)

Use the ln property $\ln (ab^\theta) = \ln (a) + \theta \ln (b)$ to isolate the $\ln(k)$ terms on both the left-hand and right-hand sides of the above equation:

$$ B \ln (k) = \theta \ln (k) + \alpha \theta \ln (k) + \beta B \theta \ln (k). $$

(9.65)

Solving for $B$:

$$ B = \frac{\theta (1 + \alpha)}{1 - \theta \beta}. $$

(9.66)

At this time, we can ignore the effects of the constant term $A$. We will get back to this shortly. Knowing the form for $W(k')$, the maximization problem can be written as the unconstrained problem:

$$ \max_{\tau^C} \ln \left( (1 - \theta \beta - \tau^C \theta (1 - \beta)) k^\theta \right) + \alpha \ln \left( \tau^C \theta k^\theta \right) + \beta A + \frac{\theta \beta (1 + \alpha)}{1 - \theta \beta} \ln \left( \theta \beta (1 - \tau^C) k^\theta \right) $$

(9.67)

Ignoring all the terms that don’t involve the variable $\tau^C$, the maximization problem is equivalent to:

$$ \max_{\tau^C} \ln \left( 1 - \theta \beta - \tau^C \theta (1 - \beta) \right) + \alpha \ln \left( \tau^C \right) + \frac{\theta \beta (1 + \alpha)}{1 - \theta \beta} \ln \left( 1 - \tau^C \right). $$

(9.68)
The first order condition is given by:

\[
\frac{-\theta (1 - \beta)}{1 - \theta \beta - \tau^C \theta (1 - \beta)} + \frac{\alpha}{\tau^C} - \frac{\theta \beta (1 + \alpha)}{1 - \tau^C} = 0. \tag{9.69}
\]

The optimal tax rate under commitment is the value \(\tau^C\) that solves the above first order condition. This equation can be solved on the computer, or it can be rewritten as a quadratic equation. I provide the steps for the latter. The first order condition can be rewritten as:

\[
\frac{\theta (1 - \beta)}{1 - \theta \beta - \tau^C \theta (1 - \beta)} = \frac{\alpha}{\tau^C} - \frac{\theta \beta (1 + \alpha)}{(1 - \theta \beta) (1 - \tau^C)}. \tag{9.70}
\]

Finding a common denominator on the right-hand side:

\[
\frac{\theta (1 - \beta)}{1 - \theta \beta - \tau^C \theta (1 - \beta)} = \frac{\alpha (1 - \theta \beta) (1 - \tau^C) - \tau^C \theta \beta (1 + \alpha)}{(1 - \theta \beta) \tau^C (1 - \tau^C)}. \tag{9.71}
\]

Cross-multiplying leads to the following long equation

\[
\theta (1 - \theta \beta) (1 - \beta) \tau^C (1 - \tau^C) = \alpha (1 - \theta \beta)^2 (1 - \tau^C) - \theta \beta (1 - \theta \beta) (1 + \alpha) \tau^C
-\alpha \theta (1 - \theta \beta) (1 - \beta) \tau^C (1 - \tau^C)
+ \theta (\theta \beta) (1 - \beta) (1 + \alpha) (\tau^C)^2. \tag{9.72}
\]

Gather all terms on the right-hand side:

\[
(\tau^C)^2 \left( \theta (1 - \theta \beta) (1 - \beta) + \alpha \theta (1 - \theta \beta) (1 - \beta) + \theta (\theta \beta) (1 - \beta) (1 + \alpha) \right)
+ \tau^C \left( -\theta (1 - \theta \beta) (1 - \beta) - \alpha (1 - \theta \beta)^2 - \theta \beta (1 - \theta \beta) (1 + \alpha) - \alpha \theta (1 - \theta \beta) (1 - \beta) \right)
+ \alpha (1 - \theta \beta)^2 = 0. \tag{9.73}
\]

The coefficient for \((\tau^C)^2\) reduces to

\[
\theta (1 + \alpha) (1 - \theta \beta) (1 - \beta) + \theta (1 + \alpha) (\theta \beta) (1 - \beta) = \theta (1 + \alpha) (1 - \beta). \tag{9.74}
\]
9.2. TIME-CONSISTENT FISCAL POLICY

The coefficient for $C$ reduces to

$$-(1 - \theta \beta) (\theta (1 - \beta) + \alpha (1 - \theta \beta) + \theta \beta (1 + \alpha) + \alpha \theta (1 - \beta))$$

(9.75)

$$= - (1 - \theta \beta) (\alpha (1 - \theta \beta) + \theta \beta (1 + \alpha) + \theta (1 + \alpha) (1 - \beta))$$

$$= - (1 - \theta \beta) (\alpha (1 - \theta \beta) + \theta (1 + \alpha)) .$$

The quadratic equation is therefore given by:

$$\{\theta (1 + \alpha) (1 - \beta)\} (\tau_C^2 - \{(1 - \theta \beta) (\alpha (1 - \theta \beta) + \theta (1 + \alpha))\} \tau_C + \alpha (1 - \theta \beta)^2 = 0. \quad (9.76)$$

Only one of the two possible solutions to the quadratic formula satisfies $\tau_C \in [0, 1]$. This is the optimal tax rate under commitment.

In our derivation, we have ignored the effects of the constant term $A$. Let’s see why we are allowed to do this. The Bellman equation is given by:

$$A + B \ln (k) = \ln \left((1 - \theta \beta - \tau_C^C \theta (1 - \beta)) k^\theta\right) + \alpha \ln (\tau_C^C \theta k)$$

$$+ \beta A + \beta B \ln \left(\theta \beta (1 - \tau_C^C) k^\theta\right) .$$

(9.77)

Use the $\ln$ property $\ln (ab^\theta) = \ln (a) + \theta \ln (b)$ to isolate the constant terms on both the left-hand and right-hand sides of the above equation:

$$A = \ln \left((1 - \theta \beta - \tau_C^C \theta (1 - \beta))\right) + \alpha \ln (\tau_C^C \theta) + \beta A + \beta B \ln \left(\theta \beta (1 - \tau_C^C)\right) . \quad (9.78)$$

We have already found the value for $B$. Solving for $A$, we see that

$$A = \frac{\ln \left((1 - \theta \beta - \tau_C^C \theta (1 - \beta))\right) + \alpha \ln (\tau_C^C \theta) + \frac{\theta \beta (1 + \alpha)}{1 - \theta \beta} \ln \left(\theta \beta (1 - \tau_C^C)\right)}{1 - \beta} . \quad (9.79)$$

With the expression for $A$, the maximization problem can be written as the unconstrained problem:

$$\max_{\tau_C^C} \ln \left((1 - \theta \beta - \tau_C^C \theta (1 - \beta)) k^\theta\right) + \alpha \ln (\tau_C^C \theta k) + \frac{\theta \beta (1 + \alpha)}{1 - \theta \beta} \ln \left(\theta \beta (1 - \tau_C^C) k^\theta\right)$$

$$+ \frac{\beta}{1 - \beta} \left\{\ln \left((1 - \theta \beta - \tau_C^C \theta (1 - \beta))\right) + \alpha \ln (\tau_C^C \theta) + \frac{\theta \beta (1 + \alpha)}{1 - \theta \beta} \ln \left(\theta \beta (1 - \tau_C^C)\right)\right\} . \quad (9.80)$$

Ignoring all the terms that don’t involve the variable $\tau_C^C$, the maximization problem is equiv-
alent to:

\[
\max_{\tau^C} \ln \left( 1 - \theta \beta - \tau^C \theta (1 - \beta) \right) + \alpha \ln \left( \tau^C \right) + \frac{\theta \beta (1 + \alpha)}{1 - \theta} \ln \left( 1 - \tau^C \right) \\
+ \frac{\beta}{1 - \beta} \left\{ \ln \left( 1 - \theta \beta - \tau^C \theta (1 - \beta) \right) + \alpha \ln \left( \tau^C \right) + \frac{\theta \beta (1 + \alpha)}{1 - \theta} \ln \left( 1 - \tau^C \right) \right\}. \tag{9.81}
\]

The unconstrained maximization problem is

\[
\max_{\tau^C} \frac{1}{1 - \beta} \left\{ \ln \left( 1 - \theta \beta - \tau^C \theta (1 - \beta) \right) + \alpha \ln \left( \tau^C \right) + \frac{\theta \beta (1 + \alpha)}{1 - \theta} \ln \left( 1 - \tau^C \right) \right\}. \tag{9.82}
\]

The optimal tax rate that solves (9.82) is the same as the optimal tax rate that solves (9.68), and the first order condition for both is given by (9.69).

**Discussion**

You are asked in an end-of-chapter exercise to calculate the optimal tax rate when \( \theta = \frac{1}{3} \), \( \beta = 0.96 \), and \( \alpha = 0.5 \). The government faces a trade-off: (i) a high tax rate allows for higher tax revenue and a greater provision of public goods, but (ii) the high tax rate discourages investment and this has cumulative effects on utility in the next period and all future periods.

Suppose that the government decides to change its tax policy at some point during the model. This could happen due to the election of a new political administration. In the current period, the capital stock is fixed as it was chosen in the previous period. The new administration finds the following policy optimal: (i) make an unannounced change in the tax rate in the current period to increase the tax revenue and (ii) set the tax rate low in future periods to continue to incentivize investment. With such a policy, the new administration is manipulating the economy by trying to turn both sides of the trade-off into benefits.

If any new administration has this optimal policy that they can deviate to, then households and firms won’t possibly believe that future administrations will keep the tax rate low in future periods. Further, households and firms in the past would never have believed that the new administration would come in and not raise taxes.

If the government can commit to a policy, then households and firms will believe that they will actually implement that policy. Without commitment, that same policy will be time-inconsistent, meaning that when it comes time for a future government to implement the policy, it will choose not to. Households and firms will only believe a time-consistent policy.
9.2.3 Time-consistent policy

A time-consistent policy is the optimal policy chosen by the government in a setting without commitment (a ‘no commitment’ or NC setting). A time-consistent policy is such that the government in each period optimally chooses the tax policy conditioning on two aspects: (i) what choices households and firms will make in the present and future conditional on present and future tax rates and (ii) how the future governments will choose taxes in response to the present government policy.

This dynamic consistency requirement for time-consistent policy closely resembles a strategic game between the current government and the future government. We will actually solve for the time-consistent optimal policy using the concept of backward induction (popularized in the field of game theory).

Backward induction

Consider a truncated version of the model with periods $t = 0, ..., T$. We truncate the model, because we know that investment will not take place in the final period $T$. We can then solve for the optimal time-consistent policy using backward induction. Backward induction proceeds by the following algorithm:

1. Solve for the optimal policy in the ultimate period $T$.

2. Knowing how optimal policy will be chosen in $T$, solve for the optimal policy in the penultimate period $T - 1$.

3. Continue until the optimal policy is found for all periods $t = 0, ..., T$.

I will do steps 1 and 2 and leave step 3 as an end-of-chapter exercise.

In period $T$, the state variable equals $k_T$, which is the capital stock at the beginning of the period. The optimization problem is given by:

$$\max_{c_T, g_T} \ln (c_T) + \alpha \ln (g_T)$$
subj. to $c_T + g_T + k_{T+1} = k_T^0$ (9.83)

It doesn’t make sense to invest for the future in the final period $T$, so $k_{T+1} = 0$. Solve the constraint for $g_T$ and insert into the objective function in order to obtain an unconstrained
maximization problem in terms of only $c_T$:

$$\max_{c_T} \ln (c_T) + \alpha \ln \left( k_T^\theta - c_T \right)$$ (9.84)

From the first order condition:

$$\frac{1}{c_T} = \frac{\alpha}{k_T^\theta - c_T}.$$

(9.85)

Solve for $c_T$:

$$c_T = \frac{1}{1 + \alpha k_T^\theta}.$$ (9.86)

From the resource constraint:

$$g_T = \frac{\alpha}{1 + \alpha k_T^\theta}.$$ (9.87)

Knowing how the government will choose policy in period $T$ as a function of $k_T$, we now consider period $T-1$. The government will choose the policy $g_{T-1}$ taking into account how this effects $k_T$ and the choice $g_T$ in period $T$. The Euler equation between periods $T-1$ and $T$ is given by:

$$\frac{1}{c_{T-1}} = \beta \theta k_T^{\theta-1} \left( 1 - \tau_T \right) \frac{1}{c_T}.$$ (9.88)

By definition, $g_T = \tau_T \theta k_T^\theta$. Multiply the numerator and denominator on the right-hand side by $k_T$:

$$\frac{1}{c_{T-1}} = \beta \theta k_T^\theta - g_T \frac{1}{k_T c_T}.$$ (9.89)

Given the expressions for $g_T$ and $c_T$ previously found:

$$\frac{1}{c_{T-1}} = \beta \theta (1 + \alpha) - \frac{\alpha}{k_T}.$$ (9.90)

Define the term

$$\Xi_T = \beta \left( \theta (1 + \alpha) - \alpha \right).$$ (9.91)

The Euler equation therefore reduces to:

$$k_T = \Xi_T c_{T-1}.$$ (9.92)

The resource constraint is given by:

$$c_{T-1} + g_{T-1} + k_T = k_T^\theta.$$ (9.93)
Using the equation $k_T = \Xi_T c_{T-1}$, we can update the resource constraint:

$$c_{T-1} (1 + \Xi_T) + g_{T-1} = k^\theta_{T-1}. \quad (9.94)$$

The optimization problem in period $T - 1$ includes the utility in both periods $T - 1$ and $T$, with the discount factor $\beta$ for period $T$:

$$\max_{c_{T-1}, g_{T-1}, c_T, g_T} \ln (c_{T-1}) + \alpha \ln (g_{T-1}) + \beta (\ln (c_T) + \alpha \ln (g_T))$$

subj. to

$$c_{T-1} (1 + \Xi_T) + g_{T-1} = k^\theta_{T-1}. \quad (9.95)$$

$$g_T = \frac{\alpha}{1+\alpha} (\Xi_T c_{T-1})^\theta. \quad (9.96)$$

$$c_T = \frac{1}{1+\alpha} (\Xi_T c_{T-1})^\theta. \quad (9.97)$$

The final two constraints are simply the expressions for $g_T$ and $c_T$ previously found and expressed in terms of $k_T = \Xi_T c_{T-1}$. The unconstrained maximization problem is given by:

$$\max_{c_{T-1}} \ln (c_{T-1}) + \alpha \ln (k^\theta_{T-1} - c_{T-1} (1 + \Xi_T)) + \beta \left( \ln \left( \frac{1}{1+\alpha} (\Xi_T c_{T-1})^\theta \right) + \alpha \ln \left( \frac{\alpha}{1+\alpha} (\Xi_T c_{T-1})^\theta \right) \right). \quad (9.98)$$

Recall that natural log has the property $\ln (ab^\theta) = \ln (a) + \theta \ln (b)$. Ignoring all the terms that don’t involve the variable $c_{T-1}$, the maximization problem is equivalent to:

$$\max_{c_{T-1}} \ln (c_{T-1}) + \alpha \ln (k^\theta_{T-1} - c_{T-1} (1 + \Xi_T)) + \theta \beta \ln (c_{T-1}) + \theta \beta \alpha \ln (c_{T-1}). \quad (9.99)$$

From the first order condition with respect to $c_{T-1}$:

$$\frac{\alpha (1 + \Xi_T)}{k^\theta_{T-1} - c_{T-1} (1 + \Xi_T)} = \frac{1 + \theta \beta (1 + \alpha)}{c_{T-1}}. \quad (9.99)$$

Solve for $c_{T-1}$:

$$c_{T-1} = \frac{1 + \theta \beta (1 + \alpha)}{(1 + \Xi_T)(1 + \alpha)(1 + \theta \beta)} k^\theta_{T-1}. \quad (9.99)$$

From the resource constraint:

$$g_{T-1} = \frac{\alpha}{(1 + \alpha)(1 + \theta \beta)} k^\theta_{T-1}. \quad (9.100)$$

You are asked in an end-of-chapter exercise to solve for the optimal policy in period $T - 2$. 
Infinite time horizon

By definition, for any time period $T - s$, public good provision is given by the equation

$$g_{T-s} = \tau_{T-s} \left( \theta k_{T-s}^\theta \right). \quad (9.101)$$

Since $g_T = \frac{\alpha}{1+\alpha} k_T^\theta$, then the tax rate $\tau_T = \frac{1}{\theta} \left( \frac{\alpha}{1+\alpha} \right)$. Since $g_{T-1} = \frac{\alpha}{(1+\alpha)(1+\theta\beta)} k_{T-1}^\theta$, the tax rate $\tau_{T-1} = \frac{1}{\theta} \left( \frac{\alpha}{1+\alpha} \right) \left( \frac{1}{1+\theta\beta} \right)$. From the end-of-chapter exercise, you should find $g_{T-2} = \frac{\alpha}{(1+\alpha)(1+\theta\beta+(\theta\beta)^2)} k_{T-2}^\theta$, which implies that the tax rate $\tau_{T-2} = \frac{1}{\theta} \left( \frac{\alpha}{1+\alpha} \right) \left( \frac{1}{1+\theta\beta+(\theta\beta)^2} \right)$. Writing down these tax rates, we begin to see a pattern:

$$\tau_T = \frac{1}{\theta} \left( \frac{\alpha}{1+\alpha} \right). \quad (9.102)$$

$$\tau_{T-1} = \frac{1}{\theta} \left( \frac{\alpha}{1+\alpha} \right) \left( \frac{1}{1+\theta\beta} \right).$$

$$\tau_{T-2} = \frac{1}{\theta} \left( \frac{\alpha}{1+\alpha} \right) \left( \frac{1}{1+\theta\beta+(\theta\beta)^2} \right).$$

$$\vdots$$

$$\lim_{s \to \infty} \tau_{T-s} = \frac{1}{\theta} \left( \frac{\alpha}{1+\alpha} \right) \left( \frac{1}{1+\theta\beta+(\theta\beta)^2 + \ldots} \right).$$

The infinite time horizon (the model we are interested in solving) is equivalent to being an infinite number of periods away from the final time period (take the limit as $s \to \infty$). By definition, $\tau^{NC} = \lim_{s \to \infty} \tau_{T-s}$. Here, $\tau^{NC}$ stands for the time-consistent policy, which by definition occurs in a setting of ‘no commitment’. The infinite sum

$$1 + \theta\beta + (\theta\beta)^2 + \ldots = \frac{1}{1-\theta\beta}. \quad (9.103)$$

This means that the optimal tax rate in the infinite time horizon model is given by:

$$\tau^{NC} = \frac{1}{\theta} \frac{\alpha}{1+\alpha} (1 - \theta\beta). \quad (9.104)$$

Discussion

You are asked in an end-of-chapter exercise to calculate the tax rate when $\theta = \frac{1}{3}$, $\beta = 0.96$, and $\alpha = 0.5$. 
Define the function

\[ \Psi (\tau) = \frac{-\theta (1 - \beta)}{1 - \theta \beta} - \frac{\alpha}{1 - \tau} + \frac{\theta \beta (1 + \alpha)}{1 - \theta \beta} = 0. \]  

(9.105)

If you recall the initial first order condition for \( \tau^C \), then you will recognize that the optimal tax rate under commitment must satisfy \( \Psi (\tau^C) = 0 \). It is easy to see that \( \Psi (\tau) > 0 \) when \( \tau < \tau^C \) and \( \Psi (\tau) < 0 \) when \( \tau > \tau^C \). An end-of-chapter exercise asks you to show that \( \Psi (\tau^{NC}) < 0 \) whenever \( \tau^{NC} < 1 \). Since the derivation of \( \tau^{NC} \) only makes sense when the value of the tax \( \tau^{NC} < 1 \), then we are comfortable making this assumption. Namely, we will only consider the values for parameters \( (\theta, \beta, \alpha) \) such that \( \tau^{NC} < 1 \). If, under these conditions, \( \Psi (\tau^{NC}) < 0 \), as you will show in the end-of-chapter exercise, then \( \tau^{NC} > \tau^C \).

In words, the mathematical expression \( \tau^{NC} > \tau^C \) means that the tax rate without commitment is higher than the tax rate with commitment. The reasons to analyze the case without commitment is because the commitment case suffers from the time inconsistency problem. The problem, as previously described, is that future governments have an incentive to increase the tax rate. Without the ability to commit, governments will always have an incentive to increase the tax rate. The optimal tax rate accounting for this lack of commitment is \( \tau^{NC} \), which is larger than \( \tau^C \).

Naturally, the welfare (utility) is lower with \( \tau^{NC} \) compared to \( \tau^C \). The governments under \( \tau^C \) have this magical power of commitment that is not granted to governments under \( \tau^{NC} \). In reality, we don’t think that governments have such a magical power and instead need to operate in a \( \tau^{NC} \) setting. In such a setting, governments are forced to set a high tax rate as this is the only policy that will be believed by the households and firms (time-consistent). This high tax rate has two effects: lower investment and higher public good expenditure. Lower investment is bad for welfare, but higher public good expenditure is good for welfare. On net, the high tax rate does more harm through the lower investment channel than benefit through the more public good channel.
9.3 Debt in the overlapping generations model

9.3.1 Sneak peek

Summary

The final section of this chapter returns to the overlapping generations model. The model is very useful at capturing long-run trends in economies. We apply the model in this section to analyze the growth paths for government debt. If government debt is sustainable in the long-run, then the current practices of the government are such that the debt level will stabilize. Overlapping generations models allow for population growth and technology growth, so when we say that the debt level stabilizes, we mean that the per-capita debt level or the debt-to-GDP ratio stabilizes (depending on the context). If the debt is not sustainable in the long run, then the debt is explosive meaning that the current practices of the government are such that the debt level (the per-capital debt level in this section) will continue to grow without bound.

The model allows us to find the conditions under which the per-capita debt level is not sustainable in the long-run. For such economies, there are a number of different alternatives to reign in debt. The first alternative is to seek an emergency loan (a bailout) or an agreement from creditors to renegotiate the terms of the current loans. Both of these alternatives are similar in that they are able to lower the debt in the current period and are labeled an emergency loan. The second alternative is to reduce the government deficit in the current period. This can be accomplished by raising taxes or reducing government expenditures. The model allows us to determine what combination of emergency loans and deficit reduction will be successful in transitioning an economy from an unsustainable debt level to a sustainable one.

The household side of the overlapping generations model is identical to what was seen in the "Money in the Overlapping Generations Model" chapter. The basics of the setup will be provided, but students are encouraged to review the previous material for a complete treatment.
9.3. DEBT IN THE OVERLAPPING GENERATIONS MODEL

Notation

The variables to be introduced in this section are given in the following table:

- $D_t$: government deficit
- $G_t$: government spending
- $B_t$: government debt
- $d_t$: per-capita government deficit
- $g_t$: per-capita government spending
- $b_t$: per-capita government debt

Main takeaways

After completing this section, you will be able to answer the following questions:

- How many stationary equilibria exist? Which ones are stable?
- What is the effective debt limit beyond which point a government is unable to repay its debts?
- What combination of emergency loans and deficit reduction is able to reduce debt to a level that is sustainable in the long run?

9.3.2 The OLG model

The model contains two groups of agents: (i) a government and (ii) households that live for 2 periods.

The economy consists of an infinite number of discrete time periods $t \in \{0, 1, 2, \ldots\}$. In each period, a single commodity is traded and consumed. In each period $t$, a cohort of homogeneous households is born. This set of households is referred to as the period $t$ cohort. These households trade and consume in periods $t$ and $t+1$. The households no longer exist in period $t+2$. In period $t$, the cohort $t$ households are referred to as "young", while in period $t+1$, the cohort $t$ households are referred to as "old".

The model allows for population growth at rate $n \geq 0$. This means that the mass of cohort $t$ households $N_t$ is larger than the mass of cohort $t-1$ households $N_{t-1}$ as follows: $N_t = (1 + n)N_{t-1}$.

The model is an endowment economy. Denote $p_t$ as the commodity price in period $t$. Denote $e^t_\tau$ as the endowment of a cohort $t$ household in period $\tau$ and denote $c^t_\tau$ as the
consumption of a cohort $t$ household in period $\tau$. The endowments are given by:

$$
e_t^t = e_y$$

$$
e_{t+1}^t = e_o$$

and all other endowments are 0.

Denote $\hat{a}_t$ as the real bonds chosen by the cohort $t$ households in period $t$, and with payouts in period $t + 1$. The real interest rate in period $t$ is given by $r_t$. Since the cohort $t$ households are no longer alive in period $t + 2$, they do not choose real bonds in period $t + 1$ (since the payouts from such assets are received after their death).

The budget constraints for cohort $t$ households are as follows:

$$
p_t c_t^t + \frac{p_t \hat{a}_t}{1 + r_t} \leq p_t e_y. \quad (9.107)
$$

$$
p_{t+1} c_{t+1}^t \leq p_{t+1} e_o + p_{t+1} \hat{a}_t.
$$

It is equivalent to define the real bond holdings as $a_t = \frac{\hat{a}_t}{1 + r_t}$. Using this convention, the budget constraints are given by:

$$
p_t c_t^t + p_t a_t \leq p_t e_y. \quad (9.108)
$$

$$
p_{t+1} c_{t+1}^t \leq p_{t+1} e_o + (1 + r_t) p_{t+1} a_t.
$$

Notice that in period $t + 1$, the interest rate is $r_t$ as this is the interest rate set by the market when the real bond choice took place (period $t$).

### 9.3.3 Government debt

The government is very simple in this model. The government does not collect taxes or make transfers. The government also does not print or distribute money. All the government does is spend and incur debt.

Denote the fixed level of government spending as $G_t$. Without any other assets or credits on the balance sheet of the government, the budget constraint of the government requires that the deficit $D_t$ satisfies:

$$
D_t = G_t. \quad (9.109)
$$

The government deficit in any time period is equal to the difference between the government
expenditures and the government revenue. Total government expenditures equal $G_t$ and total revenue equals 0.

To sustain a deficit, the government will accrue debt. To finance the debt, the government must pay interest on its previous debt obligations. The law of motion for government debt is such that government debt $B_t$ is equal to the principal plus interest on previous government debt $B_{t-1}$ plus the current period deficit $D_t$, namely:

$$B_t = (1 + r_{t-1}) B_{t-1} + D_t. \quad (9.110)$$

The interest rate is specified by the period in which the investment or borrowing took place. Since the borrowing $B_{t-1}$ took place in period $t-1$, the interest rate is $r_{t-1}$. Notice that the government pays the same interest rate $r_{t-1}$ that the households receive for their investments.

Moving forward, it is not the size of debt that matters, but rather the per-capita debt. The debt level may rise, but we are interested in determining if the debt level is rising faster or slower than the population size. Define the per-capita debt level as

$$b_t = \frac{B_t}{N_{t-1} + N_t}. \quad (9.111)$$

Notice that the total population in period $t$ is equal to $N_{t-1} + N_t$, namely the size of the cohorts born in periods $t-1$ and $t$. In a similar fashion, the per-capita deficit and per-capita government spending can be defined:

$$d_t = \frac{D_t}{N_{t-1} + N_t}, \quad (9.112)$$

$$g_t = \frac{G_t}{N_{t-1} + N_t}. \quad (9.113)$$

In per-capita terms, the transition equation for government debt is updated to:

$$b_t (N_{t-1} + N_t) = (1 + r_{t-1}) b_{t-1} (N_{t-2} + N_{t-1}) + d_t (N_{t-1} + N_t). \quad (9.113)$$

Dividing by $N_{t-1} + N_t$ yields:

$$b_t = \frac{(1 + r_{t-1}) b_{t-1}}{1 + n} + d_t. \quad (9.114)$$

Observe that $N_{t-1} + N_t = (1 + n) (N_{t-2} + N_{t-1})$ as both $N_{t-1} = (1 + n) N_{t-2}$ and $N_t =
(1 + n) N_{t-1}.

### 9.3.4 Market clearing

Consider the roles of the two types of agents in the economy: households and government. Households are using the real bond to either save or borrow \((a_t > 0\) means the household is saving and \(a_t < 0\) means the household is borrowing). The government issues debt using the exact same real asset. The government is borrowing in order to cover the deficits that it is accruing. As the model only contains two types of agents, it must be that the households are lending to the government and (equivalently) the government is borrowing from the households.

The amount of net savings for each household is the variable \(a_t\). The government is borrowing \(b_t\) on average from each household alive in period \(t\). The only households saving in period \(t\) are the young households, namely those born in period \(t\). The fraction of young households to the whole population is \(\frac{N_t}{N_{t-1} + N_t} = \frac{1+n}{2+n}\). If the government is borrowing \(b_t\) on average from each household alive, then they are borrowing 0 from the old households and \(b_t \left(\frac{2+n}{1+n}\right)\) from each of the young households. Market clearing requires that total household savings equals total government borrowing, which we express in per-capita terms:

\[
a_t = b_t \left(\frac{2+n}{1+n}\right). \tag{9.115}
\]

### 9.3.5 Equilibrium properties

Household optimal choice of the real asset satisfies the Euler equation. The household utility maximization problem is given by:

\[
\begin{align*}
\max_{c_t^y, c_{t+1}^y, a_t} & \quad u \left( c_t^y, c_{t+1}^y \right) \\
\text{subj. to} & \quad p_t e_y - p_t c_t^y - p_t a_t \geq 0 \\
& \quad p_{t+1} e_o + (1 + r_t) p_{t+1} a_t - p_{t+1} c_{t+1} \geq 0
\end{align*} \tag{9.116}
\]

The budget constraints are always written as nonnegative inequalities.

Consider the Kuhn-Tucker conditions associated with this constrained maximization problem. Denote the Lagrange multiplier for the period \(\tau\) budget constraint for a household born in period \(t\) as \(\lambda_t^\tau\). The Kuhn-Tucker conditions are given by:

- **First order conditions:**
9.3. DEBT IN THE OVERLAPPING GENERATIONS MODEL

- With respect to consumption $c_t^t$:

$$D_1 u \left( c_t^t, c_{t+1}^t \right) - p_t \lambda_t^t = 0. \quad (9.117)$$

- With respect to consumption $c_{t+1}^t$:

$$D_2 u \left( c_t^t, c_{t+1}^t \right) - p_{t+1} \lambda_{t+1}^t = 0. \quad (9.118)$$

- With respect to real bond holding $a_t$:

$$-p_t \lambda_t^t + p_{t+1} \lambda_{t+1}^t (1 + r_t) = 0. \quad (9.119)$$

- Complimentary slackness conditions:

  - For the first budget constraint

$$\lambda_t^t \left\{ p_t e_y - p_t c_t^t - p_t a_t \right\} = 0. \quad (9.120)$$

  - For the second budget constraint

$$\lambda_{t+1}^t \left\{ p_{t+1} e_o + (1 + r_t) p_{t+1} a_t - p_{t+1} c_{t+1}^t \right\} = 0. \quad (9.121)$$

The convention is that $D_1 u \left( c_t^t, c_{t+1}^t \right)$ refers to the marginal utility with respect to the first consumption term $c_t^t$, and $D_2 u \left( c_t^t, c_{t+1}^t \right)$ refers to the marginal utility with respect to the second consumption term $c_{t+1}^t$. The Euler equation is obtained by combining the three first order conditions:

$$D_1 u \left( c_t^t, c_{t+1}^t \right) = (1 + r_t) D_2 u \left( c_t^t, c_{t+1}^t \right). \quad (9.122)$$

Define the function

$$\sigma \left( 1 + r_t \right) = \left\{ a_t : D_1 u \left( c_t^t, c_{t+1}^t \right) = (1 + r_t) D_2 u \left( c_t^t, c_{t+1}^t \right) \right\}. \quad (9.123)$$

The function $\sigma$ maps from the real interest rate (taken as given by the household) into the optimal choice of real asset. The function is strictly increasing, namely an increase in the real interest rate will lead to an increase in household savings. Since the function is strictly increasing, then it is an invertible function.

An end-of-chapter exercise asks you to verify that $\sigma^{-1} \left( 0 \right) = \frac{D_1 u (e_y, e_o)}{D_2 u (e_y, e_o)}$. 
With the function $\sigma$, the per-capita debt law of motion is given by:

$$b_t = \frac{\sigma^{-1}(a_{t-1}) b_{t-1}}{1 + n} + d_t. \tag{9.124}$$

Using the market clearing condition, the per-capita transition equation can be written only in terms of the per-capita government debt:

$$b_t = \frac{\sigma^{-1} \left( \frac{b_{t-1} (2+n)}{1+n} \right) b_{t-1}}{1 + n} + d_t. \tag{9.125}$$

Suppose the per-capita government spending is stationary, meaning $g_t = g$ in all time periods. From the government budget constraint, the per-capita deficit is also stationary, meaning $d_t = d$ in all time periods. Define the equilibrium updating function $\phi$ such that

$$\phi (b_{t-1}) = \frac{\sigma^{-1} \left( \frac{b_{t-1} (2+n)}{1+n} \right) b_{t-1}}{1 + n} + d.$$

In any equilibrium, given the value for $b_{t-1}$, the updating function $\phi$ determines the value for $b_t$.

A stationary equilibrium is one in which $b_t = b_{t-1}$, namely that the per-capita debt level is constant over time. Thus a stationary equilibrium satisfies two requirements: (i) $b_t = \phi (b_{t-1})$ and (ii) $b_t = b_{t-1}$.

To find the stationary equilibria, it is essential to know the properties of the updating function $\phi$. The y-intercept of the updating function $\phi$ is equal to $d$. The updating function $\phi$ is a strictly increasing function of $b_{t-1}$. To see this, notice that an increase in $b_{t-1}$ will increase the product $\sigma^{-1} \left( \frac{b_{t-1} (2+n)}{1+n} \right)$ as both $\sigma^{-1} \left( \frac{b_{t-1} (2+n)}{1+n} \right)$ and $b_{t-1}$ will increase. The former increases as the real interest rate is directly related to $b_{t-1}$; namely an increase in $b_{t-1}$ means that households are saving more and this is only consistent with an increase in the real interest rate.

Consider the derivative $D\phi (b_{t-1})$. Using the chain rule,

$$D\phi (b_{t-1}) = \frac{\sigma^{-1} \left( \frac{b_{t-1} (2+n)}{1+n} \right) b_{t-1}}{1 + n} + \frac{b_{t-1}}{(2 + n) D\sigma \left( \sigma^{-1} \left( \frac{b_{t-1} (2+n)}{1+n} \right) \right)}.$$

Recall the property from the Inverse Function Theorem that the derivative of the inverse is equal to the inverse of the derivative. It is straightforward to observe that $D\phi (b_{t-1}) > 0$, meaning that $\phi$ is a strictly increasing function of $b_{t-1}$. 
At the y-axis, we already know \( \phi(0) = d \). The derivative

\[
D\phi(0) = \frac{\sigma^{-1}(0)}{1 + n} + \frac{0}{(2 + n) D\sigma(\sigma^{-1}(b_{t-1}\left(\frac{2+n}{1+n}\right)))} = \frac{\sigma^{-1}(0)}{1 + n}.
\]

Since the end-of-chapter exercise verified that \( \sigma^{-1}(0) = \frac{D_1u(e_y, e_o)}{D_2u(e_y, e_o)} \), then \( D\phi(0) = \frac{1}{1+n} \frac{D_1u(e_y, e_o)}{D_2u(e_y, e_o)} \).

For the analysis moving forward, we focus on economies for which \( D\phi(0) < 1 \). Recalling the chapter "Money in the Overlapping Generations Model", \( D\phi(0) < 1 \) when the endowment allocation is not Pareto efficient. If \( D\phi(0) \geq 1 \), the endowment allocation is Pareto efficient and no level of debt \( d > 0 \) is sustainable.

It can also be verified that \( \phi \) is a strictly convex function of \( b_{t-1} \).

Figure 9.1.1 contains a plot of \( \phi(b_{t-1}) \) and the 45-degree line. On the x-axis is \( b_{t-1} \) and on the y-axis is \( b_t \). Notice that the function \( \phi \) has a strictly positive intercept (equal to \( d \)) and is both strictly increasing and strictly convex. Recall that a stationary equilibrium satisfies two requirements: (i) \( b_t = \phi(b_{t-1}) \) and (ii) \( b_t = b_{t-1} \). In Figure 9.1.1, a stationary equilibrium is any point at which the function \( \phi \) intersects the 45-degree line (along the 45-degree line, \( b_t = b_{t-1} \), by definition).

As shown in Figure 9.1.1, the function \( \phi \) intersects the 45-degree line twice. The first intersection is labeled \( b_{low} \) and the second intersection is labeled \( b_{high} \). The first intersection corresponds to a low debt level and a low interest rate, while the second intersection corresponds to a high debt level and a high interest rate. Out of the two stationary equilibria, only one of them is stable. To see which one is stable, suppose that the debt level \( b_{t-1} = b_{high} - \epsilon \) for a very small value of \( \epsilon \). To determine the value for \( b_t \), find the value \( \phi(b_{t-1}) \). Notice that \( \phi(b_{t-1}) \) lies below the 45-degree line, so \( \phi(b_{t-1}) < b_{t-1} \). The current debt level is \( b_t \) and we want to use the same figure to find the value for the next debt level \( b_{t+1} \). Locate \( b_t \) on the x-axis, which is achieved by drawing a horizontal dashed line from \( \phi(b_{t-1}) \) to the 45-degree line. Knowing the value for \( b_t \), find the value \( b_{t+1} = \phi(b_t) \) on the y-axis. The value \( b_{t+1} \) on the x-axis is found by moving horizontally to the 45-degree line. Continuing this process, we observe that the sequence of debt levels converges to \( b_{low} \). The debt level \( b_{low} \) is a stable stationary equilibrium and the debt level \( b_{high} \) is an unstable stationary equilibrium.

### 9.3.6 Policy analysis

With the two stationary equilibria \( b_{low} \) and \( b_{high} \), we can gather the facts about debt transitions when the function \( \phi \) appears as in Figure 9.1.1:
1. If the initial debt level $b_{t-1} \leq b_{\text{low}}$, the sequence of debt levels converges to $b_{\text{low}}$.

2. If the initial debt level $b_{\text{low}} \leq b_{t-1} < b_{\text{high}}$, the sequence of debt levels converges to $b_{\text{low}}$.

3. If the initial debt level $b_{t-1} = b_{\text{high}}$, the sequence stays constant at the level $b_{\text{high}}$.

4. If the initial debt level $b_{t-1} > b_{\text{high}}$, the sequence of debt levels diverges to $\infty$.

The level $b_{\text{high}}$ is the effective debt limit as any debt above that level is a level of debt that cannot be sustained in the long run.

We focus attention on economies for which $d > 0$, as government debt only arises when governments run a deficit. With $d > 0$, there are two possibilities for the function $\phi$. The function $\phi$ can be as in Figure 9.1.1 in which $\phi$ intersects the 45-degree line twice, or $\phi$ can never intersect the 45-degree line. The latter possibility occurs for large values of $d$. The latter possibility also occurs for all values of $d > 0$ when $D\phi (0) \geq 1$. If the function $\phi$ never intersects the 45-degree line, then there is only one fact about the sequence of debt levels:

1. For any initial debt level, the sequence of debt levels diverges to $\infty$.

Consider the comparative statics exercise of an increase in the level of government spending. Suppose that $d$ increases, but the function $\phi$ continues to intersect the 45-degree line twice. An increase in $d$ means an increase in $b_{\text{low}}$ and a decrease in $b_{\text{high}}$. The increase in $b_{\text{low}}$ means that the stable stationary equilibrium now has a higher debt level and a higher interest rate. The decrease in $b_{\text{high}}$ means that the effective debt limit has decreased. Any initial debt levels above the lower value of $b_{\text{high}}$ will lead to a sequence of debt levels that diverges to $\infty$.

Consider the comparative statics exercise of an increase in the population growth rate. An increase in $n$ means that $D\phi (0) = \frac{1}{1+n} \frac{D_1 u(e_y, e_o)}{D_2 u(e_y, e_o)}$ decreases. The same holds true for $D\phi (b_{t-1})$. The slope of $\phi (b_{t-1})$ is smaller with a higher population growth rate. This leads to a decrease in $b_{\text{low}}$ and an increase in $b_{\text{high}}$. The decrease in $b_{\text{low}}$ means that the stable stationary equilibrium now has a lower debt level and a lower interest rate. The increase in $b_{\text{high}}$ means that the effective debt limit has increased. Higher population growth leads to a stable stationary equilibrium with a lower interest rate and a higher effective debt limit.

The following policy questions focus on economies for which the function $\phi$ intersects the 45-degree line twice. We know that this requires that the economy satisfy: $D\phi (0) < 1$ and

\[ \text{The knife-edge case in which } \phi \text{ intersects the 45-degree line only once is extremely rare as it occurs with probability zero.} \]
9.4. EXERCISES  

$ d$ not 'too' big. The policy questions require you to use Figure 9.1.1. You are asked to work through these policy questions as end-of-the-chapter exercises.

Policy question 1: Suppose that the initial debt level $b_{t-1} > b_{\text{high}}$. As a function of $b_{t-1}$, determine the size of emergency debt relief (the government is able to receive a loan from an outside party in order to reduce its debt level) required to reach a sustainable debt level.

Policy question 2: Suppose that the initial debt level $b_{t-1} > b_{\text{high}}$. Suppose that the government immediately cuts spending to a new level $d' < d$. Determine the size of the cut in the government spending $d - d'$ required to reach a sustainable debt level.

From policy questions 1 and 2, is it clear which option works best: (i) emergency debt relief only, (ii) a cut in government spending only, or (iii) a combination of both?

9.4 Exercises

1. **Tax revenue**

   Solve for the steady state of the neoclassical growth model with investment tax for the following parameter values: $\tau = 20\%$, $\theta = \frac{1}{3}$, $\beta = 0.96$, and $\delta = 0.09$.

2. **Tax revenue**

   For the revenue-maximizing tax rate $\tau^*$ in the neoclassical growth model with investment tax, verify that $\tau^* = \frac{1-\theta}{\delta}$.

3. **Tax revenue**

   Solve for the steady state of the neoclassical growth model with income tax for the following parameter values: $\tau = 20\%$, $\theta = \frac{1}{3}$, $\beta = 0.96$, and $\delta = 0.09$.

4. **Tax revenue**

   Solve for the revenue-maximizing tax rate $\tau^*$ of the neoclassical growth model with income tax for the following parameter values: $\theta = \frac{1}{3}$, $\beta = 0.96$, and $\delta = 0.09$.

5. **Tax revenue**

   For the utility-maximizing tax rate $\tau^{**}$ in the neoclassical growth model with investment tax, verify that $\tau^{**} = \frac{\alpha}{\alpha+1} \left( \frac{1-\theta}{\delta} \right)$.

6. **Tax revenue**
Solve for the utility-maximizing tax rate \( \tau^{**} \) of the neoclassical growth model with income tax for the following parameter values: \( \theta = \frac{1}{3}, \beta = 0.96, \delta = 0.09, \) and \( \alpha = 1. \)

7. Time-consistent fiscal policy

In the capital income tax model under full commitment, solve for the investment \( k_{t+1} \) as a function of the output \( k_t^\theta \). To accomplish this, you should use the guess-and-check the policy function algorithm applied to the Euler equation.

8. Time-consistent fiscal policy

Calculate the optimal tax rate with commitment for the following economic parameters: \( \theta = \frac{1}{3}, \beta = 0.96, \) and \( \alpha = 0.5. \)

9. Time-consistent fiscal policy

Given the optimal solutions for \( (c_T, g_T) \) and \( (c_{T-1}, g_{T-1}) \), solve for the optimal solutions for \( (c_{T-2}, g_{T-2}) \).

10. Time-consistent fiscal policy

Calculate the optimal tax rate without commitment (time-consistent policy) for the following economic parameters: \( \theta = \frac{1}{3}, \beta = 0.96, \) and \( \alpha = 0.5. \)

11. Time-consistent fiscal policy

Given the expression for \( \tau^{NC} \) and the definition of the function

\[
\Psi (\tau) = -\frac{\theta (1-\beta)}{1-\theta \beta - \tau \theta (1-\beta)} \frac{\alpha}{\tau} - \frac{\theta \beta (1+\alpha)}{1-\theta \beta},
\]

verify that \( \Psi (\tau^{NC}) < 0. \)

12. Debt in the overlapping generations model

Recall the function \( \sigma (1 + r_t) = \{a_t : D_1 u (c_t^t, c_{t+1}^t) = (1 + r_t) D_2 u (c_t^t, c_{t+1}^t) \} \). Verify that \( \sigma^{-1} (0) = \frac{D_1 u (e_y, e_o)}{D_2 u (e_y, e_o)}. \)

13. Debt in the overlapping generations model

Using Figure 9.1.1., suppose that the initial debt level \( b_{t-1} > b_{\text{high}} \). As a function of \( b_{t-1} \), determine the size of emergency debt relief (the government is able to receive a loan from an outside party in order to reduce its debt level) required to reach a sustainable debt level.
14. *Debt in the overlapping generations model*

Using Figure 9.1.1., suppose that the initial debt level \(b_{t-1} > b_{\text{high}}\). Suppose that the government immediately cuts spending to a new level \(d' < d\). Determine the size of the cut in the government spending \(d - d'\) required to reach a sustainable debt level.
Bibliography


Part V

New Keynesian Theory
10

New Keynesian Monetary Theory

10.1 Introducing the model

10.1.1 Sneak peek

Summary

The New Keynesian model contains nominal price rigidities so that a change in monetary policy can have real effects on equilibrium output. The model consists of both households and firms. Firms operate in a setting of monopolistic competition. This is the same setting introduced in the chapter "Endogenous Technological Change." In such a setting, firms earn positive profit by internalizing household demand into their price-setting profit maximization problem. Households solve a dynamic optimization problem in which they optimally choose the mix of consumption-savings in order to maximize their lifetime utility function. The household problem is very similar to what has been previously introduced and can be characterized by an Euler equation.

Firms play a much more important role in the New Keynesian model than they have in previous models. For simplicity, firms only use one factor of production (labor) as an input into their production function. The total cost for a firm is easy to determine as it is simply the wage rate multiplied by the amount of labor required to produce a desired output level. The marginal cost is the derivative of the total cost function with respect to output. The marginal cost is the key variable that captures the heterogeneity among firms as a result of the nominal price rigidities.
Notation

The variables to be introduced in this section are given in the following table:

- $C(t)$ consumption index in period $t$ (household)
- $C_i(t)$ consumption of the output of firm $i$
- $N(t)$ labor supplied by household in period $t$
- $N_i(t)$ amount of labor supplied to firm $i$
- $Y(t)$ total output (across all firms)
- $Y_i(t)$ output of firm $i$ in period $t$, $Y_i(t) = C_i(t)$
- $P(t)$ price index in period $t$
- $P_i(t)$ price charged by firm $i$
- $D(t)$ dividends received by households
- $A(t)$ technology available to all firms
- $W(t)$ wage rate for household labor
- $i(t)$ nominal interest rate

The parameters to be introduced in this section are given in the following table:

- $\beta$ discount factor
- $\sigma$ coefficient of relative risk aversion (household utility)
- $\gamma$ labor elasticity (household utility)
- $\epsilon$ elasticity of substitution (household consumption index)
- $\alpha$ labor share (firm production function)

Main takeaways

After completing this section, you will be able to answer the following questions:

- What is the problem for the household?
- What is the problem for the firms?

10.1.2 Household preferences

The economy consists of an infinite number of discrete time periods $t \in \{0, 1, 2, \ldots\}$. The agents in the model are a representative household and a unit mass of firms that each produce a different variety of a single good.
The representative household maximizes the objective function
\[
E_0 \sum_{t=0}^{\infty} \beta^t U (C(t), N(t)) .
\] (10.1)

The uncertainty in this model is due to monetary policy shocks and technology shocks. The expectation \(E_0\) takes into account the probabilities and all possible shocks that can be realized. The consumption index in period \(t\) is given by \(C(t)\) and the labor supply is \(N(t)\).

The economy contains a continuum of firms indexed by \(i \in [0,1]\) (mass is equal to 1), each producing a different variety. The consumption index is defined over this continuum of varieties:
\[
C(t) = \left( \int_0^1 C_i(t)^{-\frac{1}{\epsilon}} di \right)^{-\frac{1}{\epsilon}},
\] (10.2)
where the elasticity of substitution for the household is \(\epsilon > 1\).

The price of the output of firm \(i\) is \(P_i(t)\). The output prices are taken as given by the household. The household faces a nested optimization problem. The larger optimization problem is a dynamic consumption-savings problem. The smaller optimization problem determines the optimal ratio of varieties for a given consumption index value \(C(t)\). From the chapter "Endogenous Technological Change," the first order conditions with respect to \((C_i(t))_{i \in [0,1]}\) reveal that:
\[
\left( \frac{C_i(t)}{C_j(t)} \right) = \left( \frac{P_i(t)}{P_j(t)} \right)^{-\epsilon} \forall i, j \in [0,1].
\] (10.3)

The price index \(P(t)\) is defined such that:
\[
C_i(t) = \left( \frac{P_i(t)}{P(t)} \right)^{-\epsilon} C(t).
\] (10.4)

With a unit mass of varieties, if \(C_i(t) = C_j(t)\) for all varieties, then \(C(t) = C_i(t)\) for all varieties.

The utility function will have the following functional form:
\[
U (C(t), N(t)) = \frac{(C(t))^{1-\sigma}}{1-\sigma} - \frac{(N(t))^{1+\gamma}}{1+\gamma}.
\] (10.5)

The coefficient of relative risk aversion is the parameter \(\sigma > 1\). The labor elasticity is the
parameter $\gamma > 0$.

### 10.1.3 Household budget constraint

Recalling the chapter "Endogenous Technological Change," the price index is defined as:

$$P(t) = \left( \int_0^1 P_i(t)^{1-\gamma} di \right)^{\frac{1}{1-\gamma}}. \quad (10.6)$$

Additionally, the total consumption expenditure is:

$$P(t)C(t) = \left( \int_0^1 P_i(t)C_i(t) di \right). \quad (10.7)$$

The households in this model have access to a risk-free nominal bond. The household budget constraint is given by:

$$P(t)C(t) + \frac{B(t)}{1+i(t)} \leq B(t-1) + W(t)N(t) + D(t). \quad (10.8)$$

The bond position obtained in period $t$ (with payout in period $t+1$) is denoted $B(t)$. The nominal interest rate in period $t$ is $i(t)$. The wage rate is $W(t)$. The labor market is competitive, so the household takes the wage rate as given. The labor supplied by the household, a choice variable for the household, is denoted $N(t)$. Households receive dividends $D(t)$ from their ownership of the firms.

### 10.1.4 Firms’ problem

There is only one factor of production for firms: labor. The production functions for the firms are symmetric:

$$Y_i(t) = A(t) \left( N_i(t) \right)^{1-\alpha}. \quad (10.9)$$

The output of firm $i$ in period $t$ is $Y_i(t)$. The technology that is commonly available to all firms in period $t$ is $A(t)$. The technology is subject to shocks each period. The amount of labor hired by firm $i$ in period $t$ is $N_i(t)$.

The firm profit function is:

$$P_i(t)Y_i(t) - W(t)N_i(t). \quad (10.10)$$
10.1. INTRODUCING THE MODEL

The labor markets are competitive, so the firms take the wage rate $W(t)$ as given. Firms operate in a setting of monopolistic competition, meaning that firm $i$ acts as a monopolist for its output $Y_i(t)$ and chooses the output price $P_i(t)$ after internalizing the household demand function $Y_i(t) = C_i(t) = \left( \frac{P_i(t)}{P(t)} \right)^{-\epsilon} C(t)$.

10.1.5 Market clearing

Market clearing in the commodity markets requires that

$$Y_i(t) = C_i(t) \text{ for all } i \in [0, 1]. \quad (10.11)$$

Total output of the economy (average output across all firms, with a unit mass of firms) is given by:

$$Y(t) = \int_0^1 Y_i(t)di. \quad (10.12)$$

The relation between total output $Y(t)$ and the consumption index $C(t)$ is given by:

$$Y(t) = \int_0^1 Y_i(t)di = \int_0^1 \left( \frac{P_i(t)}{P(t)} \right)^{-\epsilon} C(t)di = C(t) \int_0^1 \left( \frac{P_i(t)}{P(t)} \right)^{-\epsilon} di = C(t) \Delta_p(t), \quad (10.13)$$

where $\Delta_p(t) = \int_0^1 \left( \frac{P_i(t)}{P(t)} \right)^{-\epsilon} di$ is a measure of price dispersion and the value is equal to 1 in steady state.

Market clearing in the labor markets requires that:

$$N(t) = \int_0^1 N_i(t)di. \quad (10.14)$$

Assuming that the ownership of firms is entirely in the private sector, then firms are owned only by households and not by the government. This means that $D(t) = \int_0^1 D_i(t)di$, where

$$D_i(t) = \Pi_i(t) = P_i(t)Y_i(t) - W(t)N_i(t) \quad (10.15)$$
is the profit of firm $i$ in period $t$. This implies that

$$D(t) = \int_0^1 P_i(t) Y_i(t) di - W(t) \int_0^1 N_i(t) di.$$ 

From market clearing ($Y_i(t) = C_i(t)$ and $N(t) = \int_0^1 N_i(t) di$) and the definition of the price index:

$$D(t) = P(t) C(t) - W(t) N(t).$$

From the household budget constraint this implies:

$$\frac{B(t)}{1 + i(t)} = B(t - 1). \quad (10.16)$$

Higher interest rate $i(t)$ leads to higher bond holdings $B(t)$.

Bond market clearing requires that the total bond holdings of households $B(t)$ must equal to the government debt. Governments lean against the wind, meaning that they have high debt when the interest rate is high (recession) and low debt when the interest rate is low (expansion), which is counter to what a self-interested agent would choose. We do not model the government budget constraints explicitly. The government uses open market operations to jointly choose debt and the nominal interest rate. We assume that the government can achieve whatever interest rate it targets, and it does so through a Taylor rule (to be introduced shortly). For any interest rate chosen by the government, the debt level can be solved using the government budget constraints. This debt level must satisfy the bond market clearing condition. Walras’ Law ensures that this is so: if all other markets clear, then the household budget constraint implies that the bond market clearing holds automatically.

### 10.1.6 Steady state

The equilibrium equations in the New Keynesian model are obtained as log-linear deviations around the steady state. In theory, this is a local approximation so any predictions can only be considered "reasonable" if the equilibrium is close to the steady state. The steady state has zero inflation. With zero inflation, firms choose the price $P_i(t) = P_{ss}$ for all $i \in [0, 1]$. 
10.2 Household optimization

10.2.1 Sneak peek

Summary

As we have just seen, the household problem is to maximize an infinite discounted utility function subject to a series of budget constraints being satisfied in every period. This is really no different than the household problem in the real business cycle model, with the exception that investment occurs via a risk-free bond instead of capital stock. The households are price-takers and take both the interest rate and the wage rate as given. The household problem is a constrained maximization problem in which the household chooses three variables: consumption, bond holdings, and labor supply. To solve this problem, we write down equilibrium equations for bond holdings and labor supply. The consumption choice can then be determined such that the budget constraint holds with equality.

The equilibrium equation for bond holdings is an Euler equation and captures the dynamic trade-off between current and future consumption. The solution to the Euler equation is the optimal consumption-savings blend for the household. The equilibrium equation for labor is the first order condition, which captures the labor-leisure trade-off. The first order condition for labor requires that the marginal cost of labor equals the marginal benefit of labor. The marginal cost is determined from the utility function as households receive a disutility from labor. The marginal benefit is equal to the wage rate multiplied by the utility value of an extra unit of income. The utility value of an extra unit of income is the marginal utility of consumption.

The Euler equation is a nonlinear equilibrium equation. The steady state was previously introduced and occurs at zero inflation. The Euler equation is log-linearized around the steady state. This process results in a linear equilibrium equation, albeit one that only approximates the true equilibrium equation. The linear approximation works "well" when the equilibrium inflation rate is close to the steady state inflation value of zero. The log-linearized Euler equation is sometimes called the dynamic IS equation, in homage to the IS curve from the original Keynesian theory.
10. NEW KEYNESIAN MONETARY THEORY

Notation

The variables to be introduced in this section are given in the following table:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Pi(t) )</td>
<td>inflation rate ( \Pi(t) = \frac{P(t)}{P(t-1)} )</td>
</tr>
<tr>
<td>( y(t) )</td>
<td>( y(t) = \ln(Y(t)) ), the natural log of the output.</td>
</tr>
<tr>
<td>( p(t) )</td>
<td>( p(t) = \ln(P(t)) ), the natural log of price</td>
</tr>
<tr>
<td>( \pi(t) )</td>
<td>( \pi(t) = \ln(\Pi(t)) ), the natural log of inflation</td>
</tr>
<tr>
<td>( \delta &gt; 0 )</td>
<td>( \delta = -\ln(\beta) ), the discount rate</td>
</tr>
</tbody>
</table>

Main takeaways

After completing this section, you will be able to answer the following questions:

- What is the household Euler equation?
- After log-linearizing around the zero inflation steady state, what does the household Euler equation look like?

10.2.2 Equilibrium price equations

In the dynamic consumption-savings problem for the household, the first order condition with respect to \( C(t) \) is:

\[
\beta^t (C(t))^\gamma - \lambda(t)P(t) = 0, \tag{10.17}
\]

where \( \lambda(t) \) is the Lagrange multiplier associated with the budget constraint in period \( t \). A similar equation holds for period \( t + 1 \).

The first order condition with respect to \( N(t) \) is:

\[
-\beta^t (N(t))\gamma + \lambda(t)W(t) = 0. \tag{10.18}
\]

The first order condition with respect to the bond position \( B(t) \) is given by:

\[
-\frac{\lambda(t)}{1 + i(t)} + E_t \{ \lambda(t + 1) \} = 0. \tag{10.19}
\]

The expectation \( E_t \) captures that we are taking the expected value of \( \lambda(t + 1) \) given the shock we observe in period \( t \).
Combining the first order conditions with respect to $C(t)$, $C(t+1)$, and $B(t)$ results in the Euler equation:

$$
\frac{(C(t))^{-\sigma}}{P(t)} = \beta(1 + i(t))E_t \left\{ \frac{(C(t+1))^{-\sigma}}{P(t+1)} \right\}.
$$

(10.20)

Solving for the nominal interest rate yields:

$$
\frac{1}{1 + i(t)} = E_t \left\{ \beta \left( \frac{C(t+1)}{C(t)} \right)^{-\sigma} \frac{P(t)}{P(t+1)} \right\}.
$$

(10.21)

Combining the first order conditions with respect to $C(t)$ and $N(t)$ yields the wage rate equation:

$$
W(t) = P(t) (N(t))^\gamma (C(t))^{\sigma}.
$$

In steady state, $C(t) = Y(t)$. The labor supply $N(t)$ must satisfy the production function $Y(t) = A(t) (N(t))^{1-\alpha}$, meaning that $N(t) = \left( \frac{Y(t)}{A(t)} \right)^{\frac{1}{1-\alpha}}$. These two facts together allow us to update the wage rate equation:

$$
W(t) = P(t) \left( \frac{1}{A(t)} \right)^{\frac{\gamma}{1-\alpha}} (Y(t))^{\frac{\gamma}{1-\alpha} + \sigma}.
$$

(10.22)

### 10.2.3 Household Euler equation

The Euler equation from (10.20) can be expressed as:

$$
(C(t))^{-\sigma} = \beta(1 + i(t))E_t \left( (C(t+1))^{-\sigma} \frac{P(t)}{P(t+1)} \right).
$$

(10.23)

Let’s introduce the discount rate $\delta = -\ln (\beta)$ . Additionally, let’s make the approximation

$$
i(t) = \ln (1 + i(t)).
$$

(10.24)

The inflation rate is defined as $\Pi(t+1) = \frac{P(t+1)}{P(t)}$. The updated Euler equation is given by:

$$
(C(t))^{-\sigma} = \exp[-\delta + i(t)]E_t \left( (C(t+1))^{-\sigma} \frac{1}{\Pi(t+1)} \right).
$$

(10.25)

The process of log-linearization around the zero inflation steady state simply involves taking the natural log of equation (10.25) and using the fact that $C(t) = Y(t)$ in steady state. We adopt the following notation for the natural logs of the variables in the previous
342

10. NEW KEYNESIAN MONETARY THEORY

equation:

\[ y(t) = \ln(Y(t)). \]
\[ \pi(t) = \ln(\Pi(t)). \]

Using the property of natural log, we know that \( \ln \{ (Y(t))^{-\sigma} \} = -\sigma y(t) \). The process of log-linearization around the zero inflation steady state leads to the linear equation:

\[ -\sigma y(t) = -\delta + i(t) - \sigma E_t \{ y(t + 1) \} - E_t \{ \pi(t + 1) \}. \] (10.26)

Dividing the Euler equation by \(-\sigma\) yields:

\[ y(t) = E_t \{ y(t + 1) \} - \frac{1}{\sigma} (i(t) - E_t \{ \pi(t + 1) \} - \delta). \] (10.27)

This is sometimes referred to as the dynamic IS equation, drawing a connection to the IS equation from the IS-LM model inspired by Keynes.

10.3 Firm optimization

10.3.1 Sneak peek

Summary

Firms in the New Keynesian model also have a dynamic optimization problem. Firms maximize profit by choosing the price to sell output at. The model contains nominal price rigidities that may prevent a firm from changing its price in every period. Given that a firm may not be able to change its price every period, it must maximize the discounted future profit, where the discount factor includes the nominal interest rate as in net present value discounting and the probability that the firm is unable to change its price in each period in the future. The first order condition for the firm’s problem can be found. The solution to this first order condition is the price that a firm will charge for its output in the current period.

Recall that there are a continuum (of unit mass) of firms, each producing a different variety of the good. The overall effect on the aggregate price level is determined by the fraction of firms that change their price in any period and the amount that they decide to change it. In this model, inflation and expected inflation play an important and symbiotic
role. If firms expect high inflation in the future, then the possibility of not being able to change their prices in the future raises concern about lost profit in the future. This leads firms to preemptively raise their prices in the current period when they have the opportunity to do so. High expected inflation leads to high current inflation. The high expected inflation can then be self-fulfilling as the economy moves to the next period.

The outcome of firm optimization is an equilibrium equation called the New Keynesian Phillips curve. The New Keynesian Phillips curve is a linear relation (obtained using the same method of log-linearization around the steady state as implemented for the household Euler equation) that relates the current inflation to future expected inflation and the average markup firms charge over marginal cost. The New Keynesian Phillips predicts a direct relation between firm output and the inflation rate, a property that is similar to the inverse relation between the unemployment rate and the inflation rate in the original Phillips curve.

**Notation**

The variables to be introduced in this section are given in the following table:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q^k(t)$</td>
<td>$k$–period discount factor (beginning in period $t$)</td>
</tr>
<tr>
<td>$P^*(t)$</td>
<td>price chosen by firm allowed to reset its price</td>
</tr>
<tr>
<td>$Y^k(t)$</td>
<td>output for firm in $t+k$ if last price reset was in $t$</td>
</tr>
<tr>
<td>$MC_k^k(t)$</td>
<td>marginal cost to produce $Y^k(t)$</td>
</tr>
<tr>
<td>$MC(Y)$</td>
<td>marginal cost for each firm to produce $Y$</td>
</tr>
<tr>
<td>$p^*(t)$</td>
<td>$p^<em>(t) = \ln (P^</em>(t))$</td>
</tr>
<tr>
<td>$mc(t+k)$</td>
<td>$mc(t+k) = \ln (MC(Y(t+k)))$</td>
</tr>
<tr>
<td>$\mu$</td>
<td>$\mu = \ln \left( \frac{\epsilon}{\epsilon-1} \right)$, the markup under flexible prices</td>
</tr>
<tr>
<td>$\theta$</td>
<td>probability that a firm cannot change its price in each period</td>
</tr>
<tr>
<td>$\Theta$</td>
<td>a combined parameter, $\Theta = \frac{1-\alpha}{1-\alpha+\alpha\epsilon}$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>a combined parameter, $\lambda = \frac{(1-\theta)(1-\theta\beta)}{\theta} \Theta$</td>
</tr>
</tbody>
</table>

**Main takeaways**

After completing this section, you will be able to answer the following questions:

- Why is it useful to take log-linear approximations around the steady state in the New Keynesian model?
• The New Keynesian Phillips curve is derived as the optimal solution to which problem in the model?

• According to the New Keynesian Phillips curve, which variables affect the inflation rate in the current period?

10.3.2 Nominal rigidities

Each firm may reset its price with probability $1 - \theta$ in each period. This is how the model captures nominal price rigidities.

Under monopolistic competition, each firm selects the price at which it will sell its output. If the firm is permitted to change its price every period, it will choose the price to solve a one-period profit maximization problem. In the current setup, however, since a firm is unable to change its price every period, it will solve a dynamic problem. Firms only have a problem to solve in periods in which they are allowed to change their price. This dynamic problem takes into account the following probabilities:

<table>
<thead>
<tr>
<th>Price chosen in current period unchanged for</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>at least the next period</td>
<td>$\theta$</td>
</tr>
<tr>
<td>at least the next 2 periods</td>
<td>$\theta^2$</td>
</tr>
<tr>
<td>at least the next 3 periods</td>
<td>$\theta^3$</td>
</tr>
<tr>
<td>\vdots</td>
<td></td>
</tr>
<tr>
<td>at least the next $k$ periods</td>
<td>$\theta^k$</td>
</tr>
</tbody>
</table>

As a result of the nominal rigidities, the price $P_i(t)$ charged by firm $i$ can take one of two values:

1. If firm $i$ does not reset its price, $P_i(t) = P_i(t - 1)$.

2. If firm $i$ resets its price, $P_i(t) = P^*(t)$, where $P^*(t)$ is the price set by all firms that reset their prices in period $t$. As in the "Endogenous Technological Change" chapter, since the production function is identical for all firms, the firms that change their price will always choose the same price $P^*(t)$. 
10.3. FIRM OPTIMIZATION

10.3.3 Writing the dynamic profit maximization problem

Define the 1-period discount factor as \( q^1(t) = \frac{1}{1+i(t)} \). We can also define the \( k \)-period discount factor as \( q^k(t) = \left( \frac{1}{1+i(t)} \right) \times \cdots \times \left( \frac{1}{1+i(t+k-1)} \right) \). Using the nominal interest rate equation (10.21):

\[
q^k(t) = E_t \left\{ \beta^k \left( \frac{C(t+k)}{C(t)} \right)^{\frac{-1}{\alpha}} \frac{P(t)}{P(t+k)} \right\}.
\]

The firm profit in period \( t + k \) (with a price chosen in period \( t \)) is

\[
P_i(t)Y_i(t + k) - W(t + k)N_i(t + k).
\]

From the firm production function, labor demand equals \( N_i(t + k) = \left( \frac{Y_i(t+k)}{A(t+k)} \right)^{\frac{1}{1-\alpha}} \). If a firm resets its price in period \( t \), it will choose the price \( P_i(t) \) to solve the following dynamic optimization problem:

\[
\max_{P_i(t)} \sum_{k=0}^{\infty} \theta^k E_t \left\{ q^k(t) \left( P_i(t)Y_i(t + k) - W(t + k) \left( \frac{Y_i(t+k)}{A(t+k)} \right)^{\frac{1}{1-\alpha}} \right) \right\},
\]

while internalizing the household demand function:

\[
Y_i(t + k) = C_i(t + k) = \left( \frac{P_i(t)}{P(t+k)} \right)^{-\frac{1}{\alpha}} C(t + k).
\]

Recall that \( \theta \) is the probability that the firm cannot reset its price in any given period, so \( \theta^k \) is the probability that a firm is stuck with the price \( P_i(t) \) in period \( t + k \). The term \( q^k(t) \) is the \( k \)-period discount factor.

10.3.4 Firms’ marginal costs

The production cost for firm \( i \) is the cost of hiring the labor required to produce the output \( Y_i(t + k) \). The labor market is competitive, meaning that each firm takes the wage rate \( W(t + k) \) as given. The cost of producing \( Y_i(t + k) \) units is equal to \( W(t + k) \left( \frac{Y_i(t+k)}{A(t+k)} \right)^{\frac{1}{1-\alpha}} \). Taking the derivative yields the marginal cost of production:

\[
MC(Y_i(t + k)) = \frac{1}{1-\alpha} W(t + k) \left( \frac{Y_i(t+k)}{A(t+k)} \right)^{\frac{-1}{1-\alpha}} \left( \frac{1}{A(t+k)} \right)^{\frac{1}{1-\alpha}}.
\]
10.3.5 First order condition

The first order condition with respect to $P_i(t)$ is given by:

$$\sum_{k=0}^{\infty} \theta^k E_t \left\{ q^k(t) Y_i(t+k) \left( P_i(t) - \frac{\epsilon}{\epsilon - 1} MC_i(Y_i(t+k)) \right) \right\} = 0, \quad (10.34)$$

after using the definition of the marginal cost $MC_i(Y_i(t+k))$ equation from (10.33).

Observe that in a model without nominal rigidities ($\theta = 0$, meaning a firm can change its price every period), the optimal solution is simply:

$$P_i(t) = \frac{\epsilon}{\epsilon - 1} MC_i(Y_i(t)).$$

The markup is $\frac{\epsilon}{\epsilon - 1} > 1$. A model without nominal rigidities is also referred to as a flexible price model. The flexible price outcome will be a benchmark that we return to later in the chapter.

All firms that last reset their price in period $t$ will have identical marginal cost in period $t + k$, which is denoted as $MC^k(t)$. All firms that last reset their price in period $t$ will therefore choose identical output, which is denoted as $Y^k(t)$. The marginal cost $MC^k(t) = MC(Y^k(t))$. Recall that a firm that resets its price chooses $P_i(t) = P^*(t)$. Using these conventions, the first order conditions are given by:

$$\sum_{k=0}^{\infty} \theta^k E_t \left\{ q^k(t) Y^k(t) \left( P^*(t) - \frac{\epsilon}{\epsilon - 1} MC^k(t) \right) \right\} = 0. \quad (10.35)$$

10.3.6 Average marginal cost

The average marginal cost $MC(Y(t+k))$ is the marginal cost required to produce the average output $Y(t+k)$. Using the marginal cost expression (10.33):

$$\frac{MC^k(t)}{MC(Y(t+k))} = \left( \frac{Y^k(t)}{Y(t+k)} \right)^{\frac{\alpha}{1-\alpha}}. \quad (10.36)$$

The demand function $Y^k(t) = \left( \frac{P^*(t)}{P(t+k)} \right)^{-\epsilon} Y(t+k)$ implies that this ratio is given by:

$$\frac{MC^k(t)}{MC(Y(t+k))} = \left( \frac{P(t+k)}{P^*(t)} \right)^{\frac{\alpha}{1-\alpha}}. \quad (10.37)$$
Returning to the first order conditions (10.35), replace $MC^k(t)$ with the equivalent expression from (10.37):

$$\sum_{k=0}^{\infty} \theta^k E_t \left\{ q^k(t) Y^k(t) \left( P^*(t) - \frac{\epsilon}{\epsilon - 1} MC (Y(t + k)) \left( \frac{P(t + k)}{P^*(t)} \right)^{\frac{\alpha}{1 - \alpha}} \right) \right\} = 0. \tag{10.38}$$

Multiply all terms by $P^*(t)^{\frac{\alpha}{1 - \alpha}}$:

$$\sum_{k=0}^{\infty} \theta^k E_t \left\{ q^k(t) Y^k(t) \left( P^*(t)^{\frac{1 - \alpha + \alpha \epsilon}{1 - \alpha}} - \frac{\epsilon}{\epsilon - 1} MC (Y(t + k)) (P(t + k))^{\frac{\alpha}{1 - \alpha}} \right) \right\} = 0. \tag{10.39}$$

Then multiply each term $\frac{\epsilon}{\epsilon - 1} MC (Y(t + k)) (P(t + k))^{\frac{\alpha}{1 - \alpha}}$ by $\frac{P(t + k)}{P^*(t)}$:

$$\sum_{k=0}^{\infty} \theta^k E_t \left\{ q^k(t) Y^k(t) \left( P^*(t)^{\frac{1 - \alpha + \alpha \epsilon}{1 - \alpha}} - \frac{\epsilon}{\epsilon - 1} MC (Y(t + k)) (P(t + k))^{\frac{1 - \alpha + \alpha \epsilon}{1 - \alpha}} \right) \right\} = 0. \tag{10.40}$$

Bring the summation $\sum_{k=0}^{\infty} \theta^k E_t \left\{ q^k(t) Y^k(t) \frac{\epsilon}{\epsilon - 1} MC (Y(t + k)) (P(t + k))^{\frac{1 - \alpha + \alpha \epsilon}{1 - \alpha}} \right\}$ to the right-hand side of the equation:

$$\sum_{k=0}^{\infty} \theta^k E_t \left\{ q^k(t) Y^k(t) P^*(t) \frac{1 - \alpha + \alpha \epsilon}{1 - \alpha} \right\} = \sum_{k=0}^{\infty} \theta^k E_t \left\{ q^k(t) Y^k(t) \frac{\epsilon}{\epsilon - 1} MC (Y(t + k)) (P(t + k))^{\frac{1 - \alpha + \alpha \epsilon}{1 - \alpha}} \right\}. \tag{10.41}$$

### 10.3.7 Log-linearization around the steady state

In steady state, $Y^k(t) = Y_{ss}$ (constant across all time periods) and $q^k(t) = \beta^k$ (from (10.29)). Define $p^*(t) = \ln (P^*(t))$, $p(t + k) = \ln (P(t + k))$, $mc(t + k) = \ln (MC (Y(t + k)))$, and $\mu = \ln \left( \frac{\epsilon}{\epsilon - 1} \right)$. Log-linearizing the equation (10.41) around the zero inflation steady state leads to the following equation:

$$\left( \frac{1 - \alpha + \alpha \epsilon}{1 - \alpha} \right) \sum_{k=0}^{\infty} (\theta \beta)^k p^*(t) = \sum_{k=0}^{\infty} (\theta \beta)^k E_t \{ \mu + mc(t + k) - p(t + k) \} \tag{10.42}$$

$$+ \left( \frac{1 - \alpha + \alpha \epsilon}{1 - \alpha} \right) \sum_{k=0}^{\infty} (\theta \beta)^k E_t \{ p(t + k) \}.$$
The variable \( p^*(t) \) does not depend on \( k \) and can be factored out of the summation \( \sum_{k=0}^{\infty} (\theta \beta)^k \). The remaining infinite sum \( \sum_{k=0}^{\infty} (\theta \beta)^k = \frac{1}{1-\theta \beta} \). Define \( \Theta = \frac{1-\alpha}{1-\alpha+\alpha \epsilon} \). Algebraically, equation (10.42) can be written as an infinite sum for \( p^*(t) \):

\[
p^*(t) = (1 - \theta \beta) \Theta \sum_{k=0}^{\infty} (\theta \beta)^k E_t \{\mu + mc(t+k) - p(t+k)\}
+ (1 - \theta \beta) \sum_{k=0}^{\infty} (\theta \beta)^k E_t \{p(t+k)\}.
\]

Observe that the infinite sum (10.43) can be written recursively as the following difference equation:

\[
p^*(t) = (1 - \theta \beta) \Theta (\mu + mc(t) - p(t)) + (1 - \theta \beta) p(t) + \theta \beta E_t \{p^*(t+1)\}.
\] (10.44)

### 10.3.8 Law of large numbers

From the law of large numbers, the fraction \( 1 - \theta \) of firms set the price \( P^*(t) \) in period \( t \) and the remaining fraction \( \theta \) (since they are chosen at random) have average price equal to the previous period’s price index \( P(t-1) \). Therefore, the price index from (10.6) is given by:

\[
P(t) = \left[ \theta (P(t-1))^{1-\epsilon} + (1 - \theta) (P^*(t))^{1-\epsilon} \right]^{\frac{1}{1-\epsilon}}.
\] (10.45)

Raising both sides of the equation to the exponent \( 1 - \epsilon \) and then dividing all terms by \( P(t-1) \) provides an expression for \( \Pi(t) \):

\[
(\Pi(t))^{1-\epsilon} = \theta + (1 - \theta) \left( \frac{P^*(t)}{P(t-1)} \right)^{1-\epsilon}.
\] (10.46)

By definition, \( \pi(t) = \ln(\Pi(t)) = p(t) - p(t-1) \). Log-linearizing the equation (10.46) around the zero inflation steady state leads to:

\[
(1 - \epsilon) \pi(t) = (1 - \theta) (1 - \epsilon) (p^*(t) - p(t-1))
\] (10.47)
Solving for \( p^*(t) \) leads to:

\[
p^*(t) = p(t - 1) + \frac{\pi(t)}{1 - \theta}.
\] (10.48)

### 10.3.9 New Keynesian Phillips curve

Use the inflation rate equation (10.48) to insert \( p^*(t) = p(t - 1) + \frac{\pi(t)}{1 - \theta} \) and \( E_t \{ p^*(t + 1) \} = p(t) + E_t \left\{ \frac{\pi(t+1)}{1-\theta} \right\} \) into the difference equation (10.44):

\[
p(t - 1) + \frac{\pi(t)}{1 - \theta} = (1 - \theta \beta) \Theta (\mu + mc(t) - p(t)) + (1 - \theta \beta) p(t) + \theta \beta \left( p(t) + E_t \left\{ \frac{\pi(t+1)}{1-\theta} \right\} \right).
\] (10.49)

By definition, \( \pi(t) = p(t) - p(t - 1) \), implying

\[
\frac{\pi(t)}{1 - \theta} = (1 - \theta \beta) \Theta (\mu + mc(t) - p(t)) + \pi(t) + \theta \beta E_t \left\{ \frac{\pi(t+1)}{1-\theta} \right\}.
\] (10.50)

Solving the equation for \( \pi(t) \) yields the following equation:

\[
\pi(t) = \beta E_t \{ \pi(t + 1) \} - \frac{(1 - \theta)(1 - \theta \beta)}{\theta} \Theta (p(t) - mc(t) - \mu).
\] (10.51)

Define the parameter \( \lambda = \frac{(1-\theta)(1-\theta \beta)}{\theta} \Theta \). We have arrived at the New Keynesian Phillips curve:

\[
\pi(t) = \beta E_t \{ \pi(t + 1) \} - \lambda (p(t) - mc(t) - \mu).
\] (10.52)

The term \( p(t) - mc(t) \) is the average price markup (in equilibrium) and the term \( \mu \) is the price markup in the flexible price setting.

The key observation about the New Keynesian Phillips curve is that it determines inflation as a function of the expected inflation next period and the difference between the equilibrium price markup and the flexible price markup. Inflation today is caused by firms’ decisions to set prices higher in future periods. Firms know that they cannot change their prices in every period. This inability to change prices every period means that firms are adversely affected by inflation in the future. If \( E_t \{ \pi(t + 1) \} \) is high, then \( \pi(t) \) will be high too.

The model has a stabilization mechanism involving the markups. The firms have an incentive to choose markups as close as possible to the flexible price markup \( \mu \). If \( p(t) - mc(t) > \mu \), then the equilibrium markups are too high. Firms respond by decreasing their price, which
increases demand, increases labor costs, and increases the marginal costs (marginal costs are strictly increasing in labor). The aggregate effect is a lower inflation rate $\pi(t)$ and a markup closer to the flexible price markup $\mu$. If $p(t) - mc(t) < \mu$, then the markups are too low. Firms respond by increasing their price, which decreases demand, decreases labor costs, and decreases the marginal costs. The markup increases. The aggregate effect is a high inflation rate $\pi(t)$ and a markup closer to the flexible price markup $\mu$.

### 10.3.10 The two key equations

The two key equations for the New Keynesian model are:

1. Household Euler equation (dynamic IS equation)

$$ y(t) = E_t \{y(t + 1)\} - \frac{1}{\sigma} (i(t) - E_t \{\pi(t + 1)\} - \delta). \quad (10.53) $$

2. New Keynesian Phillips curve

$$ \pi(t) = \beta E_t \{\pi(t + 1)\} - \lambda (p(t) - mc(t) - \mu). \quad (10.54) $$

### 10.4 Output gap

#### 10.4.1 Sneak peek

**Summary**

The household Euler equation and the New Keynesian Phillips curve are the two main equations that characterize an equilibrium in the New Keynesian model. There are two main variables of interest: output and inflation.

We will soon close the model by specifying a monetary policy rule. This rule is called a Taylor rule and is an equation that determines the nominal interest rate as a function of other contemporaneous variables in the model (notably, inflation). When specifying the Taylor rule, it is more useful to speak in terms of the output gap instead of the output itself.

The output gap is the difference between the natural log of the equilibrium output level and the natural log of the flexible price output level. The flexible price output level is called the natural output level. Using the properties of natural log, the output gap is approximately equal to the percent difference between the equilibrium output level and the natural output.
level. Since nominal rigidities lead to reduced output, the output gap has a negative value. An increase in the output gap means that the gap between the output level and the natural output level is shrinking and getting closer to zero.

**Notation**

The variables to be introduced in this section are given in the following table:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y^n(t)$</td>
<td>natural output level, output when $\theta = 0$</td>
</tr>
<tr>
<td>$\bar{y}(t)$</td>
<td>output gap, defined as $\bar{y}(t) = y(t) - y^n(t)$</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>combined parameter, defined as $\kappa = \lambda \left( \frac{\gamma + \alpha}{1 - \alpha} + \sigma \right)$</td>
</tr>
<tr>
<td>$\phi_a^n$</td>
<td>slope coefficient for the natural output equation $y^n(t) = \phi^n + \phi_a^n a(t)$</td>
</tr>
<tr>
<td>$\phi^n$</td>
<td>intercept for the natural output equation $y^n(t) = \phi^n + \phi^n a(t)$</td>
</tr>
</tbody>
</table>

**Main takeaways**

After completing this section, you will be able to answer the following questions:

- In terms of the output gap, what are the two equilibrium equations of the New Keynesian model?
- Why is it useful to express these equations in terms of the output gap?

**10.4.2 Output gap definition**

Define the natural level of output $Y^n(t)$ as the equilibrium level of output under flexible prices ($\theta = 0$). This is the output level that occurs in a setting without nominal rigidities: $P^n(t) = \frac{\kappa}{\kappa - 1} MC(Y^n(t))$. Define the natural log variables $y^n(t) = \ln(Y^n(t))$, $p^n(t) = \ln(P^n(t))$, and $mc^n(t) = \ln(MC(Y^n(t)))$. Recall that $\mu = \ln \left( \frac{\kappa}{\kappa - 1} \right)$. The natural output is such that $\frac{P^n(t)}{MC^n(t)} = \frac{\kappa}{\kappa - 1}$, which log-linearizes to $p^n(t) - mc^n(t) = \mu$.

There is an important difference between the steady state and the natural output level. The one steady state of the model is the zero inflation steady state. By definition, the shocks are turned off for the steady state, so the outcome is only determined by the monopolistic competition price distortion. The natural output level has both the monopolistic price distortion and also the effects of the values of the shock changing over time, but (by definition) not the distortion from the nominal price rigidities.
The difference between the equilibrium output level \( y(t) \) and the natural output level \( y^n(t) \) is the output gap:

\[
\ddot{y}(t) = y(t) - y^n(t).
\]  

(10.55)

The output gap has negative values. Ideally the economy wants to achieve an output gap \( \ddot{y}(t) = 0 \). When the output gap \( \ddot{y}(t) \) increases, the difference between \( y(t) \) and \( y^n(t) \) actually decreases, which is better for the economy as equilibrium output is closer to the natural output level.

### 10.4.3 Marginal cost log linearization

Having determined the wage rate equation (10.22), we can now insert this into the firm’s marginal cost equation (10.33) to obtain:

\[
MC(Y(t)) = \frac{1}{1 - \alpha} \left\{ P(t) \left( \frac{1}{A(t)} \right)^{\frac{\gamma}{1-\alpha}} (Y(t))^{\frac{\gamma}{1-\alpha} + \sigma} \right\} (Y(t))^{\frac{\alpha}{1-\alpha}} \left( \frac{1}{A(t)} \right)^{\frac{1}{1-\alpha}}.
\]  

(10.56)

After combining the terms involving \( \left( \frac{1}{A(t)} \right) \) and the terms involving \( Y(t) \), we obtain

\[
MC(Y(t)) = \frac{1}{1 - \alpha} P(t) \left( \frac{1}{A(t)} \right)^{\frac{\gamma + \alpha}{1-\alpha}} (Y(t))^{\frac{\gamma + \alpha}{1-\alpha} + \sigma}.
\]  

(10.57)

The process of log-linearization around the steady state is equivalent to taking the natural log of both sides:

\[
mc(t) = -\ln(1 - \alpha) + p(t) + \left( \frac{\gamma + \alpha}{1 - \alpha} + \sigma \right) y(t) - \left( \frac{1 + \gamma}{1 - \alpha} \right) a(t).
\]  

(10.58)

The natural log variables were previously defined, with the exception of \( a(t) = \ln(A(t)) \), the natural log of technology.

### 10.4.4 Natural output properties

The marginal cost equation (10.58) must be satisfied at the natural output level. An end-of-chapter exercise asks you to find coefficients \( (\phi^n, \phi^{a}) \) such that the natural output equation is given by:

\[
y^n(t) = \phi^n + \phi^{a} a(t).
\]  

(10.59)
By definition, steady state output is such that $a(t) = 0$, so $y_{ss} = \phi^n$.

### 10.4.5 Output gap and the household Euler equation

An end-of-chapter exercise asks you to use the natural output equation (10.59) to rewrite the household Euler equation in terms of the output gap. The result is as follows:

$$\ddot{y}(t) = E_t \{\ddot{y}(t+1)\} - \frac{1}{\sigma} (i(t) - E_t \{\pi(t+1)\} - \delta - \sigma \phi_a^n E_t \{a(t+1) - a(t)\}). \quad (10.60)$$

### 10.4.6 Output gap and New Keynesian Phillips curve

An end-of-chapter exercise asks you to use the marginal cost equation, both (10.58) and evaluated under the flexible price setting, to rewrite the New Keynesian Phillips curve in terms of the output gap. The result is as follows:

$$\pi(t) = \beta E_t \{\pi(t+1)\} + \kappa \ddot{y}(t), \quad (10.61)$$

where the new parameter $\kappa = \lambda \left(\frac{\gamma + \alpha}{1-\alpha} + \sigma\right)$.

### 10.4.7 The two key equations with the output gap

The two key equations (in terms of the output gap) are:

1. Household Euler equation

$$\ddot{y}(t) = E_t \{\ddot{y}(t+1)\} - \frac{1}{\sigma} (i(t) - E_t \{\pi(t+1)\} - \delta - \sigma \phi_a^n E_t \{a(t+1) - a(t)\}). \quad (10.62)$$

2. New Keynesian Phillips curve

$$\pi(t) = \beta E_t \{\pi(t+1)\} + \kappa \ddot{y}(t). \quad (10.63)$$

It is sometimes helpful to define the natural real rate of interest

$$r^n(t) = \delta + \sigma \phi_a^n E_t \{a(t+1) - a(t)\}, \quad (10.64)$$
so that the household Euler equation can be written simply as:

\[ \tilde{\gamma}(t) = E_t \{ \tilde{\gamma}(t + 1) \} - \frac{1}{\sigma} (i(t) - E_t \{ \pi(t + 1) \} - r^n(t)). \] (10.65)

10.5 

Effects of a monetary policy and technology shock

10.5.1 

Sneak peek

Summary

The nominal interest rate enters the household Euler equation. Households must form beliefs about the value of the nominal interest rate. The nominal interest rate is chosen according to a monetary policy rule. The form of monetary policy rule adopted in this section is commonly referred to as a Taylor rule. This section considers the simplest possible Taylor rule in which the nominal interest rate is an affine function of the inflation rate. The Taylor principle requires that the nominal interest rate responds positively and more than 1:1 to a change in the inflation rate. The Taylor principle is a condition required to guarantee a unique solution to the system of equilibrium equations.

With the Taylor rule, the system of two equations, consisting of the household Euler equation and the New Keynesian Phillips curve, can be written in recursive matrix form. It is then possible to quantify how a shock will affect the equilibrium variables of interest, the output gap and the inflation rate. There are two possible shocks in this simple model: a monetary policy shock and a technology shock. For each shock, the effect on the output gap and the inflation rate can be quantified. Additional variables of interest include the output and the labor supply. The effect of a shock on these variables can also be quantified.

The computations in this section are tractable given that we are dealing with a linear system of equilibrium equations. We can solve a recursive system of linear equations using the tools of matrix algebra. With the linear system, the equilibrium variables will be linear functions of the shocks. This linear analysis gives us an approximation of the effects of a shock to the true equilibrium system. If the economy is "close" to the zero inflation steady state, then the approximation is very good.
10.5. EFFECTS OF A MONETARY POLICY AND TECHNOLOGY SHOCK

Notation

The variables to be introduced in this section are given in the following table:

- $\phi_\pi$: Taylor rule coefficient
- $\nu(t)$: monetary policy shock
- $\rho_\nu$: persistence of monetary policy shock
- $\epsilon_\nu(t)$: error term for the monetary policy shock
- $\rho_a$: persistence of technology shock
- $\epsilon_a(t)$: error term for the technology shock

Main takeaways

After completing this section, you will be able to answer the following questions:

- What form of monetary policy is needed in order to close the model, and what is the significance of the Taylor principle?

- Why is it appropriate to guess that the equilibrium variable are linear functions of the shocks in the model, either a monetary policy shock or a technology shock?

- What effect does a monetary policy shock have on the inflation, output gap, output, and labor supply?

- What effect does a technology shock have on the inflation, output gap, output, and labor supply?

10.5.2 Taylor rule

The monetary authority is assumed to adopt an interest rate rule of the following form:

$$i(t) = \delta + \phi_\pi \pi(t) + \nu(t), \quad (10.66)$$

where the parameter $\phi_\pi > 1$ and $\nu(t)$ is the monetary policy shock (with zero mean). This is the simplest version of the Taylor rule. The fact that $\phi_\pi > 1$ is referred to as the Taylor principle. It requires that any change in the inflation rate must be answered with an even larger change in the nominal interest rate.

Insert the interest rate rule into the household Euler equation. This implies that the two equilibrium equations are given by:
1. Household Euler equation

\[ \ddot{y}(t) = E_t \{ \ddot{y}(t + 1) \} - \frac{1}{\sigma} (\phi_{\pi} \pi(t) - E_t \{ \pi(t + 1) \} - \sigma \phi_{\alpha}^n E_t \{ a(t + 1) - a(t) \} - \nu(t)) . \]  

(10.67)

2. New Keynesian Phillips curve

\[ \pi(t) = \beta E_t \{ \pi(t + 1) \} + \kappa \ddot{y}(t) . \]  

(10.68)

Multiply the household Euler equation by \( \sigma \) (both sides) and rearrange both equations to have the terms involving \( \ddot{y}(t) \) and \( \pi(t) \) on the left-hand side:

1. Household Euler equation

\[ \sigma \ddot{y}(t) + \phi_{\pi} \pi(t) = \sigma E_t \{ \ddot{y}(t + 1) \} + E_t \{ \pi(t + 1) \} + (\sigma \phi_{\alpha}^n E_t \{ a(t + 1) - a(t) \} - \nu(t)) . \]  

(10.69)

2. New Keynesian Phillips curve

\[ \pi(t) - \kappa \ddot{y}(t) = \beta E_t \{ \pi(t + 1) \} . \]  

(10.70)

These two equations can be written in matrix form as:

\[
A_0 \begin{pmatrix} \ddot{y}(t) \\ \pi(t) \end{pmatrix} = A_1 \begin{pmatrix} E_t \{ \ddot{y}(t + 1) \} \\ E_t \{ \pi(t + 1) \} \end{pmatrix} + B (\sigma \phi_{\alpha}^n E_t \{ a(t + 1) - a(t) \} - \nu(t)) ,
\]

(10.71)

where \( A_0 \) and \( A_1 \) are \( 2 \times 2 \) matrices and \( B \) is a \( 2 \times 1 \) vector given as follows:

\[
A_0 = \begin{bmatrix} \sigma & \phi_{\pi} \\ -\kappa & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} \sigma & 1 \\ 0 & \beta \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

(10.72)

There exists a unique solution to this system of equations provided that \( \phi_{\pi} > 1 \). This is why the Taylor principle is so important.

### 10.5.3 Effects of a monetary policy shock

This section focuses on the effects of a monetary policy shock. For that reason, we shut off the technology shock, meaning that \( a(t) \) is constant. The monetary policy shock is an \( AR(1) \)
process
\[ \nu(t) = \rho_\nu \nu(t - 1) + \epsilon_\nu(t), \] (10.73)

where \( \rho_\nu \in (0, 1) \) and \( \epsilon_\nu(t) \) is a random shock drawn from a zero-mean normal distribution. If the shock \( \epsilon_\nu(t) > 0 \), then the monetary policy is contractionary as the nominal interest rate \( i(t) = \delta + \phi_\pi \pi(t) + \nu(t) \) is high. If the shock \( \epsilon_\nu(t) < 0 \), then the monetary policy is expansionary as the nominal interest rate \( i(t) \) is low. Due to the AR(1) process, monetary policy shocks are persistent.

To solve for the effects of the monetary policy shock, we guess that the output gap \( \tilde{y}(t) \) and inflation rate \( \pi(t) \) satisfy:

\[ E_t \{ \tilde{y}(t + 1) \} = \rho_\nu \tilde{y}(t). \] (10.74)
\[ E_t \{ \pi(t + 1) \} = \rho_\nu \pi(t). \]

This occurs if the output gap and inflation rate are linear functions of the monetary policy shock \( \nu(t) \), a property that we soon verify.

The system of equations with constant technology is

\[ A_0 \begin{pmatrix} \tilde{y}(t) \\ \pi(t) \end{pmatrix} = A_1 \begin{pmatrix} E_t \{ \tilde{y}(t + 1) \} \\ E_t \{ \pi(t + 1) \} \end{pmatrix} + B (-\nu(t)), \] (10.75)

where the matrices \( A_0, A_1, \) and \( B \) were introduced in the previous subsection. Notice that without the technology shock, \( E_t \{ a(t + 1) - a(t) \} = 0 \).

Given our guess, the system of equations is

\[ A_0 \begin{pmatrix} \tilde{y}(t) \\ \pi(t) \end{pmatrix} = A_1 \begin{pmatrix} \rho_\nu \tilde{y}(t) \\ \rho_\nu \pi(t) \end{pmatrix} + B (-\nu(t)). \] (10.76)

Matrix algebra allows us to solve for the variables \( \begin{pmatrix} \tilde{y}(t) \\ \pi(t) \end{pmatrix} \) as a function of the shock \( \nu(t) \):

\[ \begin{pmatrix} \tilde{y}(t) \\ \pi(t) \end{pmatrix} = -(A_0 - \rho_\nu A_1)^{-1} B \nu(t). \] (10.77)
An end-of-chapter exercise asks you to show that

\[(A_0 - \rho_\nu A_1)^{-1} B = \frac{1}{\sigma (1 - \rho_\nu) (1 - \rho_\nu \beta) + \kappa (\phi_\pi - \rho_\nu)} \left( \frac{1 - \rho_\nu \beta}{\kappa} \right). \quad (10.78)\]

Given this result, the equilibrium response functions are given by:

\[
\begin{pmatrix} \tilde{y}(t) \\ \pi(t) \end{pmatrix} = -\frac{\nu(t)}{\sigma (1 - \rho_\nu) (1 - \rho_\nu \beta) + \kappa (\phi_\pi - \rho_\nu)} \left( \frac{1 - \rho_\nu \beta}{\kappa} \right). \quad (10.79)
\]

We already know that \(\sigma > 0\) and \(\kappa > 0\). The terms \((1 - \rho_\nu \beta) > 0\), \((1 - \rho_\nu) > 0\), and \((\phi_\pi - \rho_\nu) > 0\), the latter holding since the Taylor principle requires \(\phi_\pi > 1\). Therefore \(\frac{\partial \tilde{y}(t)}{\partial \nu(t)} < 0\) and \(\frac{\partial \pi(t)}{\partial \nu(t)} < 0\).

Since \(\tilde{y}(t) < 0\), a contractionary monetary policy (an increase in \(\nu(t)\)) means that the output gap \(\tilde{y}(t)\) decreases, meaning it lies even further away from 0.

Additionally, a contractionary monetary policy shock results in a decrease in the inflation rate \(\pi(t)\). In this model, the nominal interest rate \(i(t)\) and the inflation rate \(\pi(t)\) are inversely related. If the central bank misses its nominal interest rate target by setting \(i(t)\) too high (an increase in the monetary policy shock \(\nu(t)\)), the inflation rate \(\pi(t)\) will fall.

Moreover, all of the effects of a monetary policy shock are persistent, since the shock follows an AR(1) process, and the effects are more persistent for higher values of \(\rho_\nu\).

**Effect on output**

The output is given by:

\[y(t) = y^n(t) + \tilde{y}(t).\quad (10.80)\]

We wish to evaluate the derivative

\[
\frac{\partial y(t)}{\partial \nu(t)} = \frac{\partial y^n(t)}{\partial \nu(t)} + \frac{\partial \tilde{y}(t)}{\partial \nu(t)}. \quad (10.81)
\]

Recall that the expression for the natural output level was found to be \(y^n(t) = \phi^n + \phi^n_a(t)\) from an end-of-chapter exercise. The natural output level is independent of the monetary policy. Thus, \(\frac{\partial y^n(t)}{\partial \nu(t)} = 0\), meaning that

\[
\frac{\partial y(t)}{\partial \nu(t)} = \frac{\partial \tilde{y}(t)}{\partial \nu(t)} = -\frac{(1 - \rho_\nu \beta)}{\sigma (1 - \rho_\nu) (1 - \rho_\nu \beta) + \kappa (\phi_\pi - \rho_\nu)} < 0. \quad (10.82)
\]
This implies that a contractionary monetary policy shock (an increase in $\nu(t)$) will decrease output $y(t)$.

**Effect on labor supply**

From the production function, the labor supply is given by $N(t)^{1-\alpha} = \frac{Y(t)}{A(t)}$, so the natural log of the labor supply $n(t) = \ln(N(t))$ satisfies:

$$
(1 - \alpha) n(t) = y(t) - a(t).
$$

We wish to evaluate the derivative

$$
\frac{\partial n(t)}{\partial \nu(t)} = \frac{1}{1 - \alpha} \left\{ \frac{\partial y(t)}{\partial \nu(t)} - \frac{\partial a(t)}{\partial \nu(t)} \right\}.
$$

By definition, $\frac{\partial a(t)}{\partial \nu(t)} = 0$, meaning that the derivative of labor supply is given by:

$$
\frac{\partial n(t)}{\partial \nu(t)} = \frac{1}{1 - \alpha} \left\{ \frac{\partial y(t)}{\partial \nu(t)} - 0 \right\}
= -\frac{1}{1 - \alpha} \left[ (1 - \rho_{\nu}\beta) \left(1 - \sigma(1 - \rho_{\nu})(1 - \rho_{\nu}\beta) + \kappa(\phi_{\pi} - \rho_{\nu}) \right) \right] < 0.
$$

This implies that a contractionary monetary policy shock (an increase in $\nu(t)$) will decrease labor supply $n(t)$.

**10.5.4 Effects of a technology shock**

This section focuses on the effects of a technology shock. For that reason, we shut off the monetary policy shock, meaning that

$$
\nu(t) = 0.
$$

The technology shock is an $AR(1)$ process

$$
a(t) = \rho_{a}a(t - 1) + \epsilon_{a}(t),
$$

where $\rho_{a} \in (0, 1)$ and $\epsilon_{a}(t)$ is a random shock drawn from a zero-mean normal distribution. Now that we have specified the process for the technology shock, we can go ahead and solve
for the equilibrium variables as a function of the technology shock $a(t)$.

Recall the system of equations (without the monetary policy shock):

$$A_0 \begin{pmatrix} \tilde{y}(t) \\ \pi(t) \end{pmatrix} = A_1 \begin{pmatrix} E_t \{\tilde{y}(t + 1)\} \\ E_t \{\pi(t + 1)\} \end{pmatrix} + B (\sigma \phi_n E_t \{a(t + 1) - a(t)\}). \quad (10.88)$$

Given the $AR(1)$ process for the technology shock, the term

$$\sigma \phi^n_a E_t \{a(t + 1) - a(t)\} = \sigma \phi^n_a E_t \{\rho_a a(t) - a(t)\} \quad (10.89)$$

$$= -\sigma \phi^n_a (1 - \rho_a) a(t),$$

since $\epsilon_a(t)$ has zero mean.

As in the previous subsection, we guess that the solution $(\tilde{y}(t), \pi(t))$ satisfy

$$E_t \{\tilde{y}(t + 1)\} = \rho_a \tilde{y}(t). \quad (10.90)$$

$$E_t \{\pi(t + 1)\} = \rho_a \pi(t).$$

Two end-of-chapter exercises ask you to complete the analysis. After solving these exercises, we observe that an increase in technology $a(t)$ will have two effects. First, higher technology will increase output $y(t)$ (even though the output gap becomes smaller and further away from 0). This occurs because both the output level and the natural output level increase, with a larger increase for the natural output level (hence a fall in the output gap). Second, higher technology will decrease the labor supply $n(t)$. With the simple production function used in this model, technology and labor are substitutes, meaning that technology replaces labor in the production process. More technology means lower labor demand.

### 10.6 Exercises

1. **Output gap**

   The marginal cost equation

   $$mc(t) = -\ln(1 - \alpha) + p(t) + \left(\frac{\gamma + \alpha}{1 - \alpha} + \sigma\right) y(t) - \left(\frac{1 + \gamma}{1 - \alpha}\right) a(t)$$

   must be satisfied at the natural output level characterized by $p^*(t) = \mu + mc^*(t)$. Use these equations to express the natural output level $y^*(t)$ as a linear function of the
technology \( a(t) \):

\[ y^n(t) = \phi^n + \phi^n_a a(t). \]

What are the expressions for the coefficients \( \phi^n_a \) and \( \phi^n \)? Show that \( \phi^n_a \in (0, 1) \) (using the assumption that \( \sigma > 1 \)).

2. Output gap

Use the natural output equation \( y^n(t) = \phi^n + \phi^n_a a(t) \) to rewrite the household Euler equation in terms of the output gap.

3. Output gap

Use the marginal cost equation

\[ mc(t) = -\ln(1 - \alpha) + p(t) + \left( \frac{\gamma + \alpha}{1 - \alpha} + \sigma \right) y(t) - \left( \frac{1 + \gamma}{1 - \alpha} \right) a(t), \]

which must be satisfied at the natural output level characterized by \( p^n(t) = \mu + mc^n(t) \), to rewrite the New Keynesian Phillips curve in terms of the output gap.

4. Effects of a monetary policy and technology shock

Given the matrices

\[
A_0 = \begin{bmatrix} \sigma & \phi_\pi \\ -\kappa & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} \sigma & 1 \\ 0 & \beta \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\]

show that

\[
(A_0 - \rho_\nu A_1)^{-1} B = \frac{1}{\sigma (1 - \rho_\nu) (1 - \rho_\nu \beta) + \kappa (\phi_\pi - \rho_\nu)} \begin{bmatrix} 1 - \rho_\nu \beta \\ \kappa \end{bmatrix}.
\]

5. Effects of a monetary policy and technology shock

For a positive technology shock (an increase in \( a(t) \)), will the output gap \( \bar{y}(t) \) increase or decrease? What about the inflation rate \( \pi(t) \)? For this question, use the fact that \( \phi^n_a \in (0, 1) \).

6. Effects of a monetary policy and technology shock
For a positive technology shock (an increase in $a(t)$), will the output $y(t)$ increase or decrease? What about the labor supply $n(t)$? For this question, use the fact that $\phi_n^a \in (0, 1)$. 
Bibliography


11

New Keynesian Labor Market Theory

11.1 Extending the model

11.1.1 Sneak peek

Summary

This chapter extends the New Keynesian model from the previous chapter to include predictions about how monetary policy affects the unemployment rate. In the New Keynesian model introduced in the previous chapter, labor market frictions were not included. Aggregate employment could be calculated knowing the production function and the output of the economy, but there was no sense in which some fraction of the households would be employed and the remaining fraction would be unemployed.

To incorporate unemployment, an additional nominal rigidity will be introduced. We maintain the setting introduced in the previous chapter in which firms face a nominal price rigidity in the setting of the price that they sell output at. In this chapter, I introduce a setting in which households form labor unions and face a nominal wage rigidity in the setting of the wage that they are willing to work for. There are now two price rigidities in the model: one for the nominal price of output and one for the wage rate. Consequently, there are two elasticities of substitution: one for households on the product markets as they consume firm output and one for firms on the labor markets as they demand labor. There are also two inflation levels: one for the commodity price level and one for the wage rate.
Notation

The variables to be introduced in this section are given in the following table:

- $N(t)$: aggregate employment
- $N^d(t)$: aggregate labor demand index
- $N_i(t)$: labor demand index for firm $i$
- $N_{i,j}(t)$: labor supplied by type $j$ laborer to firm $i$
- $N_j(t)$: fraction of type $j$ laborers employed
- $W_j(t)$: wage rate for type $j$ laborers
- $W(t)$: wage index
- $\Pi_w(t)$: wage inflation rate
- $\Xi(t)$: labor disutility parameter (all households)
- $\epsilon_p > 1$: elasticity of substitution (product markets)
- $\epsilon_w > 1$: elasticity of substitution (labor markets)

Main takeaways

After completing this section, you will be able to answer the following questions:

- What is the problem for the household?
- What is the problem for the firms?

11.1.2 Households

There are a continuum of households (of unit mass) that are differentiated according to the vector $(j, h)$. The index $j \in [0, 1]$ refers to the type of labor that a particular household specializes in. The index $h \in [0, 1]$ refers to a household's disutility from work. Households are uniformly distributed on the index $h \in [0, 1]$. The disutility from labor is defined by

$$\Xi(t)h^\gamma,$$

where $\Xi(t)$ is a preference shock that is common to all households and $\gamma > 0$ is the labor elasticity (as in the previous chapter). The distribution of disutilities will generate unemployment in the model. From the disutility function, notice that higher $h$ households are less likely to accept employment. Define $N_j(t)$ as the fraction of type $j$ households that are
employed in period $t$. By construction, the households $h \leq N_j(t)$ of type $j$ will be employed and the households $h > N_j(t)$ of type $j$ will be unemployed (recall that higher $h$ households are less willing to accept employment).

For each $j \in [0, 1]$, there are a continuum of households (distributed by $h$). All households $(j, h)_{h \in [0,1]}$ form a union in order to negotiate the wage rate $W_j(t)$ (as we will see) and the union requires that all members receive the exact same consumption (and implement transfers to support this). The households decide to join the union before their disutility parameter is realized. There is a single objective function for each labor union $j \in [0, 1] :

$$E_0 \sum_{t=0}^{\infty} \beta^t U (C_j(t), N_j(t)).$$ (11.2)

As in the previous chapter, the economy contains a continuum of firms indexed by $i \in [0, 1]$ (mass is equal to 1). The consumption index for type $j$ households is defined as:

$$C_j(t) = \left( \int_0^1 C_{i,j}(t)^{\epsilon_p^{-1} \gamma_p} \, dt \right)^{\frac{\epsilon_p}{\gamma_p}} ,$$ (11.3)

where the elasticity of substitution among product varieties is $\epsilon_p > 1$. Here, $C_{i,j}(t)$ is the consumption of firm $i$ output by a household of type $j$.

The utility function has the following functional form:

$$U (C_j(t), N_j(t)) = \frac{(C_j(t))^{1-\sigma}}{1 - \sigma} - \Xi(t) \int_0^{N_j(t)} h^\gamma dh.$$ (11.4)

The utility function is evaluated for a household of type $j \in [0, 1]$. The coefficient of relative risk aversion is the parameter $\sigma > 1$. Since $\int_0^{N_j(t)} h^\gamma dh = \frac{N_j(t)^{1+\gamma}}{1+\gamma}$, the utility function is updated as:

$$U \left( C_j(t), (N_j(t))_{j \in [0,1]} \right) = \frac{(C_j(t))^{1-\sigma}}{1 - \sigma} - \Xi(t) \frac{N_j(t)^{1+\gamma}}{1+\gamma}.$$ (11.5)

The functional form is identical to the one in the previous chapter.

The price of the output of firm $i$ is $P_i(t)$. The output prices are taken as given by households. The households face a nested optimization problem. The larger optimization problem is a dynamic wage setting and consumption-savings problem. The smaller optimization problem asks, for a given consumption index value $C_j(t)$, what is the optimal ratio of varieties chosen by the household? From the "Endogenous Technological Change" chapter, the first
order conditions with respect to \((C_{i,j}(t))_{i \in [0,1]}\) reveal that:

\[
\left( \frac{C_{i,j}(t)}{C_{k,j}(t)} \right) = \left( \frac{P_i(t)}{P_k(t)} \right)^{-\epsilon_p} \forall i, k \in [0,1].
\] (11.6)

The price index \(P(t)\) is defined such that:

\[
C_{i,j}(t) = \left( \frac{P_i(t)}{P(t)} \right)^{-\epsilon_p} C_j(t).
\] (11.7)

Given the definition of the price index, the price index is derived as (see the "Endogenous Technological Change" chapter):

\[
P(t) = \left( \int_0^1 P_i(t)^{1-\epsilon_p} di \right)^{\frac{1}{1-\epsilon_p}}.
\] (11.8)

Additionally, the total consumption expenditure is given by:

\[
P(t)C_j(t) = \int_0^1 P_i(t)C_{i,j}(t)di.
\] (11.9)

Similar to the previous chapter, the household budget constraint is given by:

\[
P(t)C_j(t) + \frac{B_j(t)}{1 + i(t)} \leq B_j(t - 1) + W_j(t)N_j(t) + D_j(t).
\] (11.10)

Here, \(W_j(t)\) is the wage earned by households of type \(j\) in period \(t\). In this model, the wage rate \(W_j(t)\) is set by the households of type \(j\) who pool together to form a union with the ability to set wages in labor contracts. Households receive dividends \(D_j(t)\) from the ownership of firms.

### 11.1.3 Firms

The economy contains a continuum of firms indexed by \(i \in [0,1]\). Denote \(N_{i,j}(t)\) as the number of type \(j\) laborers that work for firm \(i\) in period \(t\). The production function for firm \(i\) is given by:

\[
Y_i(t) = A(t) \left\{ \left( \int_0^1 N_{i,j}(t)^{\frac{\alpha - 1}{\epsilon_p - 1}} dj \right)^{-\frac{\epsilon_p}{\epsilon_p - 1}} \right\}^{1-\alpha},
\] (11.11)
where the labor share is \( 1 - \alpha \) and the elasticity of substitution among labor varieties is \( \epsilon_w > 1 \). As in the previous chapter, \( A(t) \) is a technology common to all firms. For simplicity, the labor demand index for firm \( i \in [0, 1] \) is defined by

\[
N_i(t) = \left( \int_0^1 N_{i,j}(t)^{\frac{\epsilon_w - 1}{\epsilon_w}} d\tilde{y} \right)^{\frac{\epsilon_w}{\epsilon_w - 1}}. \tag{11.12}
\]

Using the labor demand index definition, output of firm \( i \in [0, 1] \) is represented simply as:

\[
Y_i(t) = A(t)N_i(t)^{1-\alpha}. \tag{11.13}
\]

The wage rate for type \( j \) laborers is \( W_j(t) \). The wages are taken as given by the firms. Firm \( i \) acts as a monopolist for variety \( i \) and chooses the price \( P_i(t) \) after internalizing the household demand function. The firm profit function is given by:

\[
P_i(t)Y_i(t) - \int_0^1 W_j(t)N_{i,j}(t)d\tilde{y}. \tag{11.14}
\]

The firms face a nested optimization problem. The larger optimization problem is a dynamic price setting problem. The smaller optimization problem asks, for a given labor demand index \( N_i(t) \), what is the optimal ratio of labor varieties to minimize production cost \( \int_0^1 W_j(t)N_{i,j}(t)d\tilde{y} \)? Using the production cost \( \int_0^1 W_j(t)N_{i,j}(t)d\tilde{y} \) as the objective function with constraint \( N_i(t) = \left( \int_0^1 N_{i,j}(t)^{\frac{\epsilon_w - 1}{\epsilon_w}} d\tilde{y} \right)^{\frac{\epsilon_w}{\epsilon_w - 1}} \), the first order conditions with respect to \( (N_{i,j}(t))_{j \in [0,1]} \) reveal that:

\[
\left( \frac{N_{i,j}(t)}{N_i(t)} \right) = \left( \frac{W_j(t)}{W_k(t)} \right)^{-\epsilon_w} \forall j, k \in [0,1]. \tag{11.15}
\]

The wage index \( W(t) \) is defined such that:

\[
N_{i,j}(t) = \left( \frac{W_j(t)}{W(t)} \right)^{-\epsilon_w} N_i(t). \tag{11.16}
\]

Given the definition of the wage index, the wage index can be derived as (the same algebra
as was used to derive the price index):

$$W(t) = \left( \int_0^1 W_j(t)^{1-\varepsilon_w} dj \right)^{\frac{1}{1-\varepsilon_w}}. \quad (11.17)$$

Additionally, the total wage expenditure is given by:

$$W(t)N_i(t) = \int_0^1 W_j(t)N_{i,j}(t) dj. \quad (11.18)$$

### 11.1.4 Market clearing

Market clearing in the commodity markets requires that

$$Y_i(t) = \int_0^1 C_{i,j}(t) dj \text{ for all } i \in [0,1]. \quad (11.19)$$

This says that the total production of variety $i \in [0,1]$ must be equal to the total consumption of variety $i$ across all household types $j \in [0,1]$. Aggregate output of the economy is defined by $Y(t) = \int_0^1 Y_i(t) di$, meaning that:

$$Y(t) = \int_0^1 Y_i(t) di = \int_0^1 \int_0^1 C_{i,j}(t) dj di. \quad (11.20)$$

Define the aggregate consumption index $C(t) = \int_0^1 C_j(t) dj$. The relation between aggregate output $Y(t)$ and the aggregate consumption index $C(t)$ is given by:

$$Y(t) = \int_0^1 \int_0^1 C_{i,j}(t) dj di = \int_0^1 \int_0^1 \left( \frac{P_i(t)}{P(t)} \right)^{-\varepsilon_p} C_j(t) dj di$$

$$= \left( \int_0^1 C_j(t) dj \right) \left( \int_0^1 \left( \frac{P_i(t)}{P(t)} \right)^{-\varepsilon_p} di \right) = C(t) \Delta_p(t), \quad (11.21)$$

where $\Delta_p(t) = \int_0^1 \left( \frac{P_i(t)}{P(t)} \right)^{-\varepsilon_p} di$ is a measure of price dispersion, which is equal to 1 in steady state.
The labor market clearing requires that:

\[ N_j(t) = \int_0^1 N_{i,j}(t)\,di. \]  

(11.22)

This states that the total supply of type \( j \) labor equals the total demand for type \( j \) labor (across all firms \( i \in [0,1] \)). Aggregate employment is defined as the aggregate across all household types \( j \in [0,1] \):

\[ N(t) = \int_0^1 N_j(t)\,dj = \int_0^1 \int_0^1 N_{i,j}(t)\,dj\,di. \]

(11.23)

Define the aggregate labor demand index as \( N^d(t) = \int_0^1 N_i(t)\,di \). Aggregate employment is related to the aggregate labor demand index by:

\[ N(t) = \int_0^1 \int_0^1 N_{i,j}(t)\,dj\,di = \int_0^1 \int_0^1 \left( \frac{W_j(t)}{W(t)} \right)^{-\epsilon_w} N_i(t)\,dj\,di \]

\[ = N^d(t) \left( \int_0^1 \left( \frac{W_j(t)}{W(t)} \right)^{-\epsilon_w} \,dj \right) = N^d(t) \Delta_w(t), \]

(11.24)

where \( \Delta_w(t) = \int_0^1 \left( \frac{W_i(t)}{W(t)} \right)^{-\epsilon_w} \,dj \) is a measure of wage dispersion, which is equal to 1 in steady state.

From the above equation:

\[ N(t) = \left( \int_0^1 N_i(t)\,di \right) \Delta_w(t). \]

(11.25)

Since \( Y_i(t) = A(t)N_i(t)^{1-\alpha} \) from the production function, then \( N_i(t) = \left( \frac{Y_i(t)}{A(t)} \right)^{\frac{1}{1-\alpha}} \). Multiplying and dividing by \( Y(t) \), aggregate employment is given by:

\[ N(t) = \left( \frac{Y(t)}{A(t)} \right)^{\frac{\epsilon_w}{1-\alpha}} \left( \int_0^1 \left( \frac{Y_i(t)}{Y(t)} \right)^{\frac{1}{1-\alpha}} \,di \right) \left( \int_0^1 \left( \frac{W_j(t)}{W(t)} \right)^{-\epsilon_w} \,dj \right). \]

(11.26)
11. NEW KEYNESIAN LABOR MARKET THEORY

By definition and using the household demand functions:

\[ \frac{Y_i(t)}{Y(t)} = \frac{\int_0^1 C_{i,j}(t) dj}{\int_0^1 \int_0^1 C_{i,j}(t) dj di} = \frac{\int_0^1 \left( \frac{P_i(t)}{P(t)} \right)^{-\epsilon_p} C_j(t) dj}{\int_0^1 \int_0^1 \left( \frac{P_i(t)}{P(t)} \right)^{-\epsilon_p} C_j(t) dj di} = \frac{\left( \int_0^1 \left( \frac{P_i(t)}{P(t)} \right)^{-\epsilon_p} di \right) \left( \int_0^1 C_j(t) dj \right)}{\left( \int_0^1 \left( \frac{P_i(t)}{P(t)} \right)^{-\epsilon_p} di \right) \left( \int_0^1 C_j(t) dj \right)} = \left( \frac{P_i(t)}{P(t)} \right)^{-\epsilon_p}. \]  

(11.27)

This implies that \( \int_0^1 \left( \frac{Y_i(t)}{Y(t)} \right)^{\frac{1}{1-\alpha}} di \) = 1 and aggregate employment simplifies to:

\[ N(t) = \left( \frac{Y(t)}{A(t)} \right)^{\frac{1}{1-\alpha}} \left( \int_0^1 \left( \frac{W_j(t)}{W(t)} \right)^{-\epsilon_w} dj \right). \]  

(11.29)

Return to the household problem and the dividend term \( D_j(t) \). In actuality, \( D_j(t) \) is the net income from firm ownership, where

\[ D_j(t) = \int_0^1 \{ S_{i,j} (t - 1) (Q_i(t) + \Pi_i(t)) - S_{i,j} (t) Q_i(t) \} di. \]  

(11.30)

Here \( S_{i,j} (t) \) is the number of shares of firm \( i \) held in period \( t \). The price of a share is \( Q_i(t) \). The payout of a share next period (in period \( t+1 \)) equals the next period share price \( Q_i(t+1) \) plus the firm profit \( \Pi_i(t+1) \). Households of type \( j \) choose \( (S_{i,j}(t))_{i \in [0,1]} \) while taking the asset prices as given. An Euler equation for share holdings states:

\[ Q_i(t) = \beta E_t \left\{ \left( \frac{C_j(t + 1)}{C_j(t)} \right)^{-\sigma} \left( \frac{P_i(t)}{P_i(t + 1)} \right) (Q_i(t + 1) + \Pi_i(t + 1)) \right\} . \]

This equation determines the share prices.

Assuming that the ownership of firms is entirely in the private sector, then firms are owned only by households and not by the government. Share market clearing requires that \( \int_0^1 S_{i,j} (t) dj = 1 \) for all \( i \in [0,1] \). This means that \( \int_0^1 D_j(t) dj = \int_0^1 \Pi_i(t) di \), where \( \Pi_i(t) \) is the profit of firm \( i \) in period \( t \):

\[ \Pi_i(t) = P_i(t)Y_i(t) - W(t)N_i(t). \]
11.1. EXTENDING THE MODEL

Since \( P(t)C_j(t) = \int_0^1 P_i(t)C_{i,j}(t)\,di \) and \( Y_i(t) = \int_0^1 C_{i,j}(t)\,dj \), then Walras’ Law implies that the summed household budget constraints (summed over all types \( j \in [0, 1] \)):

\[
\int_0^1 B_j(t)\,dj \quad \frac{1}{1 + i(t)} = \int_0^1 B_j(t - 1)\,dj.
\] (11.31)

Higher interest rate \( i(t) \) leads to higher bond holdings \( \int_0^1 B_j(t)\,dj \).

Bond market clearing requires that the total bond holdings of households \( \int_0^1 B_j(t)\,dj \) must equal to the government debt. Governments lean against the wind, meaning that they have high debt when the interest rate is high (recession) and low debt when the interest rate is low (expansion), which is counter to what a self-interested agent would choose. We do not model the government budget constraints explicitly. The government uses open market operations to jointly choose debt and the nominal interest rate. We directly assume that the government can set whatever interest rate it chooses, and it does so through a Taylor rule (to be introduced shortly). For any interest rate chosen by the government, the debt level can be solved using the government budget constraints. This debt level must satisfy the bond market clearing condition. Walras’ Law ensures that this is so (if all other markets clear, then the household budget constraint implies that the bond market clearing holds automatically).

Households are characterized by the wage rate \( W_j(t) \) that their labor union negotiates for them. The sticky wages are such that in any given period, only a random fraction of the labor unions are able to change their wage. Households face risk with two future states: wage rate changed by union or wage rate unable to be changed. Households have two assets with which to diversify their risk: bonds and share holdings. There are no restrictions on bond trade. Assume that there are no restrictions on the trading of shares, which technically allows for the short-selling of these assets. Assume that implicit debt constraints are imposed on households to rule out Ponzi schemes (the implicit debt constraint for bonds is \( \lim_{t \to \infty} \left( \frac{B_j(t)}{P(t)} \right) > -\infty \)).

Share holdings of firms \( i_1 \) and \( i_2 \) are linearly independent iff those two firms receive a different nominal price rigidity shock (1 firm can change its price and the other cannot). Households are not able to select winners and losers. They hold a mutual fund consisting of the same share holdings of all firms. A mutual fund delivers payouts that vary with \( P(t) \), while bonds have payouts that are constant across \( P(t) \). Bond holdings and the mutual
fund over all firms are two linearly independent assets. Given two unrestricted and linearly
independent assets, households are able to perfectly smooth consumption across the two
states of uncertainty. In equilibrium, \( C_j(t) = C(t) \).

### 11.1.5 Steady state

The equilibrium equations in the New Keynesian model are obtained as log-linear deviations
around the steady state. Define \( \Pi_p(t) = \frac{P(t)}{P(t-1)} \) as the commodity price inflation and \( \Pi_w(t) = \frac{W(t)}{W(t-1)} \) as the wage rate inflation. There exists a unique steady state in which \( \Pi_p(t) = \Pi_w(t) = 0 \) in all time periods. I refer to this as the zero inflation steady state.

### 11.2 Wage setting by the households

#### 11.2.1 Sneak peek

**Summary**

With the household problem of wage determination specified, we can solve for the optimal
solution. The solution method will be similar to what was used in the previous chapter to
find the optimal solution of the firm problem facing a nominal price rigidity. We evaluate
the first order condition and then log-linearize the equation around the zero inflation steady
state in order to derive a linear equation for the wage inflation rate. The equation for the
wage inflation rate has a similar form to the New Keynesian Phillips curve equation derived
in the previous chapter.

**Notation**

The variables to be introduced in this section are given in the following table:

<table>
<thead>
<tr>
<th>( W^CE_j(t) )</th>
<th>competitive equilibrium wage</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_w )</td>
<td>probability that a labor union cannot set its wage</td>
</tr>
<tr>
<td>( W^*(t) )</td>
<td>wage set by labor unions in period ( t )</td>
</tr>
<tr>
<td>( N^k(t) )</td>
<td>labor supply for unions in period ( t + k ) who set wage in ( t )</td>
</tr>
<tr>
<td>( \lambda_w )</td>
<td>a combined parameter ( \lambda_w = \frac{(1-\theta_w)(1-\beta_w)}{\theta_w(1+\gamma_w)} )</td>
</tr>
</tbody>
</table>
11.2. WAGE SETTING BY THE HOUSEHOLDS

Main takeaways

After completing this section, you will be able to answer the following questions:

- What is the dynamic problem that is solved by the labor unions under nominal rigidities?
- How is wage inflation optimally determined by the labor unions under nominal rigidities?

11.2.2 Wage determination

Competitive benchmark

Recall the household problem, focusing on period \( t \):

\[
\max_{C_j(t), N_j(t)} \frac{(C_j(t))^{1-\sigma}}{1 - \sigma} - \Xi(t) \frac{N_j(t)^{1+\gamma}}{1 + \gamma}.
\]

subject to \( P(t)C_j(t) + \frac{B_j(t)}{1 + i(t)} \leq B_j(t - 1) + W_j(t)N_j(t) + D_j(t) \). \hfill (11.32)

The first order condition of the household problem with respect to consumption \( C_j(t) \) is given by:

\[
(C_j(t))^{-\sigma} - \lambda_j(t)P(t) = 0,
\]

where \( \lambda_j(t) \) is the Lagrange multiplier for the household budget constraint.

Taking the first order condition of the household problem with respect to \( N_j(t) \) and using the above first order condition, we arrive at the competitive equilibrium (CE) wage rate:

\[
W_j^{CE}(t) = \Xi(t) (N_j(t))^\gamma (C_j(t))^{\sigma} P(t).
\]

A competitive equilibrium is a benchmark without the two key frictions: (i) labor unions set wages in a monopolistically competitive environment and (ii) labor unions face nominal wage rigidities. If the labor supply is the aggregate employment \( N(t) \), then the competitive wage equation is given by:

\[
W^{CE}(t) = \Xi(t) (N(t))^\gamma (C(t))^{\sigma} P(t).
\]
View $W^{CE}(t)$ as the average marginal labor supply cost for aggregate employment $N(t)$ when wages are set competitively.

**Flexible wage benchmark**

In this model, wages $W_j(t)$ for type $j \in [0,1]$ are not competitive, but are chosen by the type $j$ labor union. The type $j$ labor union consists of the continuum of type $j$ laborers and these laborers pool together in order to set the wage rate $W_j(t)$. Consider the wage setting with only the monopolistic competition friction (and no nominal wage rigidity friction). This setting is referred to as a flexible wage setting.

The type $j$ labor union chooses the labor supply $N_j(t)$. Using the market clearing conditions and the labor demand function $N_{i,j}(t) = \left(\frac{W_j(t)}{W(t)}\right)^{-\epsilon_w} N_i(t)$:

$$N_j(t) = \left(\frac{W_j(t)}{W(t)}\right)^{-\epsilon_w} N^d(t),$$

where recall the definition of the aggregate labor demand index $N^d(t) = \int_0^1 N_i(t) di$. The labor demand function $N_j(t) = \left(\frac{W_j(t)}{W(t)}\right)^{-\epsilon_w} N^d(t)$ is internalized into the household problem (focusing on period $t$):

$$\max_{C_j(t),W_j(t)} \frac{(C_j(t))^{1-\sigma}}{1-\sigma} - \Xi(t) \left(\frac{\left(\frac{W_j(t)}{W(t)}\right)^{-\epsilon_w} N^d(t)}{1+\gamma}\right)^{1+\gamma}.$$

sub. to $P(t)C_j(t) + \frac{B_j(t)}{1+i(t)} \leq B_j(t-1) + W_j(t) \left(\left(\frac{W_j(t)}{W(t)}\right)^{-\epsilon_w} N^d(t)\right) + D_j(t)$.

The first order condition of the household problem with respect to the wage choice $W_j(t)$ is given by:

$$\epsilon_w \Xi(t) (N_j(t))^{\gamma} (W_j(t))^{-\epsilon_w-1} = \lambda(t) (\epsilon_w - 1) (W_j(t))^{-\epsilon_w}.$$

Solving for the wage rate leads to the expression:

$$W_j(t) = \frac{\epsilon_w}{\epsilon_w - 1} W^{CE}_j(t).$$

The wage set by the labor union of type $j$ workers is a constant markup $\frac{\epsilon_w}{\epsilon_w - 1} > 1$ over the competitive wage.
11.2. WAGE SETTING BY THE HOUSEHOLDS

If the labor supply is the aggregate employment \(N(t)\), then the natural wage level is given by:

\[
W(t) = \frac{\epsilon_w}{\epsilon_w - 1} W^{CE}(t). \tag{11.39}
\]

View the ratio \(\frac{W(t)}{W^{CE}(t)} = \frac{\epsilon_w}{\epsilon_w - 1}\) as the markup that would prevail in the flexible wage benchmark. We seek to compare this to the equilibrium ratio (or markup) that arises under nominal rigidities.

11.2.3 Nominal rigidities

There are a continuum of labor unions, one for each labor type \(j \in [0, 1]\). Each labor union may reset its wage with probability \(1 - \theta_w\) in each period. The wage rate \(W_j(t)\) can take one of two values:

1. If labor union \(j\) does not reset its price, \(W_j(t) = W_j(t - 1)\).

2. If labor union \(j\) resets its price, \(W_j(t) = W^*(t)\), where \(W^*(t)\) is the wage rate chosen by all labor unions that reset their prices in period \(t\). Given the symmetry of the model, the labor unions that change their wage will always choose the same wage \(W^*(t)\).

For labor union \(j \in [0, 1]\), insert the relative components of the budget constraint into the utility function in period \(t + k\):

\[
\max_{W_j(t)} \left\{ \sum_{k=0}^{\infty} (\beta \theta_w)^k E_t \left[ \frac{\frac{1}{P(t+k)} W_j(t) N_j(t+k)}{1 - \sigma} - \Xi(t+k) \frac{N_j(t+k)^{1+\gamma}}{1+\gamma} \right] \right\}. \tag{11.40}
\]

If a labor union resets its wage in period \(t\), it will choose the wage \(W_j(t)\) to solve the following dynamic optimization problem:

\[
\max_{W_j(t)} \sum_{k=0}^{\infty} (\beta \theta_w)^k E_t \left[ \frac{\frac{1}{P(t+k)} W_j(t) N_j(t+k)}{1 - \sigma} - \Xi(t+k) \frac{N_j(t+k)^{1+\gamma}}{1+\gamma} \right].
\]

Recall that \(\theta_w\) is the probability that the labor union cannot reset its wage in any given period, so \((\theta_w)^k\) is the probability that a labor union is stuck with the wage \(W_j(t)\) in period \(t + k\).
The labor unions operate under monopolistic competition, so they internalize the firm labor demand function:

\[ N_j(t+k) = \left( \frac{W_j(t)}{W(t+k)} \right)^{-\epsilon_w} N^d(t+k). \]  

(11.41)

Notice that the wage for labor union \( j \) in period \( t+k \) equals \( W_j(t) \) as the labor union \( j \) is solving the dynamic problem as if it is stuck with the wage rate \( W_j(t) \) chosen in the current period \( t \).

### 11.2.4 First order conditions

The first order condition for the type \( j \) labor union dynamic maximization problem is given by (after rearranging and cancelling terms):

\[ \sum_{k=0}^{\infty} (\beta \epsilon_w)^k E_t \left\{ \frac{\epsilon_w}{\epsilon_w - 1} \Xi(t+k) P(t+k) (N_j(t+k))^\gamma (C_j(t+k))^\sigma - W_j(t) \right\} = 0. \]  

(11.42)

Notice that if \( \theta_w = 0 \) (flexible wage setting), then \( W_j(t) = \frac{\epsilon_w}{\epsilon_w - 1} W_j^{CE}(t) \), which is a constant markup over the competitive equilibrium wage.

By definition, \( W_j(t) = W^*(t) \). For all labor unions that last reset their wage in period \( t \), the identical labor input choice will be made, which is denoted \( N^k(t) \). Recall that households are able to completely diversify their risk and maintain constant consumption across the nominal rigidity shock. This means that the first order condition is given by:

\[ \sum_{k=0}^{\infty} (\beta \epsilon_w)^k E_t \left\{ \frac{\epsilon_w}{\epsilon_w - 1} \Xi(t+k) P(t+k) (N^k(t))^\gamma (C(t))^\sigma - W^*(t) \right\} = 0. \]  

(11.43)

### 11.2.5 Log linearization

Define \( w^*(t) = \ln (W^*(t)) \), \( \xi(t) = \ln (\Xi(t)) \), \( n^k(t) = \ln (N^k(t)) \), \( c(t) = \ln (C(t)) \), and \( p(t) = \ln (P(t)) \). Additionally, define \( \mu_w = \ln \left( \frac{\epsilon_w}{\epsilon_w - 1} \right) \), where \( \frac{\epsilon_w}{\epsilon_w - 1} \) is the markup in the absence of wage rigidities (flexible wage setting). Log-linearizing the first order condition around the zero inflation steady state leads to the following equation:

\[ w^*(t) = (1 - \beta \epsilon_w) \sum_{k=0}^{\infty} (\beta \epsilon_w)^k E_t \left\{ \mu_w + \xi(t+k) + \gamma n^k(t) + \sigma c(t) + p(t+k) \right\}. \]  

(11.44)
Define \( n^d(t+k) = \ln \left( N^d(t+k) \right) \), \( n(t+k) = \ln \left( N(t+k) \right) \), and \( w(t+k) = \ln \left( W(t+k) \right) \), where \( N^d(t+k) \) is the aggregate labor demand index, \( N(t+k) \) is aggregate employment, and \( W(t+k) \) is the wage index (all in period \( t+k \)). In steady state, \( n^d(t+k) = n(t+k) \). Log-linearizing the firm labor demand function (11.41) leads to:

\[
    n^k(t) = n(t+k) - \epsilon_w (w^*(t) - w(t+k)).
\]  

(11.45)

For the labor supply \( N(t+k) \), the competitive equilibrium wage rate is given by:

\[
    W^{CE}(t+k) = \Xi(t+k) P(t+k) (N(t+k))^\gamma (C(t+k))^\sigma.
\]

(11.46)

Define \( w^{CE}(t) = \ln \left( W^{CE}(t) \right) \) as the natural log of the competitive equilibrium wage:

\[
    w^{CE}(t+k) = \xi(t+k) + \gamma n(t+k) + \sigma c(t+k) + p(t+k).
\]

(11.47)

Inserting the expressions for \( n^k(t) \) and \( w^{CE}(t+k) \) into the log-linearized equation and solving for \( w^*(t) \) leads to:

\[
    w^*(t) = \frac{1 - \beta \theta_w}{1 + \gamma \epsilon_w} \sum_{k=0}^{\infty} (\beta \theta_w)^k E_t \left\{ \mu_w + w^{CE}(t+k) + \gamma \epsilon_w w(t+k) \right\}.
\]

(11.48)

The equation can be written recursively:

\[
    w^*(t) = \beta \theta_w E_t \{ w^*(t+1) \} + (1 - \beta \theta_w) w(t) + \frac{(1 - \beta \theta_w) (\mu_w + w^{CE}(t) - w(t))}{1 + \gamma \epsilon_w}.
\]

(11.49)

**Law of large numbers**

From the law of large numbers, the fraction \( 1 - \theta_w \) of labor unions set the wage rate \( W^*(t) \) in period \( t \) and the remaining fraction \( \theta_w \) (since they are chosen at random) have average wage rate equal to the previous period’s wage index \( W(t-1) \). Therefore, the wage index from the equation \( W(t) = \left( \int_0^1 W_j(t)^{1-\epsilon_w} dj \right)^{1/\epsilon_w} \) is given by:

\[
    W(t) = \left[ \theta_w (W(t-1))^{1-\epsilon_w} + (1 - \theta_w) (W^*(t))^{1-\epsilon_w} \right]^{1/(1-\epsilon_w)}.
\]

(11.50)
11. NEW KEYNESIAN LABOR MARKET THEORY

Denote the wage inflation as the rate of change for the wage index:

\[ \Pi_w(t) = \frac{W(t)}{W(t-1)}. \]  (11.51)

Raising both sides of the wage index to the exponent \(1 - \epsilon_w\) and dividing by \(W(t-1)\) yields an expression for \(\Pi_w(t)\):

\[ (\Pi_w(t))^{1-\epsilon_w} = \theta_w + (1 - \theta_w) \left( \frac{W^*(t)}{W(t-1)} \right)^{1-\epsilon_w}. \]  (11.52)

Log-linearizing the equation around the zero wage inflation steady state and using the definition \(\pi_w(t) = \ln(\Pi_w(t))\) leads to:

\[ w^*(t) = w(t-1) + \frac{\pi_w(t)}{1 - \theta_w}. \]  (11.53)

Using this and a similar expression for \(w^*(t + 1)\), the recursive wage rate equation can be written as:

\[
w(t-1) + \frac{\pi_w(t)}{1 - \theta_w} = \beta \theta_w w(t) + \beta \theta_w E_t \left\{ \frac{\pi_w(t + 1)}{1 - \theta_w} \right\} + (1 - \beta \theta_w) w(t) + \frac{(1 - \beta \theta_w) (\mu_w + w^{CE}(t) - w(t))}{1 + \gamma \epsilon_w}.
\]  (11.54)

Using the definition \(\pi_w(t) = w(t) - w(t-1)\) and solving for \(\pi_w(t)\) yields the wage inflation equation:

\[ \pi_w(t) = \beta E_t \left\{ \pi_w(t + 1) \right\} - \frac{(1 - \theta_w) (1 - \beta \theta_w)}{\theta_w (1 + \gamma \epsilon_w)} (w(t) - w^{CE}(t) - \mu_w). \]  (11.55)

Define \(\lambda_w = \frac{(1 - \theta_w)(1 - \beta \theta_w)}{\theta_w (1 + \gamma \epsilon_w)}\) and we obtain a wage inflation equation that looks very similar to the New Keynesian Phillips curve derived in the previous chapter:

\[ \pi_w(t) = \beta E_t \left\{ \pi_w(t + 1) \right\} - \lambda_w (w(t) - w^{CE}(t) - \mu_w). \]  (11.56)

The difference \(w(t) - w^{CE}(t)\) is the actual wage markup (over the competitive equilibrium wage) and \(\mu_w\) is the markup without nominal wage rigidities (flexible wage setting).
11.3 Price setting by the firms

11.3.1 Sneak peek

Summary

The firm problem is very similar to what we analyzed in the previous chapter. The solution to the firm problem with the nominal price rigidity provides an equation for the commodity price inflation rate. This equation for the commodity price inflation rate is called the New Keynesian Phillips curve. The derivation is identical to what was shown in the previous chapter, so this section only gathers the conclusion of that prior analysis.

Notation

The variables to be introduced in this section are given in the following table:

- $\theta_p$: probability that a firm can reset its price in any period
- $P^*(t)$: price chosen by firms in period $t$
- $\Pi_p(t)$: commodity price inflation rate
- $\pi_p(t)$: log of the commodity price inflation rate $\pi_p(t) = \ln(\Pi_p(t))$
- $\mu_p$: natural log of the markup under flexible prices, $\mu_p = \ln \left( \frac{\pi_p}{\varepsilon_p} \right)$
- $\lambda_p$: a combined parameter $\lambda_p = \frac{(1-\theta_p)(1-\theta_p)(1-\alpha)}{\theta_p(1-\alpha+\alpha_r)}$

Main takeaways

After completing this section, you will be able to answer the following questions:

- What is the dynamic problem that is solved by the firms under nominal rigidities?
- How is commodity price inflation optimally determined according to the New Keynesian Phillips curve?

11.3.2 Recap of findings from previous chapter

Exactly as in the previous chapter, firms in this model choose the commodity prices in a setting of monopolistic competition with nominal rigidities. Each firm may reset its price with probability $1 - \theta_p$ in each period. The price $P_i(t)$ charged by firm $i$ can take one of two values:
1. If firm $i$ does not reset its price, $P_i(t) = P_i(t-1)$.

2. If firm $i$ resets its price, $P_i(t) = P^*(t)$, where $P^*(t)$ is the price set by all firms that reset their prices in period $t$.

If a firm resets its price in period $t$, it will choose the price $P_i(t)$ to solve the following dynamic optimization problem:

$$
\max_{P_i(t)} \sum_{k=0}^{\infty} \theta^k E_t \left\{ q^k(t) \left( P_i(t) Y_i(t + k) - W(t + k) \left( \frac{Y_i(t + k)}{A(t + k)} \right)^{1-\alpha} \right) \right\}.
$$

By market clearing, $Y_i(t + k) = \int_0^1 C_{i,j}(t + k) dj$ and using the household demand function $C_{i,j}(t + k) = \left( \frac{P_i(t)}{P(t)} \right)^{-\epsilon_p} C_j(t)$:

$$
Y_i(t + k) = \left( \frac{P_i(t)}{P(t)} \right)^{-\epsilon_p} C(t).
$$

This demand function is internalized into the firm profit maximization problem.

As in the previous chapter, the solution is the New Keynesian Phillips curve equation:

$$
\pi_p(t) = \beta E_t \{ \pi_p(t + 1) \} - \lambda_p (p(t) - mc(t) - \mu_p).
$$

Here, the variable $\pi_p(t) = \ln \left( \frac{P_i(t)}{P_i(t-1)} \right)$, the natural log of the inflation rate. The parameter

$$
\lambda_p = \frac{(1 - \theta_p)(1 - \theta_p \beta)(1 - \alpha)}{\theta_p (1 - \alpha + \alpha \epsilon_p)}
$$

is a function of the underlying parameters of the model. The variable $mc(t) = \ln \left( MC \left( Y(t) \right) \right)$, the natural log of the average marginal cost across all firms. And finally, the parameter $\mu_p = \ln \left( \frac{\epsilon_p}{\epsilon_p - 1} \right)$, which is the natural log of the markup under flexible prices.
11.4 Output gap and wage gap

11.4.1 Sneak peek

Summary

There are two equations for the price levels in the economy, one for the commodity price inflation rate and one for the wage inflation rate. The model is more tractable when these equations are specified in terms of the output gap (as introduced in the previous chapter) and the wage gap.

Additionally, with two inflation rate equations (one for commodity prices and one for wages), there is an identity that must be satisfied for the wage gap. The identity is an equation relating variables across two time periods. However, unlike the equations for the inflation rates and the Euler equation, the wage gap identity does not hold in expectation over all possible shocks in the following period. The identity must hold for all possible shocks that can occur in the following period. This is an important distinction, and the wage gap identity will be a constant companion as we proceed through the solution of this model.

Notation

The variables to be introduced in this section are given in the following table:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \ddot{y}(t) )</td>
<td>output gap ( \ddot{y}(t) = y(t) - y^n(t) )</td>
</tr>
<tr>
<td>( rw(t) )</td>
<td>real wage rate ( rw(t) = w(t) - p(t) )</td>
</tr>
<tr>
<td>( \ddot{w}(t) )</td>
<td>real wage gap ( \ddot{w}(t) = rw(t) - rw^n(t) )</td>
</tr>
<tr>
<td>( \tilde{\lambda}_p )</td>
<td>a combined parameter ( \tilde{\lambda}_p = \frac{\sigma}{1-\sigma} \lambda_p )</td>
</tr>
<tr>
<td>( \tilde{\lambda}_w )</td>
<td>a combined parameter ( \tilde{\lambda}_w = \left( \sigma + \frac{\sigma - 1}{1-\sigma} \right) \lambda_w )</td>
</tr>
<tr>
<td>( \phi^n )</td>
<td>constant term in the ( y^n(t) ) equation</td>
</tr>
<tr>
<td>( \phi^a_n )</td>
<td>marginal product of ( y^n(t) ) with respect to ( a(t) )</td>
</tr>
<tr>
<td>( \phi^\xi_n )</td>
<td>marginal product of ( y^n(t) ) with respect to ( \xi(t) )</td>
</tr>
<tr>
<td>( \phi_p )</td>
<td>Taylor rule coefficient for commodity inflation</td>
</tr>
<tr>
<td>( \phi_w )</td>
<td>Taylor rule coefficient for wage inflation</td>
</tr>
<tr>
<td>( \nu(t) )</td>
<td>monetary policy shock</td>
</tr>
<tr>
<td>( \Omega(t) )</td>
<td>constant term in the wage gap identity</td>
</tr>
</tbody>
</table>
Main takeaways

After completing this section, you will be able to answer the following questions:

- How can the New Keynesian Phillips curve and wage inflation equation be written in terms of the output gap and the wage gap?
- How can the household Euler equation be written in terms of the output gap and the wage gap?
- What form will the Taylor rule take?
- What role does the wage gap identity play in equilibrium determination?

11.4.2 Flexible price and wage properties

Under flexible prices and wages, the natural output level (and natural employment and natural real wage) are supported. Under flexible prices:

\[ P^n(t) = \frac{\epsilon_p}{\epsilon_p - 1} MC (Y^n(t)). \]  

(11.61)

Define \( p^n(t) = \ln (P^n(t)) \), \( y^n(t) = \ln (Y^n(t)) \), and \( mc^n(t) = \ln (MC (Y^n(t))) \) and log-linearize:

\[ p^n(t) = \mu_p + mc^n(t). \]  

(11.62)

At the natural output level, there are two equations that can be used to derive the natural real wage rate. Recall from the household wage setting problem:

\[ W^n(t) = \frac{\epsilon_w}{\epsilon_w - 1} W^{CE}(t). \]  

(11.63)

We have already found that \( w^{CE}(t) = \xi(t) + \gamma n(t) + \sigma c(t) + p(t) \). The natural real wage is \( \frac{w^n(t)}{P^n(t)} \), with natural log \( rw^n(t) = \ln \left( \frac{W^n(t)}{P^n(t)} \right) = w^n(t) - p(t) \). Log-linearization yields:

\[ rw^n(t) = \mu_w + \xi(t) + \gamma n^n(t) + \sigma y^n(t). \]  

(11.64)

The second equation for the natural real wage comes from the marginal cost of the firm
11.4. OUTPUT GAP AND WAGE GAP

problem, where marginal cost is given by:

\[
MC(Y(t)) = \frac{1}{1 - \alpha} \frac{W(t)}{A(t)(N(t))^{-\alpha}}.
\] (11.65)

Log-linearizing the marginal cost equation and solving for the real wage:

\[
rw(t) = mc(t) - p(t) + \ln(1 - \alpha) + a(t) - \alpha n(t).
\] (11.66)

Evaluated at the natural output level, and using the equation \(p^n(t) = \mu_p + mc^n(t)\), we obtain our second expression for \(rw^n(t)\):

\[
rw^n(t) = -\mu_p + \ln(1 - \alpha) + a(t) - \alpha n^n(t).
\] (11.67)

11.4.3 Natural output level

From the natural real wage equations from both the household and the firm problems, we obtain the following equality:

\[
\mu_w + \xi(t) + \gamma n^n(t) + \sigma y^n(t) = -\mu_p + \ln(1 - \alpha) + a(t) - \alpha n^n(t).
\] (11.68)

Recall the equation for total employment:

\[
N(t) = \left(\frac{Y(t)}{A(t)}\right)^{\frac{1}{1-\alpha}} \Delta_w(t).
\] (11.69)

Log-linearize the above equation around the zero inflation steady state:

\[
n(t) = \frac{y(t) - a(t)}{1 - \alpha}.
\] (11.70)

Evaluated at the natural output level:

\[
n^n(t) = \frac{y^n(t) - a(t)}{1 - \alpha}.
\] (11.71)

Using the expression for \(n^n(t)\), we can solve (11.68) for \(y^n(t)\):

\[
y^n(t) = \frac{1 + \gamma}{\gamma + \alpha + \sigma(1 - \alpha)}a(t) + \frac{1 - \alpha}{\gamma + \alpha + \sigma(1 - \alpha)} \{\ln(1 - \alpha) - \mu_p - \mu_w - \xi(t)\}.
\] (11.72)
Define the coefficients

\[
\phi_a^n = \frac{1 + \gamma}{\gamma + \alpha + \sigma (1 - \alpha)},
\]

\[
\phi_\xi^n = -\frac{1 - \alpha}{\gamma + \alpha + \sigma (1 - \alpha)},
\]

\[
\phi^n = \frac{1 - \alpha}{\gamma + \alpha + \sigma (1 - \alpha)} \{\ln(1 - \alpha) - \mu_p - \mu_w\}.
\]

The natural output level can be written as a linear function of the technology shock \(a(t)\) and the preference shock \(\xi(t)\):

\[
y^n(t) = \phi^n + \phi_a^n a(t) + \phi_\xi^n \xi(t).
\]

Notice that \(\phi_a^n\) is identical to what we found in the previous chapter. In an end-of-chapter exercise from the previous chapter, you verified that \(0 < \phi_a^n < 1\) (for \(\sigma > 1\)). Steady state output \(y_{SS} = \phi^n\) as both shocks \(a(t) = \xi(t) = 0\) in steady state.

### 11.4.4 Euler equation

The Euler equation for households is given by:

\[
\frac{(C(t))^{-\sigma}}{P(t)} = \beta(1 + i(t)) E_t \left\{ \frac{(C(t + 1))^{-\sigma}}{P(t + 1)} \right\}.
\]

The log-linearization of this Euler equation leads to the dynamic IS equation:

\[
y(t) = E_t \{y(t + 1)\} - \frac{1}{\sigma} \left( i(t) - E_t \{\pi_p(t + 1)\} - \delta \right).
\]

This is identical to equation found in the previous chapter.

Define the output gap as the difference between actual output and natural output level:

\[
\tilde{y}(t) = y(t) - y^n(t).
\]

Using the definition of the output gap, the household Euler equation is given by:

\[
\tilde{y}(t) = E_t \{\tilde{y}(t + 1)\} - \frac{1}{\sigma} \left( i(t) - E_t \{\pi_p(t + 1)\} - \delta \right) + E_t \{y^n(t + 1) - y^n(t)\}.
\]

---

1The labor union for type \(j \in [0, 1]\) pools together all households \((j, h)_{h \in [0, 1]}\) and sets the wage rate \(W_j(t)\) such that \(C_j(t)\) is equal for all \((j, h)_{h \in [0, 1]}\). Recall that households can diversify risk such that \(C_j(t) = C(t)\) for all \(j \in [0, 1]\).
Using the equation for the natural output level, \( y^n(t) = \phi^n + \phi^n a(t) + \phi^a_\xi \xi(t) \), the household Euler equation is updated as

\[
\bar{y}(t) = E_t \{ \bar{y}(t + 1) \} - \frac{1}{\sigma} (i(t) - E_t \{ \pi_p(t + 1) \} - \delta)
+ \phi^a_n E_t \{ a(t + 1) - a(t) \} + \phi^\xi_n E_t \{ \xi(t + 1) - \xi(t) \}.
\]

### 11.4.5 Wage gap identity

Define the real wage and the natural real wage as \( rw(t) = \frac{w(t)}{p(t)} \) and \( rw^n(t) = \frac{w^n(t)}{p^n(t)} \), respectively. Define the real wage gap as the difference between the real wage and the natural real wage:

\[
rw(t) = rw(t) - rw^n(t).
\]  

(11.79)

A basic identity for the wage gap is given by:

\[
\bar{rw}(t) - \bar{rw}(t - 1) = w(t) - w(t - 1) - (p(t) - p(t - 1)) - (rw^n(t) - rw^n(t - 1))
= \pi_w(t) - \pi_p(t) - (rw^n(t) - rw^n(t - 1)).
\]  

(11.80)

This identity does not hold in expectation, but most hold for all shocks in period \( t \).

Using the expression for the natural real wage from the firm marginal cost equation, the relation for \( n^a(t) \), and the natural output level equation to obtain:

\[
rw^n(t) = -\mu_p + \ln(1 - \alpha) + \frac{(1 - \alpha) a(t) - \alpha \phi^\xi_\xi (\xi(t) - \xi(t - 1)) - \alpha \phi^n}{1 - \alpha}.
\]  

(11.81)

For simplicity, define the constant

\[
\Omega(t) = \frac{(1 - \alpha) a(t) - \alpha \phi^\xi_\xi (\xi(t) - \xi(t - 1))}{1 - \alpha}.
\]  

(11.82)

The wage gap identity is therefore given by:

\[
\bar{rw}(t) - \bar{rw}(t - 1) = \pi_w(t) - \pi_p(t) - \Omega(t).
\]  

(11.83)

In period \( t \), the value of the constant \( \Omega(t) \) is known as all shocks in periods \( t - 1 \) and \( t \) have been realized.
11.4.6 Updating New Keynesian Phillips curve

Recall the New Keynesian Phillips curve

\[ \pi_p(t) = \beta E_t \{ \pi(t + 1) \} - \lambda_p (p(t) - mc(t) - \mu_p). \]  
(11.84)

Using the expressions for \( rw(t) \) and \( rw^n(t) \) from the firm marginal cost equation:

\[ p(t) - mc(t) - \mu_p = -\alpha (n(t) - n^n(t)) - (rw(t) - rw^n(t)). \]  
(11.85)

From the expression for \( n(t) \) and \( n^n(t) \) and the definitions of the output gap and wage gap:

\[ p(t) - mc(t) - \mu_p = -\frac{\alpha}{1 - \alpha} \bar{y}(t) - \bar{w}(t). \]  
(11.86)

Defining \( \tilde{\lambda}_p = \frac{\alpha}{1 - \alpha} \lambda_p \), the New Keynesian Phillips curve can be written in terms of both the output gap and the real wage gap:

\[ \pi_p(t) = \beta E_t \{ \pi_p(t + 1) \} + \tilde{\lambda}_p \bar{y}(t) + \lambda_p \bar{w}(t). \]  
(11.87)

11.4.7 Updating wage inflation equation

Recall the wage inflation equation:

\[ \pi_w(t) = \beta E_t \{ \pi_w(t + 1) \} - \lambda_w (w(t) - w^{CE}(t) - \mu_w). \]  
(11.88)

Using the definition of \( w^{CE}(t) \) :

\[ w(t) - w^{CE}(t) - \mu_w = rw(t) - \xi(t) - \gamma n(t) - \sigma y(t) - \mu_w. \]  
(11.89)

Using the expression for \( rw^n(t) \) from the household wage setting problem:

\[ w(t) - w^{CE}(t) - \mu_w = rw(t) - rw^n(t) - \gamma (n(t) - n^n(t)) - \sigma (y(t) - y^n(t)). \]  
(11.90)

From the expression for \( n(t) \) and \( n^n(t) \) :

\[ w(t) - w^{CE}(t) - \mu_w = rw(t) - rw^n(t) - \left( \sigma + \frac{\gamma}{1 - \alpha} \right) (y(t) - y^n(t)). \]  
(11.91)
Defining $\tilde{\lambda}_w = (\sigma + \frac{\gamma}{1-\alpha}) \lambda_w$ and the definitions of the output gap and the wage gap, the wage inflation equation can be written in terms of the output gap and the real wage gap:

$$\pi_w(t) = \beta E_t \{\pi_w(t+1)\} + \tilde{\lambda}_w \tilde{y}(t) - \lambda_w \tilde{r}w(t). \quad (11.92)$$

## 11.5 Introducing unemployment

### 11.5.1 Sneak peek

**Summary**

The model can be used to analyze the effects of policy on the unemployment rate. Thus far, the model allows us to solve for the wage inflation and to solve for the employment rate. This section will define the unemployment rate and use the previous equilibrium equations in order to express the unemployment rate in terms of variables that we already know how to solve. The unemployment rate can be considered an auxiliary variable. When solving the model, we first solve the simultaneous system of equations for our primary variables. Once these values are determined, we can solve for the unemployment rate.

**Notation**

The variables to be introduced in this section are given in the following table:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_j(t)$</td>
<td>labor force participation for type $j$ laborers</td>
</tr>
<tr>
<td>$L(t)$</td>
<td>total labor force participation</td>
</tr>
<tr>
<td>$U(t)$</td>
<td>unemployment rate</td>
</tr>
<tr>
<td>$u^n(t)$</td>
<td>natural rate of unemployment</td>
</tr>
</tbody>
</table>

**Main takeaways**

After completing this section, you will be able to answer the following questions:

- How is the wage inflation rate related to the unemployment rate?
- How is the natural unemployment rate related to the wage markup without nominal rigidities?
- How is the unemployment rate related to the output and wage gap?
11.5.2 Decomposing labor force into employed and unemployed

The role of the labor unions is to generate an endogenous unemployment rate. Thus far, we have only spoken about employed households. There is a larger set of households called labor market participants. Labor market participants consists of two groups: employed and unemployed. Households of type \( j \) and labor disutility index \( h \) are willing to enter the labor force when:

\[
W_j(t) \geq \Xi(t) P(t) (C_j(t))^\sigma h^\gamma. \tag{11.93}
\]

Given that \( h \in [0, 1] \) is uniformly distributed, the marginal household of type \( j \) (the household that is indifferent between labor force participation and nonparticipation) has index \( h = L_j(t) \) such that

\[
W_j(t) = \Xi(t) P(t) (C_j(t))^\sigma (L_j(t))^\gamma. \tag{11.94}
\]

The labor force \( L_j(t) \) is greater than the type \( j \) employment \( N_j(t) \). Households \( h \in [0, N_j(t)] \) are employed, while households \( h \in [N_j(t), L_j(t)] \) are unemployed. The labor unions in the type \( j \) labor market choose the best course of action for all type \( j \) households. This involves choosing a wage markup. The labor union supports this markup by withholding labor supply and keeping employment \( N_j(t) \) below the labor force participation \( L_j(t) \). With a markup equal to 0, employment and labor force participation are equal. The unemployed households do not incur the disutility of labor.

On average, the labor force participation is such that

\[
W(t) = \Xi(t) P(t) (C(t))^\sigma (L(t))^\gamma. \tag{11.95}
\]

Define \( l(t) = \ln (L(t)) \). Log-linearizing the above equation yields:

\[
w(t) = \xi(t) + \gamma l(t) + \sigma c(t) + p(t). \tag{11.96}
\]

Using the definition of the competitive wage rate (at \( N(t) \)):

\[
w(t) - w^{CE}(t) = \gamma (l(t) - n(t)). \tag{11.97}
\]
11.5. **INTRODUCING UNEMPLOYMENT**

### 11.5.3 Defining unemployment

The unemployment rate is defined by

\[ u(t) = l(t) - n(t), \quad (11.98) \]

which for low values of unemployment is approximately equal to

\[ u(t) \approx 1 - \frac{N(t)}{L(t)} = \frac{L(t) - N(t)}{L(t)}, \quad (11.99) \]

which is the precise definition of unemployment. Using the definition of the competitive wage rate:

\[ w(t) - w^{CE}(t) = \gamma (l(t) - n(t)) = \gamma u(t). \quad (11.100) \]

Inserting this expression into the original wage inflation rate equation (without the output and wage gap) yields

\[ \pi_w(t) = \beta E_t \{ \pi_w(t + 1) \} - \lambda_w (\gamma u(t) - \mu_w). \quad (11.101) \]

In the flexible wage setting, the natural unemployment rate \( u^n(t) \) is such that \( \gamma u^n(t) = w^n(t) - w^{CE}(t) = \mu_w \), or

\[ u^n(t) = \frac{\mu_w}{\gamma}. \quad (11.102) \]

Given this natural unemployment rate, the wage inflation rate equation is equivalently written as an equation that is commonly referred to as the New Keynesian wage Phillips curve:

\[ \pi_w(t) = \beta E_t \{ \pi_w(t + 1) \} - \lambda_w \gamma (u(t) - u^n(t)). \quad (11.103) \]

Using the wage inflation equation in terms of both the output gap and the real wage gap (from the previous section), the unemployment rate is given by:

\[ u(t) - u^n(t) = \frac{1}{\gamma} \left( \tilde{\gamma} w(t) - \left( \sigma + \frac{\gamma}{1 - \alpha} \right) \tilde{y}(t) \right). \quad (11.104) \]

When the output gap is a large negative value (recall \( \tilde{y}(t) < 0 \)) and the wage gap is large, the unemployment rate will be high.
11.6 Solving the model

11.6.1 Sneak peek

Summary

We solve the model by focusing on the three shocks independently. The shocks to be analyzed, in order, are a monetary policy shock, a technology shock, and a preference shock.

Notation

The variables to be introduced in this section are given in the following table:

\[\begin{align*}
\rho_\nu & \text{ autoregressive coefficient for monetary shock} \\
\rho_a & \text{ autoregressive coefficient for technology shock} \\
\rho_\xi & \text{ autoregressive coefficient for preference shock} \\
\epsilon_\nu(t) & \text{ zero-mean error term for monetary shock} \\
\epsilon_a(t) & \text{ zero-mean error term for technology shock} \\
\epsilon_\xi(t) & \text{ zero-mean error term for preference shock} \\
x(t) & \text{ vector of 3 equilibrium variables } x(t) = (\hat{y}(t), \pi_p(t), \pi_w(t))^T \\
\psi(t) & \text{ vector of 2 shocks} \\
A_0 & \text{ matrix that premultiplies } x(t) \\
A_1 & \text{ matrix that premultiplies } E_t \{x(t + 1)\} \\
B & \text{ matrix that premultiplies the shock vector}
\end{align*}\]

Main takeaways

After completing this section, you will be able to answer the following questions:

- What are the effects of a monetary policy shock on the output gap, the two inflation rates, the output, aggregate employment, labor force participation, and the unemployment rate?

- What are the effects of a technology shock on the output gap, the two inflation rates, the output, aggregate employment, labor force participation, and the unemployment rate?
11.6. SOLVING THE MODEL

- What are the effects of a preference shock on the output gap, the two inflation rates, the output, aggregate employment, labor force participation, and the unemployment rate?

11.6.2 Setting up the system of equations

A Taylor rule must be included to determine the interest rate \( i(t) \). The Taylor rule only enters into the household Euler equation. Extending the Taylor rule from the previous chapter, the Taylor rule in the current model is of the form:

\[
i(t) = \delta + \phi_p \pi_p(t) + \phi_w \pi_w(t) + \nu(t),
\]

(11.105)

where the parameters satisfy \( \phi_p + \phi_w > 1 \) and \( \nu(t) \) is the monetary policy shock (with zero mean). The Taylor principle in this model with two types of inflation (commodity and wage) requires that \( \phi_p + \phi_w > 1 \). This strict inequality is required in order to have a solution to the system of equations.

Assume that the monetary policy shock follows an AR(1) process as

\[
\nu(t) = \rho_\nu \nu(t - 1) + \epsilon_\nu(t),
\]

(11.106)

where \( \rho_\nu \in (0, 1) \) and \( \epsilon_\nu(t) \) is a random variable drawn from a zero-mean normal distribution.

Assume that the technology shock follows an AR(1) process as

\[
a(t) = \rho_a a(t - 1) + \epsilon_a(t),
\]

(11.107)

where \( \rho_a \in (0, 1) \) and \( \epsilon_a(t) \) is a random variable drawn from a zero-mean normal distribution.

Assume that the preference shock follows an AR(1) process as

\[
\xi(t) = \rho_\xi \xi(t - 1) + \epsilon_\xi(t),
\]

(11.108)

where \( \rho_\xi \in (0, 1) \) and \( \epsilon_\xi(t) \) is a random variable drawn from a zero-mean normal distribution.

Recall the wage gap identity:

\[
\hat{w}(t) = \hat{w}(t - 1) + \pi_w(t) - \pi_p(t) - \Omega(t).
\]

(11.109)

Inserting the Taylor rule and the wage gap identity into the Euler equation, the New Keyne-
sian Phillips curve, and the wage inflation equation, we arrive at the 3 equilibrium equations:

\[
\sigma \tilde{y}(t) = \sigma E_t \{ \tilde{y}(t + 1) \} - \phi_p \pi_p(t) - \phi_w \pi_w(t) - \nu(t) + E_t \{ \pi_p(t + 1) \} \tag{11.110}
\]

\[
- \sigma \phi_a^n (1 - \rho_a) a(t) - \sigma \phi_\xi^n (1 - \rho_\xi) \xi(t).
\]

\[
\pi_p(t) = \beta E_t \{ \pi_p(t + 1) \} + \tilde{\lambda}_p \tilde{y}(t) + \lambda_p (\tilde{r}w(t - 1) + \pi_w(t) - \pi_p(t) - \Omega(t)). \tag{11.112}
\]

\[
\pi_w(t) = \beta E_t \{ \pi_w(t + 1) \} + \tilde{\lambda}_w \tilde{y}(t) - \lambda_w (\tilde{r}w(t - 1) + \pi_w(t) - \pi_p(t) - \Omega(t)). \tag{11.113}
\]

Define \( x(t) = \begin{bmatrix} \tilde{y}(t) \\ \pi_p(t) \\ \pi_w(t) \end{bmatrix} \) as the equilibrium variables. The shock vector is given by

\[
z(t) = \begin{bmatrix} \nu(t) \\ a(t) \\ \xi(t) \\ \tilde{r}w(t - 1) - \Omega(t) \end{bmatrix}. \tag{11.114}
\]

The 3 equations can be written recursively as:

\[
A_0 x(t) = A_1 E_t \{ x(t + 1) \} + B z(t), \tag{11.115}
\]

where the matrices are given by:

\[
A_0 = \begin{bmatrix} \sigma & \phi_p & \phi_w \\ -\tilde{\lambda}_p & 1 + \lambda_p & -\lambda_p \\ -\tilde{\lambda}_w & -\lambda_w & 1 + \lambda_w \end{bmatrix}. \tag{11.116}
\]

\[
A_1 = \begin{bmatrix} \sigma & 1 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{bmatrix}.
\]

\[
B = \begin{bmatrix} -1 & -\sigma \phi_a^n (1 - \rho_a) & -\sigma \phi_\xi^n (1 - \rho_\xi) & 0 \\ 0 & 0 & 0 & \lambda_p \\ 0 & 0 & 0 & \lambda_w \end{bmatrix}.
\]

Given the matrix form \( A_0 x(t) = A_1 E_t \{ x(t + 1) \} + B z(t) \), the solutions \( x(t) \) must be linear functions of the shocks \( (\nu(t), a(t), \xi(t), \tilde{r}w(t - 1) - \Omega(t)) \). Rather than analyzing all shocks at the same time, it is instructive to focus on the shocks independently.
If the term $\tilde{r}w(t-1) - \Omega(t) = 0$, then analyzing each shock in turn, we would start with $\nu(t)$ and recognize that $E_t\{x(t+1)\} = \rho_x x(t)$, meaning that
\[ A_0 x(t) = A_1 E_t\{x(t+1)\} + B z (t) \]  
(11.117)
can be solved for $x(t)$:
\[ x(t) = (A_0 - \rho_x A_1)^{-1} B z(t), \]  
(11.118)
which is very similar to the approach followed in the previous chapter. However, since the term $\tilde{r}w(t-1) - \Omega(t)$ must be accounted for, a more robust solution method must be employed.

### 11.6.3 Effects of a monetary policy shock

Shut off the technology shock and the preference shock, meaning that $a(t)$ and $\xi(t)$ are constant in every period. Recalling the definition of the term $\Omega(t)$, then $\Omega(t) = 0$. The variables $(\tilde{y}(t), \pi_p(t), \pi_w(t))$ are linear functions of $(\nu(t), \tilde{r}w(t-1))$. Denote these linear functions as:
\[ \tilde{y}(t) = \eta_y \nu(t) + \tilde{\eta}_y \tilde{r}w(t-1). \]  
(11.119)
\[ \pi_p(t) = \eta_p \nu(t) + \tilde{\eta}_p \tilde{r}w(t-1). \]  
\[ \pi_w(t) = \eta_w \nu(t) + \tilde{\eta}_w \tilde{r}w(t-1). \]

There are 6 unknowns ($\eta_y, \eta_p, \eta_w, \tilde{\eta}_y, \tilde{\eta}_p, \tilde{\eta}_w$). We will solve for the values of these unknowns using the 3 equilibrium equations.

#### New Keynesian Phillips curve

By definition:
\[ \pi_p(t) = \eta_p \nu(t) + \tilde{\eta}_p \tilde{r}w(t-1). \]  
(11.120)
\[ E_t\{\pi_p(t+1)\} = \eta_p E_t\{\nu(t+1)\} + \tilde{\eta}_p \tilde{r}w(t). \]

Since the shock follows an AR(1) process, this implies
\[ \pi_p(t) - \beta E_t\{\pi_p(t+1)\} = \eta_p (1 - \beta \rho_x) \nu(t) + \tilde{\eta}_p \tilde{r}w(t-1) - \beta \tilde{\eta}_p (\tilde{r}w(t-1) + \pi_w(t) - \pi_p(t)). \]  
(11.121)
11. NEW KEYNESIAN LABOR MARKET THEORY

From the New Keynesian Phillips curve:

\[ \pi_p(t) - \beta E_t \{ \pi_p(t + 1) \} = \tilde{\lambda}_p \tilde{y}(t) + \lambda_p (\tilde{\bar{w}}(t - 1) + \pi_w(t) - \pi_p(t)) \, . \]  

(11.122)

Using the linear functions for \((\tilde{y}(t), \pi_p(t), \pi_w(t))\) and setting the above equations equal yields:

\[ \nu(t) \left\{ \eta_p (1 - \beta \rho_w) + (\beta \tilde{\eta}_p + \lambda_p) (\eta_p - \eta_w) - \tilde{\lambda}_p \eta_y \right\} \]

+ \tilde{\bar{w}}(t - 1) \left\{ \tilde{\eta}_p (1 - \beta) + (\beta \tilde{\eta}_p + \lambda_p) (\tilde{\eta}_p - \tilde{\eta}_w) - \lambda_p - \tilde{\lambda}_p \tilde{\eta}_y \right\} = 0. \]

(11.123)

Since the equation must hold for all shock values \((\nu(t), \tilde{\bar{w}}(t - 1))\), we have two equations:

- **Equation 1:** \( \eta_p (1 - \beta \rho_w) + (\beta \tilde{\eta}_p + \lambda_p) (\eta_p - \eta_w) - \tilde{\lambda}_p \eta_y = 0. \)
- **Equation 2:** \( \tilde{\eta}_p (1 - \beta) + (\beta \tilde{\eta}_p + \lambda_p) (\tilde{\eta}_p - \tilde{\eta}_w) - \lambda_p - \tilde{\lambda}_p \tilde{\eta}_y = 0. \)

**Wage inflation equation**

By definition:

\[ \pi_w(t) = \eta_w \nu(t) + \tilde{\eta}_w \tilde{\bar{w}}(t - 1). \]

(11.124)

\[ E_t \{ \pi_w(t + 1) \} = \eta_w E_t \{ \nu(t + 1) \} + \tilde{\eta}_w \tilde{\bar{w}}(t). \]

Since the shock follows an AR(1) process, this implies

\[ \pi_w(t) - \beta E_t \{ \pi_w(t + 1) \} = \eta_w (1 - \beta \rho_w) \nu(t) + \tilde{\eta}_w \tilde{\bar{w}}(t - 1) - \beta \tilde{\eta}_w (\tilde{\bar{w}}(t - 1) + \pi_w(t) - \pi_p(t)) \, . \]

(11.125)

From the wage inflation equation:

\[ \pi_w(t) - \beta E_t \{ \pi_w(t + 1) \} = \tilde{\lambda}_w \tilde{y}(t) + \lambda_w (\tilde{\bar{w}}(t - 1) + \pi_w(t) - \pi_p(t)) \, . \]

(11.126)

Using the linear functions for \((\tilde{y}(t), \pi_p(t), \pi_w(t))\) and setting the above equations equal yields:

\[ \nu(t) \left\{ \eta_w (1 - \beta \rho_w) + (\beta \tilde{\eta}_w + \lambda_w) (\eta_p - \eta_w) - \tilde{\lambda}_w \eta_y \right\} \]

+ \tilde{\bar{w}}(t - 1) \left\{ \tilde{\eta}_w (1 - \beta) + (\beta \tilde{\eta}_w + \lambda_w) (\tilde{\eta}_p - \tilde{\eta}_w) - \lambda_w - \tilde{\lambda}_w \tilde{\eta}_y \right\} = 0. \]

(11.127)

Since the equation must hold for all shock values \((\nu(t), \tilde{\bar{w}}(t - 1))\), we have two equations:
11.6. SOLVING THE MODEL

- Equation 3: \( \eta_w (1 - \beta \rho_w) + (\beta \bar{\eta}_w + \lambda_w) (\eta_p - \eta_w) - \tilde{\lambda}_w \eta_y = 0. \)

- Equation 4: \( \tilde{\eta}_w (1 - \beta) + (\beta \bar{\eta}_w + \lambda_w) (\tilde{\eta}_p - \tilde{\eta}_w) - \lambda_w - \tilde{\lambda}_w \tilde{\eta}_y = 0. \)

**Household Euler equation**

By definition:

\[
\ddot{y}(t) = \eta_y \nu(t) + \bar{\eta}_y \tilde{w}(t - 1).
\]

\[
E_t \{ \ddot{y}(t + 1) \} = \eta_y E_t \{ \nu(t + 1) \} + \bar{\eta}_y \tilde{w}(t).
\]

Since the shock follows an AR(1) process, this implies

\[
\ddot{y}(t) - E_t \{ \ddot{y}(t + 1) \} = \eta_y (1 - \rho_w) \nu(t) - \bar{\eta}_y (\pi_w(t) - \pi_p(t)).
\]

From the household Euler equation:

\[
\ddot{y}(t) - E_t \{ \ddot{y}(t + 1) \} = -\frac{1}{\sigma} \left( \phi_p \pi_p(t) + \phi_w \pi_w(t) + \nu(t) \right) + \frac{1}{\sigma} E_t \{ \pi_p(t + 1) \}.
\]

Using the linear functions for \((\ddot{y}(t), \pi_p(t), \pi_w(t))\) and \(E_t \{ \pi_p(t + 1) \}\) and setting the above equations equal:

\[
\begin{align*}
\nu(t) \left\{ \eta_y (1 - \rho_w) + \left( \ddot{\eta}_y + \frac{\tilde{\eta}_p}{\sigma} \right) (\eta_p - \eta_w) + \frac{1}{\sigma} \left( \phi_p \eta_p + \phi_w \eta_w + 1 - \eta_p \rho_w \right) \right\} \\
+ \tilde{\nu}(t - 1) \left\{ - \ddot{\eta}_p \right\} + \left( \ddot{\eta}_y + \frac{\tilde{\eta}_p}{\sigma} \right) (\tilde{\eta}_p - \tilde{\eta}_w) + \frac{1}{\sigma} \left( \phi_p \tilde{\eta}_p + \phi_w \tilde{\eta}_w \right) \right\} = 0.
\end{align*}
\]

Since the equation must hold for all shock values \((\nu(t), \tilde{\nu}(t - 1))\), we have two equations:

- Equation 5: \( \eta_y (1 - \rho_w) + \left( \ddot{\eta}_y + \frac{\tilde{\eta}_p}{\sigma} \right) (\eta_p - \eta_w) + \frac{1}{\sigma} \left( \phi_p \eta_p + \phi_w \eta_w - \eta_p \rho_w \right) = -\frac{1}{\sigma}. \)

- Equation 6: \(- \ddot{\eta}_w \right\} + \left( \ddot{\eta}_y + \frac{\tilde{\eta}_p}{\sigma} \right) (\tilde{\eta}_p - \tilde{\eta}_w) + \frac{1}{\sigma} \left( \phi_p \tilde{\eta}_p + \phi_w \tilde{\eta}_w \right) = 0. \)

We have 6 equations for the 6 unknowns \((\eta_y, \eta_p, \eta_w, \bar{\eta}_y, \tilde{\eta}_p, \tilde{\eta}_w)\). Observe that Equations 2, 4, and 6 only depend on the three unknowns \((\bar{\eta}_y, \tilde{\eta}_p, \tilde{\eta}_w)\). An end-of-chapter exercise asks you to solve for the effects of a monetary policy shock, namely to find the values for the unknowns \((\eta_y, \eta_p, \eta_w)\).
11.6.4 Effects of a technology shock

Shut off the monetary policy shock and the preference shock, meaning that \( \nu(t) = 0 \) and \( \xi(t) \) are constant in every period. Recalling the definition of the term \( \Omega(t) \), then \( \Omega(t) = \frac{(1-\alpha\phi^n_a)(a(t)-\alpha(t-1))}{1-\alpha} \). The variables \((\bar{y}(t), \pi_p(t), \pi_w(t))\) are linear functions of \((a(t), \bar{r}w(t-1) - \Omega(t))\). Denote these linear functions as:

\[
\bar{y}(t) = \eta_y a(t) + \bar{\eta}_y (\bar{r}w(t-1) - \Omega(t)). 
\]

\[
\pi_p(t) = \eta_\pi a(t) + \bar{\eta}_\pi (\bar{r}w(t-1) - \Omega(t)). 
\]

\[
\pi_w(t) = \eta_w a(t) + \bar{\eta}_w (\bar{r}w(t-1) - \Omega(t)). 
\]

We use the same notation for the 6 unknowns \((\eta_y, \eta_\pi, \eta_w, \bar{\eta}_y, \bar{\eta}_\pi, \bar{\eta}_w)\) as in the previous subsection. Do not be confused; these coefficients will have different values. We will solve for the values of these unknowns using the 3 equilibrium equations.

**New Keynesian Phillips curve**

By definition:

\[
\pi_p(t) = \eta_\pi a(t) + \bar{\eta}_\pi (\bar{r}w(t-1) - \Omega(t)). 
\]

\[
E_t \{\pi_p(t+1)\} = \eta_\pi E_t \{a(t+1)\} + \bar{\eta}_\pi (\bar{r}w(t) - E_t \{\Omega(t+1)\}). 
\]

Since the shock follows an AR(1) process, this implies

\[
E_t \{\pi_p(t+1)\} = \left( \eta_\pi \rho_a + \bar{\eta}_\pi \frac{(1-\alpha\phi^n_a)(1-\rho_a)}{1-\alpha} \right) a(t) + \bar{\eta}_\pi (\bar{r}w(t-1) + \pi_w(t) - \pi_p(t) - \Omega(t)). 
\]

The difference

\[
\pi_p(t) - \beta E_t \{\pi_p(t+1)\} = \left( \eta_\pi - \beta \left( \eta_\pi \rho_a + \bar{\eta}_\pi \frac{(1-\alpha\phi^n_a)(1-\rho_a)}{1-\alpha} \right) \right) a(t) + \bar{\eta}_\pi (\bar{r}w(t-1) - \Omega(t)) 
\]

\[
- \beta \bar{\eta}_\pi (\bar{r}w(t-1) + \pi_w(t) - \pi_p(t) - \Omega(t)). 
\]

From the New Keynesian Phillips curve:

\[
\pi_p(t) - \beta E_t \{\pi_p(t+1)\} = \bar{\lambda}_p \bar{y}(t) + \lambda_p (\bar{r}w(t-1) + \pi_w(t) - \pi_p(t) - \Omega(t)). 
\]
Using the linear functions for \((\tilde{y}(t), \pi_p(t), \pi_w(t))\) and setting the above equations equal yields:

\[
a(t) \left\{ \eta_p - \beta \left( \eta_p \rho_a + \eta_w \frac{(1 - \alpha \phi^w_a)(1 - \rho_a)}{1 - \alpha} \right) + (\beta \tilde{\eta}_p + \lambda_p) (\eta_p - \eta_w) - \tilde{\lambda}_p \eta_y \right\} + (\tilde{\tau}\tilde{w}(t-1) - \Omega(t)) \left\{ \tilde{\eta}_p (1 - \beta) + (\beta \tilde{\eta}_p + \lambda_p) (\tilde{\eta}_p - \tilde{\eta}_w) - \lambda_p - \tilde{\lambda}_p \tilde{\eta}_y \right\} = 0.
\]

(11.138)

Since the equation must hold for all shock values \((a(t), \tilde{\tau}\tilde{w}(t-1) - \Omega(t))\), we have two equations:

- **Equation 1**: \(\eta_p (1 - \beta \rho_a) + (\beta \tilde{\eta}_p + \lambda_p) (\eta_p - \eta_w) - \tilde{\lambda}_p \eta_y = \beta \tilde{\eta}_p \frac{(1 - \alpha \phi^w_a)(1 - \rho_a)}{1 - \alpha}.

- **Equation 2**: \(\tilde{\eta}_p (1 - \beta) + (\beta \tilde{\eta}_p + \lambda_p) (\tilde{\eta}_p - \tilde{\eta}_w) - \lambda_p - \tilde{\lambda}_p \tilde{\eta}_y = 0.

Observe that Equation 2 with the technology shock is identical to Equation 2 with the monetary policy shock.

**Wage inflation equation**

By definition:

\[
\pi_w(t) = \eta_w a(t) + \tilde{\eta}_w (\tilde{\tau}\tilde{w}(t-1) - \Omega(t)).
\]

(11.139)

\[
E_t \{\pi_w(t+1)\} = \eta_w E_t \{a(t+1)\} + \tilde{\eta}_w (\tilde{\tau}\tilde{w}(t) - E_t \{\Omega(t+1)\}).
\]

Since the shock follows an AR(1) process, this implies

\[
E_t \{\pi_w(t+1)\} = \left( \eta_w \rho_a + \tilde{\eta}_w \frac{(1 - \alpha \phi^w_a)(1 - \rho_a)}{1 - \alpha} \right) a(t) + \tilde{\eta}_w (\tilde{\tau}\tilde{w}(t-1) + \pi_w(t) - \pi_p(t) - \Omega(t)).
\]

(11.140)

The difference

\[
\pi_w(t) - \beta E_t \{\pi_w(t+1)\} = \left( \eta_w - \beta \left( \eta_w \rho_a + \tilde{\eta}_w \frac{(1 - \alpha \phi^w_a)(1 - \rho_a)}{1 - \alpha} \right) \right) a(t) + \tilde{\eta}_w (\tilde{\tau}\tilde{w}(t-1) - \Omega(t)) - \beta \tilde{\eta}_w (\tilde{\tau}\tilde{w}(t-1) + \pi_w(t) - \pi_p(t) - \Omega(t)).
\]

(11.141)

From the wage inflation equation:

\[
\pi_w(t) - \beta E_t \{\pi_w(t+1)\} = \tilde{\lambda}_w \tilde{y}(t) + \lambda_w (\tilde{\tau}\tilde{w}(t-1) + \pi_w(t) - \pi_p(t) - \Omega(t)).
\]

(11.143)
Using the linear functions for \( \tilde{y}(t), \pi_p(t), \pi_w(t) \) and setting the above equations equal yields:

\[
a(t) \left\{ \eta_w - \beta \left( \eta_w \rho_a + \tilde{\eta}_w \frac{(1 - \alpha \phi_a^n)(1 - \rho_a)}{1 - \alpha} \right) + (\tilde{\beta} \tilde{\eta}_w + \lambda_w) (\eta_p - \eta_w) - \tilde{\lambda}_w \eta_y \right\} + (\tilde{\rho} \tilde{w}(t - 1) - \Omega(t)) \left\{ \tilde{\eta}_w (1 - \beta) + (\tilde{\beta} \tilde{\eta}_w + \lambda_w) (\tilde{\eta}_p - \tilde{\eta}_w) - \lambda_w - \tilde{\lambda}_w \tilde{\eta}_y \right\} = 0.
\]

Since the equation must hold for all shock values \((\nu(t), \tilde{\rho} \tilde{w}(t - 1))\), we have two equations:

- **Equation 3**: \( \eta_w (1 - \beta \rho_a) + (\tilde{\beta} \tilde{\eta}_w + \lambda_w) (\eta_p - \eta_w) - \tilde{\lambda}_w \eta_y = \beta \tilde{\eta}_w \frac{(1 - \alpha \phi_a^n)(1 - \rho_a)}{1 - \alpha} \).

- **Equation 4**: \( \tilde{\eta}_w (1 - \beta) + (\tilde{\beta} \tilde{\eta}_w + \lambda_w) (\tilde{\eta}_p - \tilde{\eta}_w) - \lambda_w - \tilde{\lambda}_w \eta_y = 0 \).

Observe that Equation 4 with the technology shock is identical to Equation 4 with the monetary policy shock.

**Household Euler equation**

By definition:

\[
\tilde{y}(t) = \eta_y a(t) + \tilde{\eta}_y (\tilde{\rho} \tilde{w}(t - 1) - \Omega(t)) \quad \text{.}
\]  
\[E_t \{\tilde{y}(t + 1)\} = \eta_y E_t \{a(t + 1)\} + \tilde{\eta}_y (\tilde{\rho} \tilde{w}(t) - E_t \{\Omega(t + 1)\}) \quad \text{.}
\]

Since the shock follows an AR(1) process, this implies

\[
E_t \{\tilde{y}(t + 1)\} = \left( \eta_y \rho_a + \tilde{\eta}_y \frac{(1 - \alpha \phi_a^n)(1 - \rho_a)}{1 - \alpha} \right) a(t) + \tilde{\eta}_y (\tilde{\rho} \tilde{w}(t - 1) + \pi_w(t) - \pi_p(t) - \Omega(t)) \quad \text{.}
\]

The difference

\[
\tilde{y}(t) - E_t \{\tilde{y}(t + 1)\} = \left( \eta_y - \left( \eta_y \rho_a + \tilde{\eta}_y \frac{(1 - \alpha \phi_a^n)(1 - \rho_a)}{1 - \alpha} \right) \right) a(t) - \tilde{\eta}_y (\pi_w(t) - \pi_p(t)) \quad \text{.}
\]

From the household Euler equation:

\[
\tilde{y}(t) - E_t \{\tilde{y}(t + 1)\} = -\frac{1}{\sigma} (\phi_p \pi_p(t) + \phi_w \pi_w(t)) + \frac{1}{\sigma} E_t \{\pi_p(t + 1)\} - \phi_a^n (1 - \rho_a) a(t) \quad \text{.}
\]
From above:

\[ E_t \{ \pi_p(t + 1) \} = \left( \eta_p \rho_a + \tilde{\eta}_p \frac{(1 - \alpha \phi_a^n) (1 - \rho_a)}{1 - \alpha} \right) a(t) + \tilde{\eta}_p \left( \tilde{r} \tilde{w}(t - 1) + \pi_w(t) - \pi_p(t) - \Omega(t) \right). \]

(11.149)

Using the linear functions for \((\tilde{y}(t), \pi_p(t), \pi_w(t))\) and setting the above equations equal:

\[ a(t) \left\{ \eta_y - \left( \eta_y \rho_a + \tilde{\eta}_y \frac{(1 - \alpha \phi_a^n) (1 - \rho_a)}{1 - \alpha} \right) - \frac{1}{\sigma} \left( \eta_p \rho_a + \tilde{\eta}_p \frac{(1 - \alpha \phi_a^n) (1 - \rho_a)}{1 - \alpha} \right) \right\} + a(t) \left\{ \phi_a^n (1 - \rho_a) + \left( \tilde{\eta}_y + \frac{\tilde{\eta}_p}{\sigma} \right) (\eta_p - \eta_w) + \frac{1}{\sigma} (\phi_p \eta_p + \phi_w \eta_w) \right\} + (\tilde{r} \tilde{w}(t - 1) - \Omega(t)) \left\{ - \frac{\tilde{\eta}_p}{\sigma} + \left( \tilde{\eta}_y + \frac{\tilde{\eta}_p}{\sigma} \right) (\tilde{\eta}_p - \tilde{\eta}_w) + \frac{1}{\sigma} (\phi_p \tilde{\eta}_p + \phi_w \tilde{\eta}_w) \right\} = 0. \]

(11.150)

Since the equation must hold for all shock values \((\nu(t), \tilde{r} \tilde{w}(t - 1))\), we have two equations:

- **Equation 5:**

\[ \eta_y (1 - \rho_a) + \left( \tilde{\eta}_y + \frac{\tilde{\eta}_p}{\sigma} \right) (\eta_p - \eta_w) + \frac{1}{\sigma} (\phi_p \eta_p + \phi_w \eta_w - \eta_p \rho_a) = \tilde{\eta}_y \frac{(1 - \alpha \phi_a^n) (1 - \rho_a)}{1 - \alpha} + \frac{\tilde{\eta}_p}{\sigma} \frac{(1 - \alpha \phi_a^n) (1 - \rho_a)}{1 - \alpha} - \phi_a^n (1 - \rho_a). \]

(11.151)

- **Equation 6:**

\[ - \frac{\tilde{\eta}_p}{\sigma} + \left( \tilde{\eta}_y + \frac{\tilde{\eta}_p}{\sigma} \right) (\tilde{\eta}_p - \tilde{\eta}_w) + \frac{1}{\sigma} (\phi_p \tilde{\eta}_p + \phi_w \tilde{\eta}_w) = 0. \]

Observe that Equation 6 with the technology shock is identical to Equation 6 with the monetary policy shock.

We have 6 equations for the 6 unknowns \((\eta_y, \eta_p, \eta_w, \tilde{\eta}_y, \tilde{\eta}_p, \tilde{\eta}_w)\). Since Equations 2, 4, and 6 are identical to the case with a monetary policy shock, then the values for \((\tilde{\eta}_y, \tilde{\eta}_p, \tilde{\eta}_w)\) are identical to the values from the case with a monetary policy shock. An end-of-chapter exercise asks you to solve for the effects of a technology shock. Since \(\Omega(t)\) contains the term \(a(t)\), then the coefficient for \(a(t)\) is found from the following equations:

\[ \tilde{y}(t) = \left( \eta_y - \tilde{\eta}_y \frac{(1 - \alpha \phi_a^n)}{1 - \alpha} \right) a(t) + \tilde{\eta}_y \left( \tilde{r} \tilde{w}(t - 1) + \frac{1 - \alpha \phi_a^n}{1 - \alpha} a(t - 1) \right). \]

(11.152)

\[ \pi_p(t) = \left( \eta_p - \tilde{\eta}_p \frac{(1 - \alpha \phi_a^n)}{1 - \alpha} \right) a(t) + \tilde{\eta}_p \left( \tilde{r} \tilde{w}(t - 1) + \frac{1 - \alpha \phi_a^n}{1 - \alpha} a(t - 1) \right). \]

\[ \pi_w(t) = \left( \eta_w - \tilde{\eta}_w \frac{(1 - \alpha \phi_a^n)}{1 - \alpha} \right) a(t) + \tilde{\eta}_w \left( \tilde{r} \tilde{w}(t - 1) + \frac{1 - \alpha \phi_a^n}{1 - \alpha} a(t - 1) \right). \]
11.6.5 Effects of a preference shock

Shut off the monetary policy shock and the technology shock, meaning that \( \nu(t) = 0 \) and \( a(t) \) are constant in every period. Recalling the definition of the term \( \Omega(t) \), then \( \Omega(t) = -\frac{\alpha \phi_{\xi}^n (\xi(t) - \xi(t-1))}{1-\alpha} \). The variables \((\tilde{y}(t), \pi_p(t), \pi_w(t))\) are linear functions of \((\xi(t), \tilde{\tau}w(t - 1) - \Omega(t))\). Denote these linear functions as:

\[
\tilde{y}(t) = \eta_y \xi(t) + \tilde{\eta}_y (\tilde{\tau}w(t - 1) - \Omega(t)). 
\]

\[
\pi_p(t) = \eta_p \xi(t) + \tilde{\eta}_p (\tilde{\tau}w(t - 1) - \Omega(t)).
\]

\[
\pi_w(t) = \eta_w \xi(t) + \tilde{\eta}_w (\tilde{\tau}w(t - 1) - \Omega(t)). 
\]

The analysis proceeds exactly as with the analysis for the technology shock. The only difference is that (i) all parameters \( \rho_n \) in the equations are replaced by \( \rho_{\xi} \) and (ii) the term \( \frac{(1-\alpha \phi_{\xi}^n)(1-\rho_n)}{1-\alpha} \) is replaced by the term \( -\frac{\alpha \phi_{\xi}^n (1-\rho_{\xi})}{1-\alpha} \) in Equations 1, 3, and 5. Equations 2, 4, and 6 remain unchanged (same equations no matter whether the shock is a monetary policy shock, a technology shock, or a preference shock).

An end-of-chapter exercise asks you to solve for the effects of a preference shock. Since \( \Omega(t) \) contains the term \( \xi(t) \), then the coefficient for \( \xi(t) \) is found from the following equations:

\[
\tilde{y}(t) = \left( \eta_y + \tilde{\eta}_y \frac{\alpha \phi_{\xi}^n}{1-\alpha} \right) \xi(t) + \tilde{\eta}_y \left( \tilde{\tau}w(t - 1) - \frac{\alpha \phi_{\xi}^n}{1-\alpha} \xi(t-1) \right). 
\]

\[
\pi_p(t) = \left( \eta_p + \tilde{\eta}_p \frac{\alpha \phi_{\xi}^n}{1-\alpha} \right) \xi(t) + \tilde{\eta}_p \left( \tilde{\tau}w(t - 1) - \frac{\alpha \phi_{\xi}^n}{1-\alpha} \xi(t-1) \right). 
\]

\[
\pi_w(t) = \left( \eta_w + \tilde{\eta}_w \frac{\alpha \phi_{\xi}^n}{1-\alpha} \right) \xi(t) + \tilde{\eta}_w \left( \tilde{\tau}w(t - 1) - \frac{\alpha \phi_{\xi}^n}{1-\alpha} \xi(t-1) \right). 
\]

11.7 Exercises

1. Solving the model

For the model with only the monetary policy shock, determine the values for \((\eta_y, \eta_p, \eta_w)\) by choosing parameter values. Can you find parameter values so that values \((\eta_y, \eta_p, \eta_w)\) change sign? What does this imply about the effect of parameter values on the predictions of the model under a monetary policy shock?

2. Solving the model
For the model with only the technology shock, determine the values for \((\eta_y, \eta_p, \eta_w, \tilde{\eta}_y, \tilde{\eta}_p, \tilde{\eta}_w)\) by choosing parameter values. The effects of a technology shock for each of the variables \((\tilde{y}(t), \pi_p(t), \pi_w(t))\), respectively, are given by the coefficients

\[
\left( \eta_y - \tilde{\eta}_y \frac{1 - \alpha \phi_a^n}{1 - \alpha}, \eta_p - \tilde{\eta}_p \frac{1 - \alpha \phi_a^n}{1 - \alpha}, \eta_w - \tilde{\eta}_w \frac{1 - \alpha \phi_a^n}{1 - \alpha} \right).
\]

Can you find parameter values so that these coefficients change sign? What does this imply about the effect of parameter values on the predictions of the model under a technology shock?

3. **Solving the model**

For the model with only the preference shock, determine the values for \((\eta_y, \eta_p, \eta_w, \tilde{\eta}_y, \tilde{\eta}_p, \tilde{\eta}_w)\) by choosing parameter values. The effects of a preference shock for each of the variables \((\tilde{y}(t), \pi_p(t), \pi_w(t))\), respectively, are given by the coefficients

\[
\left( \eta_y + \tilde{\eta}_y \frac{\alpha \phi^n}{1 - \alpha}, \eta_p + \tilde{\eta}_p \frac{\alpha \phi^n}{1 - \alpha}, \eta_w + \tilde{\eta}_w \frac{\alpha \phi^n}{1 - \alpha} \right).
\]

Can you find parameter values so that these coefficients change sign? What does this imply about the effect of parameter values on the predictions of the model under a preference shock?
Bibliography


Part VI

Financial Economics
12

Asset Pricing

12.1 General financial equilibrium

12.1.1 Sneak peek

Summary

This chapter introduces the main asset pricing models: CAPM, Lucas asset pricing, bubbles, and Black-Scholes. To understand these different asset pricing models, it is essential to understand the theory of competitive financial markets. The field of general equilibrium studies competitive markets in which prices are determined in equilibrium. In the chapter 'Microfoundations', static general equilibrium models with multiple commodity markets were analyzed. In the present chapter, dynamic general equilibrium models with multiple financial markets will be introduced.

With multiple time periods, households transfer resources across time using financial assets. A financial asset is a contract that specifies a vector of future payouts. The price of a financial asset is an endogenous variable whose value must be such that the asset market clearing condition holds. The asset market clearing condition states that the number of units of the financial contract sold must be equal to the number of units purchased.

In models of asset pricing, it is important to model multiple financial assets traded in each period. This allows for an analysis of the portfolio effects that are absent in a model with only one asset.

To focus the analysis on financial markets, the models in this chapter assume that there is a single commodity traded and consumed in each period.
Notation

The variables/parameters to be introduced in this section are given in the following table:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$</td>
<td>set of all states $S = {0, ..., S}$</td>
</tr>
<tr>
<td>$c^h(s)$</td>
<td>consumption in state $s \in S$</td>
</tr>
<tr>
<td>$c^h$</td>
<td>vector of consumption in all states</td>
</tr>
<tr>
<td>$\pi_s$</td>
<td>probability that state $s$ occurs</td>
</tr>
<tr>
<td>$e^h(s)$</td>
<td>endowment in state $s \in S$</td>
</tr>
<tr>
<td>$e^h$</td>
<td>vector of endowment in all states</td>
</tr>
<tr>
<td>$r_j(s)$</td>
<td>payout of asset $j$ in state $s$</td>
</tr>
<tr>
<td>$q_j$</td>
<td>price for asset $j$</td>
</tr>
<tr>
<td>$z_j^h$</td>
<td>asset position for asset $j$ by household $h$</td>
</tr>
<tr>
<td>$\lambda^h(s)$</td>
<td>Lagrange multiplier for household $h$ in state $s$</td>
</tr>
</tbody>
</table>

Main takeaways

After completing this section, you will be able to answer the following questions:

- How is a general financial equilibrium defined?
- What are the necessary and sufficient conditions for a solution to the household problem?

12.1.2 Model with uncertainty

This section introduces the fundamental dynamic model of risk and uncertainty. In this text, I use the terms risk and uncertainty interchangeably, although behavioral economics recognizes the different economic predictions depending upon whether households know the probability of future events (risk) or do not (uncertainty). As illustrated in Figure 12.1.1, the economy consists of two time periods: $t = 0$ and $t = 1$. In period $t = 1$, there are $S$ states of uncertainty. Counting the initial time period, there are a total of $S + 1$ states. Define the set of states as $S = \{0, ..., S\}$.

The models that we consider in this chapter are pure exchange models (no firms). Households receive endowments in each state. With heterogeneous households, the endowments differ and households have an incentive to transfer wealth across the states. These transfers
are made possible using the available financial assets. For this reason, the model introduced in this section is referred to as the general financial model.

There are a finite number of households $H$. Denote the set of households as $H = \{1, ..., H\}$ and denote any element of that set as $h \in H$. In each state, a single commodity is traded and consumed. Throughout this chapter, utility functions will be of the expected utility form:

$$U^h (c^h) = u_0^h (c^h(0)) + \sum_{s=1}^{S} \pi_s u_s^h (c^h(s)).$$  

(12.1)

The probabilities $(\pi_s)_{s \in S}$ must be such that $\sum_{s=1}^{S} \pi_s = 1$. For simplicity, denote $\pi_0 = 1$ as the probability that state $s = 0$ occurs. The utility functions $u_0^h : \mathbb{R}_+ \to \mathbb{R}$ and $u_s^h : \mathbb{R}_+ \to \mathbb{R}$ (for all $s > 0$) are assumed to satisfy $C^2$, strictly increasing, and strictly concave.

The variable $c^h(s)$ denotes the consumption by household $h$ in state $s$. The column vector $c^h = (c^h(s))_{s \in S}$ is the consumption by household $h$ in all states.

Households receive endowments in all states $s \in S$. The endowment for household $h$ in state $s$ is $e^h(s) > 0$. The endowment vector for household $h$ in all states is $e^h = (e^h(s))_{s \in S}$.

The price of the commodity is normalized to 1 in every state.

### 12.1.3 Financial assets

There are a total of $J$ financial assets. The assets belong to the set $J = \{1, ..., J\}$. Asset $j$ is traded at the nominal price $q_j$ in the initial period. The payout for asset $j$ in state $s > 0$ is denoted $r_j(s) \geq 0$. The payouts of all $J$ assets in all states $s > 0$ are collected in the $S \times J$ payout matrix:

$$R = \begin{bmatrix}
    r_1(1) & \cdots & r_J(1) \\
    \vdots & \ddots & \vdots \\
    r_1(S) & \cdots & r_J(S)
\end{bmatrix}. $$

(12.2)

The row vector $q = (q_1, ..., q_J)$ is the asset price vector. The asset payouts are parameters and the asset prices are variables.

Given $R$ and $q$, households determine how much of each asset to hold. The combinations of all asset positions is called the portfolio. Household $h$ chooses the portfolio $z^h \in \mathbb{R}^J$, where the position for a particular asset $j$ is denoted $z^h_j \in \mathbb{R}$. If $z^h_j \leq 0$, the household has sold the asset (often called short-selling or holding a short position). In this case, the asset has negative payout in the states $s > 0$ as $r_j(s) z^h_j \leq 0$ (the household has borrowed). If $z^h_j \geq 0$,
the household has purchased the asset (a long position). In this case, the asset has positive payout in the states \( s > 0 \) as \( r_j(s) z^h_j \geq 0 \) (the household has saved). In this section, we assume that the assets are in zero net supply. A zero net supply asset is equivalent to an IOU. If there is a household that borrows $10, there must exist a household willing to lend $10, such that the total amount borrowed equals the total amount lent.

### 12.1.4 Household problem

The household budget constraints are given by:

\[
\begin{align*}
  c^h(0) + \sum_{j \in J} q_j z^h_j &\leq e^h(0), \\
  c^h(s) &\leq e^h(s) + \sum_{j \in J} r_j(s) z^h_j \quad \forall s > 0.
\end{align*}
\]

Portfolio expenditures equal \( \sum_{j \in J} q_j z^h_j \) and this is a cost in the initial period \( (s = 0) \). Portfolio payouts in state \( s > 0 \) are equal to \( \sum_{j \in J} r_j(s) z^h_j \). Depending upon the combination of borrowing and lending, portfolio expenditures can be positive or negative and portfolio payouts can be positive or negative.

The household utility maximization problem is such that households maximize utility subject to the \( S + 1 \) budget constraints:

\[
\begin{align*}
  \text{maximize} & \quad U^h(c^h) \\
  \text{subject to} & \quad c^h(0) + \sum_{j \in J} q_j z^h_j \leq e^h(0), \\
  & \quad \forall s > 0 \quad c^h(s) \leq e^h(s) + \sum_{j \in J} r_j(s) z^h_j.
\end{align*}
\]

### 12.1.5 General financial equilibrium

A general financial equilibrium consists of two parts: household optimization and market clearing (over both commodity and asset markets).

**Definition 12.1** A general financial equilibrium is \( \left( (c^h, z^h)_{h \in H}, q \right) \) such that
12.1. GENERAL FINANCIAL EQUILIBRIUM

1. \(\forall h \in H\), given \(q\), \((c^h, z^h)\) is an optimal solution to the household problem

\[
\begin{align*}
\text{maximize} & \quad U^h (c^h) \\
\text{subject to} & \quad c^h(0) + \sum_{j\in J} q_j z_j^h \leq e^h(0) , \forall s > 0 \quad c^h(s) \leq e^h(s) + \sum_{j\in J} r_j(s) z_j^h \\
\end{align*}
\]

(12.5)

2. Markets clear

\[
\begin{align*}
\sum_{h\in H} c^h(s) &= \sum_{h\in H} c^h(s) \quad \text{for all } s \in S. \\
\sum_{h\in H} z_j^h &= 0 \quad \text{for all } j \in J.
\end{align*}
\]

(12.6)

12.1.6 Kuhn-Tucker conditions

To solve for the optimal household choices and eventually the equilibrium variables, the Kuhn-Tucker conditions must be specified. There are two variables \(c^h\) and \(z^h\). Consumption is \((S + 1)\)-dimensional and assets are \(J\)-dimensional. There are \(S + 1\) budget constraints, where \(\lambda^h(s)\) is the Lagrange multiplier for the budget constraint in state \(s\). The row vector \(\lambda^h = (\lambda^h(0), ..., \lambda^h(S))\) collects the Lagrange multipliers in all states.

The Kuhn-Tucker conditions are given by:

\begin{itemize}
  \item First order conditions
  \begin{itemize}
    \item With respect to the consumption choices \((c^h(s))_{s\in S}\) :
    \[
    \pi_s Du^h_s (c^h(s)) - \lambda^h(s) = 0 \quad \forall s \in S.
    \]
    \item With respect to the asset choices \((z^h_j)_{j\in J}\) :
    \[
    \lambda^h(0) q_j = \sum_{s>0} \lambda^h(s) r_j(s) \quad \forall j \in J.
    \]
  \end{itemize}
  \item Complimentary slackness conditions
\end{itemize}
- For the state $s = 0$ budget constraint:

$$
\lambda^h(0) \left( e^h(0) - c^h(0) - \sum_{j \in J} q_j z^h_j \right) = 0. \quad (12.9)
$$

- For the states $s > 0$ budget constraints:

$$
\lambda^h(s) \left( e^h(s) - c^h(s) + \sum_{j \in J} r_j(s) z^h_j \right) = 0 \quad \forall s > 0.
$$

The matrix $\begin{pmatrix} -q \\ R \end{pmatrix}$ is a $(S + 1) \times J$ dimensional matrix where the first row is the $1 \times J$ dimensional row vector $-q = (-q_1, ..., -q_J)$ and the last $S$ rows are the $S \times J$ dimensional payout matrix $R$. The first order conditions with respect to the asset choices are expressed in matrix notation as:

$$
\lambda^h \begin{pmatrix} -q \\ R \end{pmatrix} = 0. \quad (12.11)
$$

The portfolio payouts across all states can be represented as

$$
\begin{pmatrix} -q \\ R \end{pmatrix} z^h \in \mathbb{R}^{S+1},
$$

where the first element is the portfolio expenditure, the second row is the portfolio payout in state $s = 1$, and so forth with the final row being the portfolio payout in state $s = S$.

These first order conditions are related to the concept of no arbitrage, which we address in the following section.

Since $u^h_s$ is strictly increasing, $\lambda^h(s) > 0$ for all states, meaning that the budget constraints are binding.
12.2 No arbitrage

12.2.1 Sneak peek

Summary

The fundamental concept in all finance models and for all asset pricing equations is the concept of no arbitrage. No arbitrage has a very intuitive economic interpretation. We will study this concept in the context of the general financial equilibrium model and use the Hens' method in order to determine the set of asset prices that satisfy no arbitrage. The concept will reappear in all future asset pricing models that we consider in this chapter.

An arbitrage opportunity exists if an investor is able to strictly increase its payout in some periods without strictly decreasing its payout in any period. This arbitrage opportunity can be exploited by scaling up this particular investment without bound. In competitive markets, the price per unit of the asset is the same regardless of the number of assets held. This means that the portfolio expenditures and portfolio payouts are linear. By scaling up an investment with arbitrage payouts, the payouts are scaled up as well. It is optimal for an investor to scale up the investment without bound and earn an infinite profit.

Since there do not exist any financial markets in which an infinite number of assets are traded, the definition of equilibrium requires that all arbitrage opportunities are ruled out. Mathematically, no arbitrage is a necessary condition for an equilibrium to exist.

The economic condition that rules out no arbitrage is introduced in this section. Though intuitive, this condition is difficult to work with. We introduce an equivalent definition using the concept of state prices. A state price is a useful variable that can measure the relative price between different states, which, similar to the relative prices across commodity markets, captures the relative trade-off for households deciding how to transfer resources across states.

Notation

The variables to be introduced in this section are given in the following table:

\[ \alpha(s) \quad \text{state price in state } s \in S \]

Main takeaways

After completing this section, you will be able to answer the following questions:
- What are two equivalent definitions for no arbitrage, and which one is more convenient to work with mathematically?
- How can we determine the set of prices that satisfy no arbitrage in an economy with 2 assets?
- How can we determine the set of prices that satisfy no arbitrage in an economy with 3 assets?

12.2.2 No arbitrage definitions

Given $R$ and $q$, an arbitrage opportunity exists if there exists $z \in \mathbb{R}^J$ such that $\sum_{j \in \mathcal{J}} q_j z_j \leq 0$ and $\sum_{j \in \mathcal{J}} r_j(s) z_j \geq 0 \quad \forall s \in \{1, \ldots, S\}$, with strict inequality for at least one state. This represents an arbitrage opportunity, because the household could hold the portfolio $\kappa z$ as $\kappa \to \infty$, and provide itself with unbounded payout in some states, without sustaining a loss in any other states. In matrix notation, an arbitrage opportunity exists if there exists $z \in \mathbb{R}^J$ such that

$$
\begin{pmatrix}
-q \\
R
\end{pmatrix} z \text{ is a vector with nonnegative elements and at least one strictly positive element. Since } \begin{pmatrix}
-q \\
R
\end{pmatrix} z \text{ is the net addition to household income in all states, an arbitrage portfolio allows households to gain income in all states and never lose income in any state.}
$$

The first no arbitrage definition is given by:

No Arbitrage (quantities): There does not exist $z \in \mathbb{R}^J$ such that $\sum_{j \in \mathcal{J}} q_j z_j \leq 0$ and $\sum_{j \in \mathcal{J}} r_j(s) z_j \geq 0 \quad \forall s \in \{1, \ldots, S\}$, with strict inequality for at least one state.

This condition is quite intuitive, but difficult to work with mathematically. What we would like to do is use an equivalent definition. This equivalent definition will involve strictly positive state prices $(\alpha(1), \ldots, \alpha(S))$:

No Arbitrage (state prices): There exists $\alpha = (\alpha(1), \ldots, \alpha(S)) \in \mathbb{R}^{S+}$ such that $(1, \alpha) \begin{pmatrix}
-q \\
R
\end{pmatrix} = 0$, or equivalently $q = \alpha R$.

State prices are 'price-like' variables, so the convention is that they are written as row vectors (similar to prices $q$). The term $(1, \alpha)$ is a $(S + 1)$–dimensional row vector. The
matrix \( \begin{pmatrix} -q \\ R \end{pmatrix} \) is a \((S + 1) \times J\) dimensional matrix. The product \((1, \alpha) \begin{pmatrix} -q \\ R \end{pmatrix} \) is a \(1 \times J\) vector of zeroes.

We will now learn a useful method to easily determine which asset prices \(q\) satisfy no arbitrage.

### 12.2.3 Example: 2 assets

Consider the payout matrix \( R = \begin{bmatrix} 1 & 2 \\ 4 & 8 \\ 3 & 6 \end{bmatrix} \). The two assets are redundant. The second asset has payouts that are exact 2 times the payouts of the first asset. It is easy to see in this instance what the relationship between the asset prices must be. The price for asset 2 must be twice as large as the asset price for asset 1: \( q_2 = 2q_1 \).

The same idea holds whenever we have redundant assets. If asset \(J\) is a linear combination of the first \(J - 1\) assets with weights given by:

\[
\begin{bmatrix} r_J(1) \\ \vdots \\ r_J(S) \end{bmatrix} = \sum_{j=1}^{J-1} \chi_j \begin{bmatrix} r_j(1) \\ \vdots \\ r_j(S) \end{bmatrix}
\]

(12.12)

for any \((\chi_j)_{j \in \{1, \ldots, J-1\}} \in \mathbb{R}^{J-1}\), then the asset prices must maintain this same linear relationship:

\[
q_J = \sum_{j=1}^{J-1} \chi_j q_j.
\]

(12.13)

To see this explicitly, consider the second definition of no arbitrage:

**No Arbitrage (state prices):** There exists \(\alpha\) so that \(q = \alpha R\).

The no arbitrage condition states that

\[
q_J = \sum_{s=1}^{S} \alpha(s)r_J(s).
\]

(12.14)
By definition, each \( r_J(s) \) is a linear combination of \( (r_1(s), ..., r_{J-1}(s)) \):

\[
q_J = \sum_{s=1}^{S} \alpha(s) \sum_{j=1}^{J-1} \chi_j r_j(s). 
\]

(12.15)

Switch the order of summation:

\[
q_J = \sum_{j=1}^{J-1} \chi_j \sum_{s=1}^{S} \alpha(s) r_j(s). 
\]

(12.16)

For each asset \( j \in \{1, ..., J-1\} \), the no arbitrage condition is \( \sum_{s=1}^{S} \alpha(s) r_j(s) = q_j \). Replacing this expression yields the result:

\[
q_J = \sum_{j=1}^{J-1} \chi_j q_j. 
\]

(12.17)

So pricing redundant assets is not particularly challenging. What we want to do is figure out relationships for asset prices that are not redundant. To do this, we use the no arbitrage (state prices) definition from above.

The method to find all possible no arbitrage prices is due to an idea by Thorsten Hens, and is referred to as the Hens’ method. The Hens’ method for economies with two assets proceeds in three steps.

1. Plot the pairs \((r_1(s), r_2(s))\) in \( \mathbb{R}_+^2 \) for all states \( s > 0 \). As you can see in Figure 12.2.1, I have plotted the pairs corresponding to the payout matrix \( R = \begin{bmatrix} 0 & 1 \\ 4 & 2 \\ 3 & 3 \end{bmatrix} \) (the x-axis corresponds to the asset \( j = 1 \) and the y-axis to asset \( j = 2 \)).

2. The convex cone of the set \( \{(r_1(1), r_2(1)), ..., (r_1(S), r_2(S))\} \) includes the area between all rays through the points \( \{(r_1(1), r_2(1)), ..., (r_1(S), r_2(S))\} \). In Figure 12.2.1, the convex cone is the shaded region (and extends in the upper-right direction without bound).

3. If the asset prices \( q = (q_1, q_2) \) belong to the interior of this convex cone, they no arbitrage prices. If the asset prices \( q = (q_1, q_2) \) lie on the boundary or outside the cone, they are arbitrage prices.
12.2. NO ARBITRAGE

If asset prices \( q = (q_1, q_2) \) belong to the interior of the cone, by definition there exists strictly positive state prices such that \( q = (\alpha(1), \alpha(2)) R \). This is equivalent to the definition of no arbitrage (state prices).

12.2.4 Example: 3 assets

Consider an economy with \( S \geq 3 \) states of uncertainty and \( J = 3 \) assets. The Hens’ method can still be applied in the case of 3 assets, but we have to work a little bit harder to get the payout matrix in the correct form. There are now 4 steps to this process.

1. Given the payout matrix \( R \), we need to perform elementary column operations on the matrix \( \begin{pmatrix} -q \\ R \end{pmatrix} \) so that the new matrix \( R^* \) has one asset (we choose the asset \( j = 1 \)) with constant payouts of 1 in every state \( s > 0 \). Elementary column operations can take one of the following three forms: (i) multiply all elements in a column by a scalar, (ii) interchange all the elements in two columns, or (iii) add a multiple of all the elements in column A to column B. These column operations will not change the set of all possible portfolio payouts.

The equivalent asset price vector and payout matrix following the linear operations are \( \begin{pmatrix} -q^* \\ R^* \end{pmatrix} \), where

\[
R^* = \begin{bmatrix} 1 & r_2^*(1) & r_3^*(1) \\ 1 & : & : \\ 1 & r_2^*(S) & r_3^*(S) \end{bmatrix}.
\]  

(12.18)

As an example, suppose that \( R = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 4 & 2 \\ 4 & 3 & 3 \end{bmatrix} \). The difference between the payouts for asset 3 and asset 1 is a constant payout of 1 in every state \( s > 0 \). The elementary column operation will be to replace column 1 with the difference (1st column) - (3rd column). Whatever elementary column operations are performed on the payout matrix must also be performed on the asset price vector.
This results in the equivalent asset price vector and payout matrix:

\[
\begin{pmatrix}
-q^* \\
R^*
\end{pmatrix} = \begin{bmatrix}
-q_1 - (-q_3) & -q_2 & -q_3 \\
1 & 0 & 1 \\
1 & 4 & 2 \\
1 & 3 & 3
\end{bmatrix}.
\] (12.19)

2. Plot the pairs \((r_2^*(s), r_3^*(s))\) in \(\mathbb{R}^2_+\) for all states \(s > 0\). As you can see in Figure 12.2.2, I have plotted the pairs corresponding to the payout matrix

\[
R^* = \begin{bmatrix}
1 & 0 & 1 \\
1 & 4 & 2 \\
1 & 3 & 3
\end{bmatrix}.
\]

3. The convex hull of the set \(\{(r_2^*(1), r_3^*(1)), \ldots, (r_2^*(S), r_3^*(S))\}\) is the area between all line segments connecting any pair of points in \(\{(r_2^*(1), r_3^*(1)), \ldots, (r_2^*(S), r_3^*(S))\}\). In Figure 12.2.2, the convex hull is the shaded triangle-shaped area.

4. If the asset prices \(\left(\frac{q_2}{q_1}, \frac{q_3}{q_1}\right)\) belong to the interior of this convex hull, then they are no arbitrage prices. Continuing with my example, as \(q_1^* = q_1 - q_3\), \(q_2^* = q_2\), and \(q_3^* = q_3\), the asset price pairs that I will actually be plotting are \(\left(\frac{q_2}{q_1-q_3}, \frac{q_3}{q_1-q_3}\right)\).

From the definition no arbitrage (state prices), the asset prices must satisfy

\[
(q_1^*, q_2^*, q_3^*) = (\alpha(1), \ldots, \alpha(S)) \begin{bmatrix}
1 & r_2^*(1) & r_3^*(1) \\
1 & : & : \\
1 & r_2^*(S) & r_3^*(S)
\end{bmatrix}
\]

for some strictly positive state prices. This means that \(q_1^* = \sum_{s=1}^{S} \alpha(s) \cdot 1\) or \(q_1^* = \sum_{s=1}^{S} \alpha(s)\) for the first asset and

\[
(q_2^*, q_3^*) = (\alpha(1), \ldots, \alpha(S)) \begin{bmatrix}
1 & r_2^*(1) & r_3^*(1) \\
1 & : & : \\
1 & r_2^*(S) & r_3^*(S)
\end{bmatrix}
\] (12.20)
for the second and third assets. We want to divide both sides of the previous equations by \(q_1^* = \sum_{s>0} \alpha(s)\):

\[
\begin{pmatrix}
q_2^* \\
q_3^*
\end{pmatrix}
= \left(\frac{\alpha(1), \ldots, \alpha(S)}{\sum_{s>0} \alpha(s)}\right)
\begin{bmatrix}
r_2^*(1) & r_3^*(1) \\
\vdots & \vdots \\
r_2^*(S) & r_3^*(S)
\end{bmatrix}.
\tag{12.21}
\]

In other words, there exist updated state prices \((\alpha^*(1), \ldots, \alpha^*(S)) = \left(\frac{\alpha(1)}{\sum_{s>0} \alpha(s)}, \ldots, \frac{\alpha(S)}{\sum_{s>0} \alpha(s)}\right)\) that are strictly positive and satisfy both \(\sum_{s>0} \alpha^*(s) = 1\) and

\[
\begin{pmatrix}
q_2^* \\
q_3^*
\end{pmatrix}
= \left(\frac{\alpha^*(1), \ldots, \alpha^*(S)}{\sum_{s>0} \alpha(s)}\right)
\begin{bmatrix}
r_2^*(1) & r_3^*(1) \\
\vdots & \vdots \\
r_2^*(S) & r_3^*(S)
\end{bmatrix}.
\tag{12.22}
\]

This is exactly the definition of the interior of the convex hull. Therefore, no arbitrage is equivalent to the interior of the convex hull.

### 12.3 Consumption asset pricing model (CAPM)

#### 12.3.1 Sneak peek

**Summary**

The first asset pricing model to introduce is the consumption asset pricing model, or CAPM, for short. The model takes place in the exact same setting as the general financial equilibrium, namely two time periods with states of uncertainty in the second time period. The model relies on two crucial assumptions: (i) the household utility functions are quadratic (satisfying mean-variance preferences) and (ii) a market portfolio exists with returns equal to the aggregate resources (GDP) in all states. With these two assumptions, the no arbitrage conditions can be used to derive an asset pricing equation for any asset in the economy.

Assuming the presence of a risk-free bond, the excess return of any asset is equal to the expected return (with expectation taken across all states) minus the risk-free return. The CAPM formula states that the excess return for any asset is a linear function of the excess return for the market portfolio. The slope coefficient in this linear function is called the CAPM beta and is proportional to the covariance between the asset and the market
If the asset and the market portfolio are perfectly correlated, the excess return for the asset is higher (than the excess return for the market portfolio) if and only if the variance for the asset is higher (than the variance for the market portfolio). As the asset and the market portfolio become less correlated, the excess return for the asset will decrease.

Expected return is defined as the ratio of expected payouts over price. If asset 1 and asset 2 have the same expected payouts, but asset 1 has a lower return, then asset 1 will have a higher asset price. The CAPM model predicts that assets with a smaller correlation with the market portfolio (leading to lower return) will have a larger asset price.

**Notation**

The variables/parameters to be introduced in this section are given in the following table:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a^h$</td>
<td>quadratic utility parameter</td>
</tr>
<tr>
<td>$b^h$</td>
<td>quadratic utility parameter</td>
</tr>
<tr>
<td>$R_{ij}(s)$</td>
<td>return of asset $j$ in state $s \in S$</td>
</tr>
<tr>
<td>$R_f$</td>
<td>return on risk-free bond</td>
</tr>
<tr>
<td>$R_M(s)$</td>
<td>return on market portfolio in state $s \in S$</td>
</tr>
<tr>
<td>$\beta_j$</td>
<td>coefficient for asset $j$ in CAPM formula</td>
</tr>
</tbody>
</table>

**Main takeaways**

After completing this section, you will be able to answer the following questions:

- What are the key assumptions of the CAPM model?
- In the CAPM model, what is the relation between the excess returns of any asset and the excess return for the market portfolio?

### 12.3.2 CAPM utility

This section introduces the consumption asset pricing model (CAPM). The utility function takes on a specific form in this model:

$$U^h(c^h) = u^h(0) + \frac{1}{S} \sum_{s>0} u^h(c^h(s)). \quad (12.23)$$
In this formulation, the probabilities are equal for all states. We could also carry out the analysis if the probabilities are not equal, but we save on notation by assuming equal probabilities.

Here, the utility functions \( u_h^0 : \mathbb{R}_+ \to \mathbb{R} \) and \( u^h : \mathbb{R}_+ \to \mathbb{R} \) are assumed to be of the quadratic form:

\[
\begin{align*}
    u_h^0 (c^h(0)) &= a^h c^h(0) - \frac{1}{2} (c^h(0))^2, \\
    u^h (c^h(s)) &= b^h c^h(s) - \frac{1}{2} (c^h(s))^2 \quad \forall s > 0.
\end{align*}
\]

The terms \((a^h, b^h)\) are preference parameters for the household. The marginal utilities are given by:

\[
\begin{align*}
    Du_h^0 (c^h(0)) &= a^h - c^h(0), \\
    Du^h (c^h(s)) &= b^h - c^h(s) \quad \forall s > 0.
\end{align*}
\]

We assume that the parameters \((a^h, b^h)\) are always big enough so that the marginal utilities are always strictly positive.

### 12.3.3 CAPM endowments

The second key assumption in the CAPM is that the asset payouts are such that a portfolio \(\theta\) exists such that

\[
\begin{pmatrix}
    \sum_{h \in \mathcal{H}} e^h(1) \\
    : \\
    \sum_{h \in \mathcal{H}} e^h(S)
\end{pmatrix} = R\theta.
\]

In words, this says that the portfolio \(\theta\) is able to exactly replicate the aggregate resources in the economy. This portfolio is called the "market portfolio."
12.3.4 No arbitrage and state prices

The first order conditions of the household problem are (with respect to consumption first and then with respect to assets):

\[
Du_0^h (c^h(0)) - \lambda^h(0) = 0. \tag{12.27}
\]

\[
\frac{1}{S} Du^h (c^h(s)) - \lambda^h(s) = 0 \quad \forall s > 0.
\]

\[
\lambda^h(0) q_j = \sum_{s > 0} \lambda^h(s) r_j(s) \quad \forall j \in J.
\]

Combining the first order conditions allows us to state the Euler equation, which is used to determine the optimal portfolio for each household:

\[
q_j Du_0^h (c^h(0)) = \frac{1}{S} \sum_{s > 0} Du^h (c^h(s)) r_j(s) \quad \forall j \in J. \tag{12.28}
\]

Using the equations for the marginal utilities determined above:

\[
q_j (a^h - c^h(0)) = \frac{1}{S} \sum_{s > 0} (b^h - c^h(s)) r_j(s) \quad \forall j \in J. \tag{12.29}
\]

Define the aggregate preference parameters as \( A = \sum_{h \in H} a^h \) and \( B = \sum_{h \in H} b^h \). Let’s add up the Euler equations for all households in the economy:

\[
q_j \left( A - \sum_{h \in H} c^h(0) \right) = \frac{1}{S} \sum_{s > 0} \left( B - \sum_{h \in H} c^h(s) \right) r_j(s) \quad \forall j \in J. \tag{12.30}
\]

From market clearing, \( \sum_{h \in H} c^h(s) = \sum_{h \in H} e^h(s) \) for all states \( s \in \{0, ..., S\} \). Using this fact:

\[
q_j = \frac{\frac{1}{S} \sum_{s > 0} \left( B - \sum_{h \in H} c^h(s) \right) r_j(s)}{A - \sum_{h \in H} c^h(0)} \quad \forall j \in J. \tag{12.31}
\]

The no arbitrage condition specifies the presence of state prices \( \alpha = (\alpha(1), ..., \alpha(S)) \) such that:

\[
q_j = \frac{1}{S} \sum_{s > 0} \alpha(s) r_j(s) \quad \forall j \in J. \tag{12.32}
\]
12.3. CONSUMPTION ASSET PRICING MODEL (CAPM)

The portfolio equation above implies that the state prices are given by:

$$\alpha(s) = \frac{\left(B - \sum_{h \in H} e^h(s)\right)}{\left(A - \sum_{h \in H} e^h(0)\right)} \quad \forall s > 0. \tag{12.33}$$

### 12.3.5 Asset pricing equation

Define the row vector $\alpha = (\alpha(1), ..., \alpha(S))$ and the column vector $r_j = \begin{pmatrix} r_j(1) \\ \vdots \\ r_j(S) \end{pmatrix}$. The asset price $q_j = E(\alpha r_j)$, since the asset price is the expectation over all states $s > 0$. Using the definition of variance and covariance,

$$E(\alpha r_j) = E(\alpha) E(r_j) + \text{cov}(\alpha, r_j). \tag{12.34}$$

Assume that the economy contains a risk-free bond with return $R_f$. For simplicity, the risk-free bond has price equal to 1 and payout equal to $R_f$. The no arbitrage condition states that:

$$q_f = 1 = \frac{1}{S} \sum_{s > 0} \alpha(s) R_f, \tag{12.35}$$

meaning that $E(\alpha) = \frac{1}{S} \sum_{s > 0} \alpha(s) = \frac{1}{R_f}$. The asset price equation $q_j = E(\alpha r_j)$ is updated to:

$$q_j = \frac{1}{R_f} E(r_j) + \text{cov}(\alpha, r_j). \tag{12.36}$$

### 12.3.6 Returns equation

In order to be consistent when comparing assets, we will speak only in terms of the returns, where $\text{return} = \frac{\text{payout}}{\text{price}}$. The return vector is denoted $R_j = \begin{pmatrix} R_j(1) \\ \vdots \\ R_j(S) \end{pmatrix}$, where each return is defined by $R_j(s) = \frac{r_j(s)}{q_j}$.

Divide the asset price equation (12.36) by $q_j$. The expectation and the covariance have
the linearity property, meaning that when we divide all terms by $q_j$ :

$$1 = \frac{1}{R_f} E(R_j) + cov(\alpha, R_j).$$

Assume that there exists a market portfolio $M$ such that the payout from holding this portfolio is

$$R_M = \left( \begin{array}{c} r_M(1) \\ \vdots \\ r_M(S) \end{array} \right) = \left( \begin{array}{c} \sum_{h \in H} e^h(1) \\ \vdots \\ \sum_{h \in H} e^h(S) \end{array} \right).$$

Denote the return on the market portfolio as $R_M$, with average return $E(R_M)$.

Since the state prices are given by

$$\alpha(s) = \left( \begin{array}{c} B - \sum_{h \in H} e^h(s) \\ A - \sum_{h \in H} e^h(0) \end{array} \right), \quad \forall s > 0,$$

the vectors $R_M$ and $\alpha$ are inversely related. By definition, $cov(\alpha, R_j) = -cov(R_M, R_j)$.

The returns equation for any asset $j$ is updated to:

$$1 = \frac{1}{R_f} E(R_j) - cov(R_M, R_j).$$

Multiply both sides by $R_f$ and re-arrange:

$$E(R_j) - R_f = R_f cov(R_M, R_j).$$

Consider the above equation for the market portfolio itself, using the fact that $cov(R_M, R_M) = var(R_M)$:

$$E(R_M) - R_f = R_f var(R_M).$$
12.3.7 CAPM formula

The CAPM formula is the solution to the two equations:

\[
E(R_j) - R_f = R_f \text{cov}(R_M, R_j). 
\]

\[
E(R_M) - R_f = R_f \text{var}(R_M).
\]

If we divide the equation \(E(R_j) - R_f = R_f \text{cov}(R_M, R_j)\) by the equation \(E(R_M) - R_f = R_f \text{var}(R_M)\), then we arrive at the CAPM formula:

\[
E(R_j) - R_f = \left(\frac{\text{cov}(R_M, R_j)}{\text{var}(R_M)}\right) (E(R_M) - R_f).
\]

(12.42)

The solution is such that there exists a coefficient \(\beta_j\), typically called the CAPM beta, such that

\[
E(R_j) - R_f = \beta_j (E(R_M) - R_f),
\]

(12.43)

where the value for the CAPM beta is

\[
\beta_j = \left(\frac{\text{cov}(R_M, R_j)}{\text{var}(R_M)}\right).
\]

(12.44)

The CAPM pricing formula says that the excess return for any asset, \(E(R_j) - R_f\), is linearly related to the excess return for the market portfolio, \(E(R_M) - R_f\).

The CAPM beta has the property that if a derivative asset is created with returns \(R^* = \sum_j \kappa_j R_j\), the CAPM beta for this new asset is \(\beta^* = \sum_j \kappa_j \beta_j\).

12.4 Lucas asset pricing model

12.4.1 Sneak peek

Summary

This section introduces the Lucas asset pricing model. The model differs from the general financial model (of which CAPM is a special case) in several important dimensions. Instead of a 2-period model, the Lucas asset pricing model is an infinite time horizon model. Instead of heterogeneous households, the Lucas asset pricing model has homogeneous households, meaning that all households have the same preferences and endowments. This can be
interpreted as a representative household (a single household).

In the general financial equilibrium, assets are in zero net supply, meaning that every unit of lending must be offset by a unit of borrowing. In the Lucas asset pricing model, the asset is in unit net supply, meaning that the initial supply of the asset is one and the total net asset holdings must remain equal to one after trading. Additionally, the asset is an infinite-lived asset.

This should be interpreted as a stock, or fractional ownership in a firm. As owners of a firm, the stock holders are entitled to dividend payouts. The stock also has value in all periods as it is infinite-lived. The expected payouts of the stock are equal to the sum of the expected price for the stock and the expected dividends.

The Lucas asset pricing model, using the no arbitrage condition, provides a closed-form solution for the stock prices in a stationary equilibrium. The stock prices depend upon the current state of the economy, so the stationary stock prices are a vector where each different state has a different stock price.

Any additional assets that are added to the system are redundant in the Lucas asset pricing model, because with the representative household the allocation is autarchic and does not change with the addition of new assets. These new assets are called derivative assets as no arbitrage conditions can be used to derive the prices of these derivative assets in terms of the underlying stock prices. Common examples of derivative assets include put options and call options.
12.4. LUCAS ASSET PRICING MODEL

Notation

The variables/parameters to be introduced in this section are given in the following table:

| I        | set of all states $I = \{1, ..., I\}$ |
| d_i      | dividend of stock in state $i$        |
| p_i      | price of stock in state $i$           |
| c_i      | consumption in state $i$              |
| $\pi_{ij}$ | probability state $j$ occurs conditional on state $i$ |
| $\theta$ | share in the stock held by household |
| a_j      | Arrow security paying out in state $j$|
| q_{ij}   | price of Arrow security $j$ in state $i$|
| $\bar{p}$ | strike price |
| $\rho$   | price for a put option |
| $\gamma$ | price for a call option |

Main takeaways

After completing this section, you will be able to answer the following questions:

- How is the vector of stationary stock prices determined in the Lucas asset pricing model?
- How are the Arrow security prices determined?
- How are the prices for derivative assets determined, including put options and call options?
- What is the difference between an American and a European option?

12.4.2 The model setup

The Lucas asset pricing model is a pure exchange model (no firms). The model contains an infinite number of discrete time periods. An infinite-lived financial asset called a stock is traded in all time periods. In each period, 1 of a possible $I$ different states can occur. The set of states are denoted $I = \{1, ..., I\}$. Each state corresponds to a different dividend value for the stock.
The parameter $\pi_{ij}$ is the probability that state $j$ will occur tomorrow given that state $i$ occurs today. Gather all of the probabilities together into the $I \times I$ square matrix $\Pi$, which is called the Markov transition matrix. The probability $\pi_{ij}$ is the element in row $i$ and column $j$ of the matrix $\Pi$. By definition, $\sum_j \pi_{ij} = 1$ for any row $i$.

The possible dividend payouts are $d \in \{d_1, ..., d_I\}$ and all dividends are assumed to be strictly positive.

The interpretation for this model is to view a stock as a fruit tree that produces a perishable fruit each period according to some random process. We will adopt this convention and refer to the stock as a tree and the dividends as fruit.

Denote $p_i$ as the price of the tree given that the current state is $i$ (i.e., the current dividend is $d_i$). There is no dividend growth in this model (or inflation), so the prices will be stationary. This simply means that in all periods in which the tree produces $d_i$ units of dividend, the price of the tree $p_i$ will be the same.

There exists a unit mass of homogeneous households. Households are infinite-lived. Households do not have endowments of fruit, but are initially endowed with 1 unit of the tree. The tree is an asset in unit net supply. This means that the total ownership for this tree must sum to 1. Households choose the share of the tree to hold. The share is denoted $\theta$. The notation of dynamic programming uses $\theta'$ to denote the share next period. Household consumption of the fruit is denoted $c$. The household utility function is $u : \mathbb{R}_+ \to \mathbb{R}$, which is assumed to be $C^2$, strictly increasing, and strictly concave. The discount factor is $\beta \in (0, 1)$.

The state variables are the current state $i$ and the current share $\theta$. The optimal consumption choice $c$ is a function of these state variables and we denote $c_i$ as the consumption when the current state is $i$. The Bellman equation for the household is given by:

$$V_i(\theta) = \max_{c_i \geq 0, \theta' \geq 0} u(c_i) + \beta \sum_{j \in I} \pi_{ij} V_j(\theta')$$

subject to $c_i + \theta' p_i = \theta (p_i + d_i)$.

(12.45)

As a function of the two state variables $i$ and $\theta$, the household chooses consumption to satisfy the budget constraint. The term $\theta' p_i$ is the value spent by the household to obtain the share $\theta'$ for next period. The income term $\theta [p_i + d_i]$ is the total payout from the household’s share holding. The value of the stock is $p_i$ and the dividend payout is $d_i$. Since the household owns the share $\theta$, the payout is $\theta [p_i + d_i]$.

With probability $\pi_{ij}$, the economy will be in state $j$ next period. Adding over all possible
states $j \in I$, we obtain the discounted weighted value function $\beta \sum_{j \in I} \pi_{ij} V_j(\theta')$. The state variables next period are $j$ and $\theta'$.

### 12.4.3 Solving the model

Solve the budget constraint for $c_i$ and insert this expression into the objective function of the Bellman equation. We are then left with an unconstrained maximization problem only in terms of the variable $\theta'$. The first order condition of this maximization problem is:

$$-p_i Du(c_i) + \beta \sum_{j \in I} \pi_{ij} DV_j(\theta') = 0.$$

(12.46)

The Envelope Theorem provides an expression for the derivative $DV_j(\theta')$:

$$DV_j(\theta') = (p_j + d_j) Du(c_j).$$

(12.47)

Combining the first order condition and the Envelope Theorem, we reach the Euler equation:

$$p_i Du(c_i) = \beta \sum_{j \in I} \pi_{ij} (p_j + d_j) Du(c_j).$$

(12.48)

The model contains both commodity market clearing conditions and asset market clearing conditions. For the commodity markets, the aggregate amount of the one commodity (the fruit) in state $i$ is $d_i$. The total household consumption in state $i$ is $c_i$. Market clearing dictates that

$$c_i = d_i.$$  

(12.49)

On the stock market, the aggregate share holdings of all households must equal 1 since the asset is in unit net supply. All households are identical, so the share holdings must be equal to 1:

$$\theta = \theta' = 1.$$  

(12.50)

Define the following vectors and matrices:

$$P = \begin{bmatrix} p_1 \\ \vdots \\ p_I \end{bmatrix}, \quad d = \begin{bmatrix} d_1 \\ \vdots \\ d_I \end{bmatrix}, \quad Du = \begin{bmatrix} Du(d_1) & 0 & 0 \\ 0 & \ldots & 0 \\ 0 & 0 & Du(d_I) \end{bmatrix}.$$  

(12.51)
The Euler equations (for all $I$ of the possible current states) can be stacked on top of each other:

\[
\begin{pmatrix}
Du(d_1) p_1 \\
\vdots \\
Du(d_I) p_I
\end{pmatrix} = 
\begin{pmatrix}
\sum_{j \in I} \pi_{1j} Du(d_j) (p_j + d_j) \\
\vdots \\
\sum_{j \in I} \pi_{Ij} Du(d_j) (p_j + d_j)
\end{pmatrix}.
\] (12.52)

These Euler equations can be written in matrix notation:

\[
DuP = \beta \Pi Du (P + d).
\] (12.53)

The objective in any asset pricing model is to obtain a formula for the asset prices, in this case the prices for the trees. These prices differ depending upon the current state and we have just found a system of equations that can be simultaneously solved for all $I$ of these prices.

We solve for this vector $P$ using basic matrix operations. First, the matrix $Du$ is invertible, since $u$ is strictly increasing (each diagonal term $Du(d_i) > 0$):

\[
P = \beta Du^{-1} \Pi Du (P + d).
\] (12.54)

We can collect the terms involving $P$:

\[
[I - \beta Du^{-1} \Pi Du] P = (\beta Du^{-1} \Pi Du) d.
\] (12.55)

The matrix $(I - \beta Du^{-1} \Pi Du)$ is always invertible since $\beta \in (0, 1)$ and $\Pi$ is a Markov transition matrix (the row sums equal 1). Thus:

\[
P = [I - \beta Du^{-1} \Pi Du]^{-1} (\beta Du^{-1} \Pi Du) d.
\] (12.56)

### 12.4.4 Adding Arrow securities

It is possible to obtain a slightly more convenient form for the stock prices $P$. Redundant assets called Arrow securities are introduced into the model. The Arrow security $a_j$ is traded today and has payout equal to 1 tomorrow only if state $j$ occurs. Denote $q_{ij}$ as the price of Arrow security $a_j$ when the current state is $i$. 
The Bellman equation is updated to include the additional choice of the Arrow securities:

\[ V_i(\theta, a) = \max_{c_i, a_i'} u(c_i) + \beta \sum_{j \in I} \pi_{ij} V_j (\theta', a_j') \]

subject to \[ c_i + \theta' p_i + \sum_{j \in I} q_{ij} a_j' = \theta (p_i + d_i) + a. \] (12.57)

The Euler equations associated with the Arrow security are:

\[ q_{ij} Du(c_i) = \beta \pi_{ij} Du(c_j). \] (12.58)

The market clearing conditions remain \[ c_i = d_i \] and \[ \theta = \theta' = 1. \] The Arrow securities are in zero net supply. Since all households make identical decisions, the market clearing condition for Arrow securities implies \[ a = a_j' = 0 \quad \forall j \in I. \]

Using the commodity market clearing condition, the price of the Arrow security satisfies:

\[ q_{ij} = \frac{\beta \pi_{ij} Du(d_j)}{Du(d_i)}. \] (12.59)

The matrix \[ Q = [q_{ij}]_{(i,j) \in I^2} \] contains all \( I^2 \) Arrow security prices, one for each combination of the current state \( i \) and the next period state \( j \). In the matrix, the current state \( i \) corresponds to row \( i \) and the next period state \( j \) corresponds to column \( j \). In matrix notation:

\[ Q = \beta Du^{-1} \Pi Du. \] (12.60)

Recall the system of equations for the tree prices in all states

\[ P = [I - \beta Du^{-1} \Pi Du]^{-1} (\beta Du^{-1} \Pi Du) d. \] (12.61)

We can express this system in terms of the matrix \( Q \):

\[ P = [I - Q]^{-1} Qd. \] (12.62)

The relationship between \( p \) and \( q \) is very simple:

\[ p_i = \sum_j q_{ij} (p_j + d_j). \] (12.63)

The Arrow security price \( q_{ij} \) is identical to the the state price \( \alpha(s) \) from the previous sections.
Recall that a state price is a relative price for transfers between two states. In this case, \( q_{ij} \) is the price for transfers between state \( i \) and state \( j \). Finance jargon refers to state prices as either the stochastic discount factor or the pricing kernel. All three terms (state prices, stochastic discount factor, pricing kernel) mean the same thing.

The stock price \( p_i \) is equal to the weighted expectation of the next period stock prices \( p_j \) and the next period dividends \( d_j \), where the weights are the state prices \( q_{ij} \).

### 12.4.5 Application: Pricing 1-period put option

Using the Lucas asset pricing model, it is straightforward to price derivative assets. The next subsections will cover the pricing of the following types of derivative assets:

1. 1-period put option.
2. 1-period call option.
3. 2-period European put option.
4. 2-period American put option.

A put option is the right, but not the obligation, to sell a share of the Lucas tree at a fixed strike price. Since it is a 1-period option, the option can only be exercised 1 period from now. The strike price is given by the parameter \( \tilde{p} \).

An option does not have any dividends. Its payouts are only financial. The price for the put option is equal to the weighted expectation of the next period payouts. The weights assigned to each of the states in the next period are the Arrow security prices \( q_{ij} = \frac{\beta p_j Du(d_j)}{Du(d_i)} \) from the previous subsection.

If the price of the Lucas tree next period is above the strike price \( (p_j > \tilde{p}) \), households are not interested in selling. They would not sell an object worth \( p_j \) and only receive \( \tilde{p} \). Households would not exercise the option in this case, meaning that the payout of the option equals 0.

If the price of the Lucas tree next period is below the strike price \( (p_j < \tilde{p}) \), households would want to sell. They would be able to sell at a higher price \( \tilde{p} \) than what the Lucas tree is currently worth. Households are not worried that, although the current price is low, the price of the stock may rebound in future periods. Households are always permitted to sell at the strike price \( \tilde{p} \) and then repurchase the same number of shares of the tree at the lower price \( p_j \). Households receive a payout equal to \( \tilde{p} - p_j \) per share of option held.
The payouts of a put option can be seen in Figure 12.4.1.

Using this figure, we can write an equation for the price of a 1-period put option (traded when the current state is $i$ and with strike price $\bar{p}$). The price of a 1-period put option is denoted $\rho_i^1(\bar{p})$ and equal to the weighted expectation of next period payouts:

$$
\rho_i^1(\bar{p}) = \sum_j q_{ij} \max\{\bar{p} - p_j, 0\}. \tag{12.64}
$$

### 12.4.6 Application: Pricing 1-period call option

A call option is the right, but not the obligation, to buy a share of the Lucas tree at a fixed strike price of $\bar{p}$. The 1-period option can only be exercised in the next period. A call option is exercised in all of those states in which a put option is not exercised (and vice versa).

If the price of the Lucas tree next period is above the strike price ($p_j > \bar{p}$), households would want to buy. They would be able to buy an object worth $p_j$ by only paying the price $\bar{p}$. Households would receive a payout equal to $p_j - \bar{p}$ per share of option held.

If the price of the Lucas tree next period is below the strike price ($p_j < \bar{p}$), households would not be interested in buying. They would not buy an object worth $p_j$ by paying the price $\bar{p}$. The option would not be exercised in this case, meaning that the payout of the option equals 0.

The payouts of a call option can be seen in Figure 12.4.2.

Using this figure, we can easily identify the payouts of the option in all states next period. The price of a 1-period call option (traded when the current state is $i$ and with strike price $\bar{p}$) is denoted by $\gamma_i^1(\bar{p})$ and equal to the weighted expectation of next period payouts:

$$
\gamma_i^1(\bar{p}) = \sum_j q_{ij} \max\{p_j - \bar{p}, 0\}. \tag{12.65}
$$

### 12.4.7 Application: Pricing 2-period European put option

When we get to multi-period put options, there are two types of options based upon when the option can be exercised. If it can only be exercised in 2 periods (but not in 1 period), then it is a European option. If it can be exercised in either period, then it is an American option. Notice that with an American option, if it is exercised in 1 period, then the asset no longer exists and the decision to exercise or not in 2 periods is moot.

A 2-period European put option is the right, but not the obligation, to sell a share of the Lucas tree 2 periods from now at a fixed strike price. This type of option cannot be
exercised early, but can only be exercised 2 periods from now. Denote the strike price as \( \tilde{p} \).

Since we only care about the states that are realized 2 periods from now, we can consider the product \( Q^2 = Q \cdot Q \), whose elements \( Q^2_{(i,j)} \) are the 2-period stochastic discount factors from state \( i \) today to state \( j \) 2 periods from now. As before, a put option is only exercised when the strike price is above the stock price.

The payouts of a 2-period European put option can be seen in Figure 12.4.3.

Using this figure, we can easily identify the payouts of the option in all states 2 periods from now. The price of a 2-period European put option (traded when the current state is \( i \) and with strike price \( \tilde{p} \)) is denoted by \( \rho^2_{i} (\tilde{p}) \) and equal to the weighted expectation of payouts 2 periods ahead:

\[
\rho^2_{i} (\tilde{p}) = \sum_{j} Q^2_{(i,j)} \max \{ \tilde{p} - p_j, 0 \}.
\]

(12.66)

12.4.8 Application: Pricing 2-period American put option

A 2-period American put option is the right, but not the obligation, to sell a share of the Lucas tree at a fixed strike price. This type of option can be exercised either 1 period from now or 2 periods from now. If the option is exercised 1 period from now, it is no longer available to be exercised 2 periods from now. Denote the strike price as \( \tilde{p} \). As before, a put option is only exercised when the strike price is above the stock price.

We can determine the price of a 2-period American put option by using backward induction. In state \( k \) 2 periods from now, households exercise the put option whenever the strike price is above the stock price. Knowing this, the interesting decisions are made in the states \( j \) in the intermediate period that occurs 1 period from now.

In state \( j \) next period, households can either exercise the option or hold the option for the period after tomorrow. By exercising the option, households receive a payout \( \tilde{p} - p_j \). By not exercising the option, the 2-period American option becomes a 1-period option with price \( \rho^1_{j} (\tilde{p}) \).

The payouts can be seen in Figure 12.4.4.

Using this figure, we can easily identify the payouts of the option in all states 2 periods from now, and backward induce the payouts of the option in all states next period. To evaluate the price of the 2-period American option (traded today), we must apply the weighted expectation over all states \( j \) tomorrow. The price of a 2-period American put option (traded when the current state is \( i \) and with strike price \( \tilde{p} \)) is denoted by \( \rho^2_{i} (\tilde{p}) \) and equal to the
weighted expectation of the equation computed for next period payouts:

\[
\rho_i^{2A}(\bar{p}) = \sum_j q_{ij} \max \{ \bar{p}_j - p_j, \rho_j^1(\bar{p}) \}
\]

\[
= \sum_j q_{ij} \max \left\{ \bar{p} - p_j, \sum_k q_{jk} \max \{ \bar{p} - p_k, 0 \} \right\}.
\] (12.67)

12.5 Bubbles

12.5.1 Sneak peek

Summary

Thus far, we have considered the general financial equilibrium model (with CAPM as a special case) and the Lucas asset pricing model. In the general financial equilibrium model, there are only two time periods with states of uncertainty in the final period. There are also heterogeneous households that trade assets in zero net supply. With only two time periods, the assets are necessarily short-lived assets.

In the Lucas asset pricing model, there are an infinite number of time periods with states of uncertainty realized in each period. There is a representative household and assets are infinite-lived and available in unit net supply.

To analyze the possibility for bubbles, we consider a model that combines elements from the general financial equilibrium model and the Lucas asset pricing model. The model for bubbles contains an infinite number of time periods, but does not contain uncertainty. A setting without uncertainty is deterministic. There are heterogeneous households that trade infinite-lived assets available in unit net supply.

In this setting, the fundamental value of an asset is defined, which is equal to the net present discounted value of future dividends. The discount factor is determined endogenously according to the discount factor parameter and the ratio of marginal utilities. The discount factor is identical to the state prices previously introduced.

A bubble, by definition, arises if the asset price exceeds the fundamental value of the asset. Such a bubble is a positive bubble. Though a negative bubble, in which the asset price is strictly less than the fundamental value, could occur in theory, we focus our attention on the more common case of a positive bubble. Bubbles may exist in models with explicit debt constraints, but cannot exist in models with implicit debt constraints.
12. ASSET PRICING

Notation

The variables/parameters to be introduced in this section are given in the following table:

- \( c^h_t \): consumption by household \( h \) in period \( t \)
- \( e^h_t \): endowment by household \( h \) in period \( t \)
- \( d_t \): dividend in period \( t \)
- \( p_t \): price of the asset in period \( t \)
- \( \theta^h_t \): asset holding by household \( h \) in period \( t \)
- \( q_t \): present value price
- \( \mu^h_t \): Lagrange multiplier for debt constraint

Main takeaways

After completing this section, you will be able to answer the following questions:

- What are the different types of debt constraints and how do they rule out Ponzi schemes?
- Can a bubble exist with explicit debt constraints?
- Can a bubble exist with implicit debt constraints?

12.5.2 No Ponzi schemes

This section considers a model with an infinite number of discrete periods \( t \in \{0, 1, 2, \ldots\} \). The future realizations of the model are known with certainty. In each time period, a single commodity is traded and consumed.

The economy contains a set of heterogeneous households \( h \in H = \{1, \ldots, H\} \). Denote \( c^h_t \) as the consumption by household \( h \) in period \( t \). The preferences of the households are:

\[
U(c^h) = \sum_{t=0}^{\infty} \beta^t u(c^h_t) .
\]  (12.68)

The utility function \( u : \mathbb{R}_+ \rightarrow \mathbb{R} \) is assumed to be \( C^2 \), strictly increasing, strictly concave, and satisfy the Inada condition:

\[
\lim_{c \to 0} Du(c) = +\infty.
\]  (12.69)

In each time period, households are endowed with \( e^h_t \) units of the commodity, where \( e^h_t > 0 \).
The financial markets contain a single infinite-lived asset, which can be re-traded in all periods. The dividends of the asset in period \( t \) are \( d_t \geq 0 \), where the dividends are made in units of the commodity. The price of the asset in period \( t \) is \( p_t \). Money is an example of an infinite-lived asset in which \( d_t = 0 \) for all time periods.

The asset is in unit net supply. The households are initially endowed with \( \theta^h_{-1} \) units of the asset such that \( \sum_{h \in H} \theta^h_{-1} = 1 \).

The household’s choice of how much of the asset to hold from period \( t \) to period \( t + 1 \) is denoted \( \theta^h_t \in \mathbb{R} \). The household budget constraint is given by:

\[
 c^h_t + p_t \theta^h_t = e^h_t + \theta^h_{t-1} (p_t + d_t) .
\]

The left-hand side is the consumption expenditure plus the amount spent to obtain the asset amount \( \theta^h_t \) to carry into period \( t + 1 \). The right-hand side is the endowment plus the value of the asset holdings. The value of an asset holding is both the market price that it can be sold at and the dividends it pays out. If a household maintains its portfolio \( \theta^h_t = \theta^h_{t-1} \), the budget constraint specifies that the household actually sells its portfolio \( \theta^h_{t-1} \) and then repurchases the same amount \( \theta^h_t \) at the same price.

The present value price \( q_t \) is the commodity price level in period \( t \). This present value price \( q_t \) is the inverse of the "price" for money. Money need not be included in the present model, as it is a redundant asset. Without uncertainty, there is only one independent asset that households need to transfer resources across time. The independent asset is the long-lived asset paying dividends. Money is the redundant asset and is omitted from the model.

Even without its explicit inclusion in the model, the possibility for money requires that the no arbitrage condition must hold as the long-lived asset and money are dependent. The no arbitrage condition states that the return on money must be equal to the return on the other infinite-lived asset:

\[
 \frac{\text{"price" for money in } t + 1}{\text{"price" for money in } t} = \frac{p_{t+1} + d_{t+1}}{p_t} .
\]

Since \( q_t \) is the inverse of the "price" for money, the equation becomes:

\[
 \frac{q_t}{q_{t+1}} = \frac{p_{t+1} + d_{t+1}}{p_t} .
\]

Normalizing the initial period price \( q_0 = 1 \), then all future present value prices can be defined
recursively:

\[ q_{t+1} = \frac{p_t}{p_{t+1} + d_{t+1}} q_t. \]  

(12.73)

The ratio \( \frac{q_{t+1}}{q_t} \) is the state price. An equivalent means to derive the state price \( \frac{q_{t+1}}{q_t} \) is to consider the Euler equation for any \( h \) associated with the long-lived asset:

\[ p_t Du (c^h_t) = (p_{t+1} + d_{t+1}) \beta Du (c^h_{t+1}). \]  

(12.74)

The discount factor times the ratio of marginal utilities was how we defined state prices in the previous sections and this product equals:

\[ \frac{\beta Du (c^h_{t+1})}{Du (c^h_t)} = \frac{p_t}{p_{t+1} + d_{t+1}}. \]

This is identical to the state price \( \frac{q_{t+1}}{q_t} \) derived above.

The state price is always of the form \( \frac{q_{t+1}}{q_t} \). When money is included and \( q_t \) is the commodity price level, this is equal to \( \frac{q_{t+1}}{q_t} \). When an infinite-lived asset paying dividends is included, this is equal to \( \frac{p_t}{p_{t+1} + d_{t+1}} \). The ratio \( \frac{p_t}{p_{t+1} + d_{t+1}} \) is easier to work with in our subsequent bubble definitions, because it only relies only observables from the financial markets and does not require information about household preferences.

Denoting the state price as \( \alpha (s) \), we can write all asset pricing equations in the general form:

\[ \text{price}(t) = \sum_{\text{all states } s \text{ in } t+1} \alpha (s) \times (\text{payout in state } s). \]  

(12.75)

In this deterministic setting, there is only one state that occurs in time period \( t + 1 \). The price of the asset is \( p_t \) and the payout in \( t + 1 \) is \( p_{t+1} + d_{t+1} \). This leads to the pricing equation

\[ p_t = \frac{q_{t+1}}{q_t} (p_{t+1} + d_{t+1}), \]  

(12.76)

which confirms our definition of the state price from above.

In this model, we allow households to hold negative positions on the asset. If \( \theta^b_t < 0 \), the household is borrowing. This is referred to as short-selling an asset, since the household is actually holding negative shares of the asset.

With an infinite time horizon, we need to worry about a trading strategy with ever-increasing debt that breaks apart an equilibrium. A Ponzi scheme is a trading scheme whereby a household begins in debt with \( \theta^b(t) < 0 \) and pays off this debt using additional
borrowing in the next period. The household then pays off this new higher level of debt with additional borrowing in the following period. This process continues throughout the infinite time horizon of the model and allows the household to infinitely delay its repayment of the debt.

Such a scheme is called a Ponzi scheme in reference to Charles Ponzi, the Italian businessman working out of Boston in the early 1920’s that promised unbelievable (literally) returns on investment, but was actually paying early investors using the investments of later investors. The scheme eventually unraveled and Ponzi spent most of the next 14 years in prison before being deported to Italy.

12.5.3 Types of debt constraints

A Ponzi scheme is inconsistent with equilibrium as financial markets can never accommodate the optimal demand of households to borrow an infinite amount. Mathematically, no Ponzi schemes is a necessary condition for the existence of an equilibrium.

There are two possible ways to rule out Ponzi schemes:

1. An explicit debt constraint

   The households face a constraint $\theta_t^h \geq -A$, where $A$ is a fixed parameter. This constraint prevents Ponzi schemes as eventually a Ponzi schemer would run up against the debt limit.

2. An implicit debt constraint

   The households face a debt constraint:
   
   $$p_t\theta_t^h \geq -\frac{1}{q_t} \sum_{\tau=t+1}^{\infty} q_\tau e^h_{\tau}.$$  

   (12.77)

   The constraint says that the household borrowing can never be larger than the net present value of the entire endowment sequence of the household. If the constraint is binding, then the household would be able to repay the debt by setting consumption equal to 0 and using all future endowments to repay the debt. This constraint prevents Ponzi schemes.

   The household problem in this model, including one of the debt constraints above, is
given by:
\[
\max_{\{c^h_t, \theta^h_t\}} \sum_{t=0}^{\infty} \beta^t u (c^h_t) \quad (12.78)
\]
subject to
\[
c^h_t + p_t \theta^h_t = c^h_{t+1} + \theta^h_{t-1} (p_t + d_t) \quad \forall t.
\]
debt constraint

The market clearing conditions are as follows, first for assets and then for commodities:
\[
\sum_{h \in H} \theta^h_t = 1. \quad (12.79)
\]
\[
\sum_{h \in H} c^h_t = \sum_{h \in H} e^h_t + d_t.
\]
The commodity market clearing condition states the total consumption equals total aggregate resources. The aggregate resources are the sum of the household endowments and the dividend of the asset.

Consider the Euler equations under the explicit debt constraint, where \(\mu^h_t\) is the Lagrange multiplier associated with the constraint \(\theta^h_t \geq -A\):
\[
p_t D u \left( c^h_t \right) = \mu^h_t + (p_{t+1} + d_{t+1}) \beta D u \left( c^h_{t+1} \right). \quad (12.80)
\]
The complimentary slackness condition requires that \(\mu^h_t (\theta^h_t + A) = 0\). If the constraint is binding, then \(\mu^h_t > 0\) and the optimal decisions of households are affected by the constraint. This is why the constraint is referred to as "explicit."

For the implicit debt constraint, households would never find it in their best interest to choose assets such that \(p_t \theta^h_t = -\frac{1}{q^h} \sum_{t=1}^{\infty} q_{t} e^h_{t-1}\). A binding constraint means that the household is setting consumption equal to 0 in all future periods and the Inada condition \(\lim_{c \to 0} D u^h(c) = +\infty\) ensures that this is never an optimal thing to do. This is why the constraint is referred to as "implicit."

The implicit debt constraint never binds (in equilibrium), meaning that the Euler equation is given by:
\[
p_t D u \left( c^h_t \right) = (p_{t+1} + d_{t+1}) \beta D u \left( c^h_{t+1} \right). \quad (12.81)
12.5. BUBBLES

12.5.4 Bubbles with explicit debt constraints

Can bubbles exist in equilibrium? It turns out that this depends on the type of debt constraints that we impose.

Beginning in period $t$, the net present discounted value of an asset’s future dividends equals $\frac{1}{q_t} \sum_{\tau=t+1}^{\infty} q_{\tau} d_{\tau}$. The term $\frac{1}{q_t}$ must be added to discount back to period $t$ (when the asset is purchased). We define the fundamental value of an asset as the net preset discounted value of its future dividends. If $p_t = \frac{1}{q_t} \sum_{\tau=t+1}^{\infty} q_{\tau} d_{\tau}$, the asset price equals the fundamental value and no bubble is present. If $p_t > \frac{1}{q_t} \sum_{\tau=t+1}^{\infty} q_{\tau} d_{\tau}$, a bubble exists and the size of the bubble equals the difference $p_t - \frac{1}{q_t} \sum_{\tau=t+1}^{\infty} q_{\tau} d_{\tau}$. We will focus exclusively on positive bubbles.

The fundamental value is equal to $\frac{1}{q_t} (q_{t+1} d_{t+1} + q_{t+2} d_{t+2} + ....)$ Using the expression for the present value price, the fundamental value is equivalently expressed as:

$$p_t \left( \frac{d_{t+1}}{p_{t+1} + d_{t+1}} + \frac{p_{t+1}}{d_{t+1}} \frac{d_{t+2}}{p_{t+2} + d_{t+2}} + .... \right)$$

$$= p_t \sum_{\tau=t+1}^{\infty} \left( \prod_{k=t+1}^{\tau-1} \frac{p_k}{p_k + d_k} \right) \frac{d_{\tau}}{p_{\tau} + d_{\tau}}.$$

Using this expression, a bubble exists if and only if

$$1 > \sum_{\tau=t+1}^{\infty} \left( \prod_{k=t+1}^{\tau-1} \frac{p_k}{p_k + d_k} \right) \frac{d_{\tau}}{p_{\tau} + d_{\tau}}.$$  \hspace{1cm} (12.83)

An example of an asset bubble is money. Recall that money is an infinite-lived asset with dividends $d_t = 0$ in all time periods. Yet the "price" for money is strictly positive (money has value) and hence strictly above the fundamental value.

Consider the following example in which a bubble exists in equilibrium. The economy contains two households with the following utility function:

$$U (c^h) = \sum_{t=0}^{\infty} \beta^t \ln (c^h_t).$$  \hspace{1cm} (12.84)

The financial markets consist of a single infinite-lived asset in unit net supply. The dividends
of the asset are \( d_t = 0 \) in all time periods (think, money). The initial endowments of the asset are \( \theta_{-1}^1 = 1 \) and \( \theta_{-1}^2 = 0 \).

In this economy, assume that the discount factor lies in the range \( \beta \in \left( \frac{2}{3}, 1 \right) \) and the households have the following endowment structure:

\[
e^1_t = \begin{cases} 
2 + \frac{3\beta - 2}{3(1+\beta)} & t = 0 \\
3 & \text{for } t \text{ odd} \\
2 & \text{for } t \text{ even}
\end{cases},
\]

(12.85)

The total endowment in this economy is 5, meaning that the endowments of the second household are:

\[
e^2_t = \begin{cases} 
3 - \frac{3\beta - 2}{3(1+\beta)} & t = 0 \\
2 & \text{for } t \text{ odd} \\
3 & \text{for } t \text{ even}
\end{cases}.
\]

(12.86)

The explicit debt constraints are identical for both households and given by \( \theta_t^h \geq -1 \).

The equilibrium conditions that must be satisfied in all time periods are the budget constraints (for both households)

\[
c_t^h + p_t \theta_t^h = e_t^h + \theta_{t-1}^h (p_t + d_t),
\]

(12.87)

the market clearing conditions (for both asset markets and commodity markets)

\[
\theta_t^1 + \theta_t^2 = 1,
\]

(12.88)

\[
c_t^1 + c_t^2 = 5,
\]

and the Euler equations (for both households). The transversality conditions also must be satisfied for both households.

With the natural log utility function and zero dividends, the Euler equations are given by:

\[
\frac{p_t}{c_t^h} = \mu_t^h + \beta \frac{p_{t+1}}{c_{t+1}^h}.
\]

(12.89)

Recall that \( \mu_t^h \) is the Lagrange multiplier associated with the constraint \( \theta_t^h \geq -1 \) and must satisfy the complimentary slackness condition:

\[
\mu_t^h (\theta_t^h + A) = 0.
\]

(12.90)
12.5. **BUBBLES**

The transversality condition is given by:

$$\lim_{t \to \infty} \beta^t Du \left( c_t^h \right) \theta_t^h = 0. \quad (12.91)$$

An end-of-chapter exercise asks you to verify that all of the equilibrium conditions (budget constraints, Euler, market clearing, and transversality) are satisfied for the following variables:

$$p_t = p = \frac{3\beta - 2}{3(1 + \beta)} \quad \forall t. \quad (12.92)$$

$$\begin{cases} (3 - 3p, 2) & \text{for } t \text{ odd} \\ (2 + 3p, -1) & \text{for } t \text{ even} \end{cases} \quad (c_t^1, \theta_t^1).$$

$$\begin{cases} (2 + 3p, -1) & \text{for } t \text{ odd} \\ (3 - 3p, 2) & \text{for } t \text{ even} \end{cases} \quad (c_t^2, \theta_t^2).$$

Since these variables satisfy all equilibrium conditions, then an equilibrium exists in which the asset price \( p_t = p = \frac{3\beta - 2}{3(1 + \beta)} > 0 \) in all time periods. This price is strictly positive since \( \beta > \frac{2}{3} \) by assumption. The asset price bubble is the entire price itself, since the fundamental value equals 0 (zero dividends). The principal cause of the bubble is the inability of households to "pop" the bubble. If an asset is over-valued, households can exploit this by short-selling the asset. The constraint \( \theta_t^h \geq -1 \) prevents enough short-selling from taking place.

### 12.5.5 No bubbles with implicit debt constraints

With implicit debt constraints, however, bubbles are not possible.

The income \( m^h \) is equal to the net present value of the entire endowment sequence plus the payout of the initial asset holding:

$$m^h = \sum_{t=0}^{\infty} q_t e_t^h + \theta_{-1}^h (p_0 + d_0). \quad (12.93)$$

Since implicit debt constraints do not affect the optimal household choices, each household can take the income \( m^h \) and use it to make consumption choices over the entire future history of the model:

$$\sum_{t=0}^{\infty} q_t e_t^h \leq m^h. \quad (12.94)$$
The price for consumption in period $t$ is the present value price $q_t$. This plays an equivalent role as the state prices introduced in the general financial equilibrium model.

Household income $m^h$ can be rewritten by splitting the asset payouts into the net present discounted value of dividends and the bubble component:

$$m^h = \sum_{t=0}^{\infty} q_t e_t^h + \theta_{-1}^h \sum_{t=1}^{\infty} q_t d_t + \theta_{-1}^h \left( p_0 + d_0 - \sum_{t=1}^{\infty} q_t d_t \right). \quad (12.95)$$

Since $q_0 = 1$ by normalization, the term

$$d_0 + \sum_{t=1}^{\infty} q_t d_t = \sum_{t=0}^{\infty} q_t d_t. \quad (12.96)$$

This means that the income $m^h$ is given by:

$$m^h = \sum_{t=0}^{\infty} q_t e_t^h + \theta_{-1}^h \sum_{t=0}^{\infty} q_t d_t + \theta_{-1}^h \left( p_0 - \frac{1}{q_0} \sum_{t=1}^{\infty} q_t d_t \right). \quad (12.97)$$

To obtain Walras’ Law, note that the budget constraints bind for all households and then sum the budget constraints over all households:

$$\sum_{h \in H} m^h = \sum_{t=0}^{\infty} q_t \sum_{h \in H} c_t^h. \quad (12.98)$$

In words, Walras’ Law states that the total income equals the total value of expenditures. Combining this with the commodity market clearing conditions yields:

$$\sum_{h \in H} m^h = \sum_{t=0}^{\infty} q_t \sum_{h \in H} c_t^h = \sum_{t=0}^{\infty} q_t \sum_{h \in H} e_t^h + \sum_{t=0}^{\infty} q_t d_t. \quad (12.99)$$

Using the asset market clearing condition, $\sum_{h \in H} \theta_{-1}^h = 1$, equations (12.97) and (12.99) imply that:

$$\sum_{h \in H} \theta_{-1}^h \left( p_0 - \frac{1}{q_0} \sum_{t=1}^{\infty} q_t d_t \right) = 0. \quad (12.100)$$

Using the asset market clearing condition again, we arrive at a relation between the initial
12.6. THE BLACK-SCHOLES MODEL

asset price and the net present discounted value of dividends:

\[ p_0 = \frac{1}{q_0} \sum_{t=1}^{\infty} q_t d_t. \]  

The initial asset price is equal to the fundamental value. There is no bubble in the initial period. If a bubble doesn’t arise beginning in the initial period, it won’t arise in any future period. In conclusion, bubbles are not possible with implicit debt constraints.

12.6 The Black-Scholes model

12.6.1 Sneak peek

Summary

The final asset pricing model is very different in structure than the previous models. Though it relies on different mathematical structure, the Black-Scholes model is important enough to warrant inclusion in these pages. Importantly, as with all asset pricing models, the solution of the Black-Scholes model relies fundamentally on the concept of no arbitrage.

The Black-Scholes model provides a formula for the price of a European call option. The model is set in continuous time, which requires a bit more of an introduction compared to the discrete-time models we have been working with thus far in the text. Continuous time has the disadvantage of an initial fixed cost required to learn the notation and the techniques. The advantage is that the mathematical tools of stochastic calculus allow for closed-form solutions in continuous-time models that would not be possible in discrete-time models. In the context of the Black-Scholes model, the asset pricing formula is the solution to a partial differential equation (PDE).

Using the concept of no arbitrage, we derive the put-call parity equation which relates the price of a call option, the price of a put option, and the price of a forward contract. The Black-Scholes formula provides an equation for the price of a European call option. The price of a forward contract is straightforward to derive. Using put-call parity, we can then determine the price of a European put option.
Notation

The variables/parameters to be introduced in this section are given in the following table:

- $r$: continuous interest rate
- $K$: strike price
- $F$: price of a forward contract
- $C$: price of a call option
- $P$: price of a put option
- $S_t$: price of stock in time $t$
- $\beta_t$: price of bond in time $t$
- $\mu$: stock price growth rate
- $\sigma$: stock price standard deviation
- $B_t$: geometric Brownian motion
- $a_t$: units of stock in the portfolio in time $t$
- $b_t$: units of bond in the portfolio in time $t$
- $f(t, S_t)$: price of redundant portfolio
- $\Phi$: cdf for the normal distribution

Main takeaways

After completing this section, you will be able to answer the following questions:

- What role does no arbitrage play in pricing a European call option?
- What partial differential equation must be solved in order to find the formula for the price of a European call option?
- In the formula for the price of a European call option, what are the effects of the various model parameters?

12.6.2 Pricing a forward contract

The Black-Scholes model provides an asset pricing formula for a European call option. The model consists of a continuous time horizon, meaning that the tools of stochastic calculus will be required to analyze the problem. The appealing aspect of the Black-Scholes model is that it has a closed form solution, meaning that we arrive at a simple formula for the price of a European call option.
Assume that there is a continuous interest rate $r$ for households. Suppose that we are currently in time $t$. The net present value of $K$ paid out in time $T > t$ is equal to $e^{-r(T-t)}K$. If the interest rate $r = 0$, the time discounting factor $e^{-r(T-t)} = 1$. If the interest rate $r > 0$, the time discounting factor $e^{-r(T-t)} < 1$.

Consider a forward contract in time $t$ to buy an asset in time $T > t$ at the price $K$. The current price of the asset in time $t$ is $S_t$. By no arbitrage, the price of this forward contract must be:

$$ F = S_t - e^{-r(T-t)}K. \quad (12.102) $$

Consider the two investment options for the household: (i) buy the forward contract at the price $F$, which allows the holder to buy the asset in time $T$ at the price $K$ or (ii) buy the asset now at price $S_t$ and hold onto it until time $T$. Both investment options lead to the same outcome in period $T$, namely 1 unit of the asset. Option (i) costs $F + e^{-r(T-t)}K$ in present value terms (the price $K$ must be discounted back to the current period $t$). Option (ii) costs $S_t$. No arbitrage requires that both options cost the same,

$$ F + e^{-r(T-t)}K = S_t, \quad (12.103) $$

so the forward contract pricing equation is confirmed.

### 12.6.3 Put-call parity

Recall that a European put option is the right, but not the obligation, to sell the stock for price $K$ in time $T$. Diametrically, a European call option is the right, but not the obligation, to buy the stock for price $K$ in time $T$. Denote the price of the call as $C$ and the price of the put as $P$.

Consider the following two investment options for the household: (i) buy the call option at the price $C$ and (ii) buy the forward contract at price $F$ and the put option at price $P$. All of these contracts are purchased in the current period $t$ and specify the strike price of $K$ in period $T$.

Denote $S_T$ as the stock price in period $T$. Under option (i), the call option is exercised whenever the stock price exceeds the strike price. The payout in each state is equal to $\max\{S_T - K, 0\}$ and the expected payout is the expected value of this payout. Under option (ii), the put option is exercised whenever the stock price is lower than the strike price. The household buys the stock at price $K$ according to the forward contract. This earns a payout
of $S_T - K$. Simultaneously, the household sells the stock whenever $S_T < K$. This earns a payout equal to $\max\{K - S_T, 0\}$. The summed payout for the portfolio equals

$$S_T - K + \max\{K - S_T, 0\} = \max\{S_T - K, 0\}. \quad (12.104)$$

This is the same payout as for option (i) with the call option. No arbitrage requires that both options cost the same:

$$C = F + P. \quad (12.105)$$

Subtracting the put option price, we get the put-call parity equation:

$$C - P = F. \quad (12.106)$$

We already know the forward contract pricing equation $F = S_t - e^{-r(T-t)}K$. The Black-Scholes model will find an equation for $C$. Given this put-call parity equation, the equation for $P$ can be easily determined.

### 12.6.4 Two underlying assets

In the Black-Scholes model, there are two assets: a stock with risky returns and a bond with risk-free returns. Let $S_t$ denote the price of a stock at time $t$ and let $\beta_t$ denote the price of a bond at time $t$. The change in the bond price follows a standard process:

$$d\beta_t = r\beta_t dt. \quad (12.107)$$

The discrete time analogue is $\beta(t) = (1 + r)\beta(t - 1)$. The change in the stock price follows a random process governed by geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dB_t. \quad (12.108)$$

$B_t$ is the geometric Brownian motion. The discrete time analogue is that $S(t) = (1 + \mu)S(t - 1) + \epsilon(t)$, where $\epsilon(t) \sim N(0, \sigma^2)$, a normal distribution with mean 0 and variance $\sigma^2$. The stock price formula says that the stock price grows with mean $\mu$ and variance $\sigma^2$ times the Brownian motion "noise" term. The term $\sigma S_t dB_t$ has expected value equal to 0.
12.6.5 Self-financing condition

We will find the equation for a European call option with strike price $K$ in time $T$. Denote $f(t, S_t)$ as the price of the European call option at time $t$ as a function of the time $t$ stock price $S_t$. The price must be $f(T, S_T) = \max\{S_T - K, 0\}$ at the termination time $T$. At the termination time, the payout of the European call option is $\max\{S_T - K, 0\}$, so this must be the price at which that option is valued. The key for the Black-Scholes formula is to be able to derive prices $f(t, S_t)$ at times before the termination time $T$.

To find a price for this option, we will construct a redundant portfolio consisting of stock and bond holdings. The price of this redundant portfolio must be equal to the price of the European call option:

$$f(t, S_t) = a_t S_t + b_t \beta_t,$$

where the portfolio consists of $a_t$ units of the stock in time $t$ and $b_t$ units of the bond in time $t$. These holdings $(a_t, b_t)$ are continuously adjusting over time. By adjusting these coefficients, we are restructuring the portfolio.

The European call option and the portfolio are redundant, meaning that any change in the price for one must be exactly equal to a change in the price for the other. The change in the price for the portfolio is $a_t dS_t + b_t d\beta_t$. The change in the price for the European call option is the derivative $df(t, S_t)$. These two changes must be identical, an equation which is called either the restructuring requirement or the self-financing condition:

$$df(t, S_t) = a_t dS_t + b_t d\beta_t \text{ for all } t.$$  

(12.110)

Given the processes previously specified ($d\beta_t = r \beta_t dt$ and $dS_t = \mu S_t dt + \sigma S_t dB_t$), the right-hand side of the self-financing condition is given by:

$$a_t (\mu S_t dt + \sigma S_t dB_t) + b_t (r \beta_t dt) = \{a_t \mu S_t + b_t r \beta_t\} dt + a_t \sigma S_t dB_t.$$  

(12.111)

For the left-hand side of the self-financing condition, we cite the Itô formula. The Itô formula for geometric Brownian motion dictates that for any function $f(t, S_t)$:

$$df(t, S_t) = f_t (t, S_t) dt + \frac{1}{2} f_{xx}(t, S_t) dS_t \cdot dS_t + f_x (t, S_t) dS_t.$$  

(12.112)

The term $df(t, S_t)$ is a total derivative, while the terms $f_t (t, S_t)$, $f_x (t, S_t)$, and $f_{xx}(t, S_t)$ are partial derivatives (partial second derivatives in the case of $f_{xx}(t, S_t)$). The geometric
Brownian motion is such that

\[
    dS_t \cdot dS_t = (\mu S_t dt + \sigma S_t dB_t) (\mu S_t dt + \sigma S_t dB_t) = \sigma^2 S_t^2 dt. \tag{12.113}
\]

This property occurs since \( dt \cdot dt = 0 \), \( dt \cdot dB_t = 0 \), \( dB_t \cdot dt = 0 \), and \( dB_t \cdot dB_t = dt \).

Given the fact that \( dS_t \cdot dS_t = \sigma^2 S_t^2 dt \) under Brownian motion, the Itô formula can be rewritten as:

\[
    df(t, S_t) = \left\{ f_t(t, S_t) + \frac{1}{2} f_{xx}(t, S_t) \sigma^2 S_t^2 + f_x(t, S_t) \mu S_t \right\} dt + f_x(t, S_t) \sigma S_t dB_t. \tag{12.114}
\]

Since the self-financing condition \( df(t, S_t) = a_t dS_t + b_t dB_t \) must hold at all instance of time (for all values of \( t \leq T \)), then it must be that the terms in front of \( dt \) must be equal and the terms in front of \( dB_t \) must be equal:

\[
    a_t \mu S_t + b_t r \beta_t = f_t(t, S_t) + \frac{1}{2} f_{xx}(t, S_t) \sigma^2 S_t^2 + f_x(t, S_t) \mu S_t. \tag{12.115}
\]

The second equation is for the coefficients in front of the Brownian motion term \( dB_t \). This equation yields:

\[
    a_t = f_x(t, S_t). \tag{12.116}
\]

The first equation is for the coefficients in front of the time term \( dt \). Using the previous conclusion that \( a_t = f_x(t, S_t) \):

\[
    f_x(t, S_t) \mu S_t + b_t r \beta_t = f_t(t, S_t) + \frac{1}{2} f_{xx}(t, S_t) \sigma^2 S_t^2 + f_x(t, S_t) \mu S_t. \tag{12.117}
\]

The term \( f_x(t, S_t) \mu S_t \) cancels from both sides, meaning that the coefficient \( b_t \) can be evaluated:

\[
    b_t = \frac{1}{r \beta_t} \left\{ f_t(t, S_t) + \frac{1}{2} f_{xx}(t, S_t) \sigma^2 S_t^2 \right\}. \tag{12.118}
\]
12.6.6 Black-Scholes partial differential equation

Having determined the holdings \((a_t, b_t)\) at each instant in time, the equilibrium equation for the price of the European call option in period \(t\) as a function of the stock price \(S_t\) is:

\[
f(t, S_t) = a_t S_t + b_t \beta_t = f_x(t, S_t) S_t + \frac{1}{r} \left\{ f_t(t, S_t) + \frac{1}{2} f_{xx}(t, S_t) \sigma^2 S^2_t \right\}.
\]  

(12.119)

We are tasked with solving a partial differential equation (PDE) satisfying:

1. \[ f(t, S_t) = f_x(t, S_t) S_t + \frac{1}{r} \left\{ f_t(t, S_t) + \frac{1}{2} f_{xx}(t, S_t) \sigma^2 S^2_t \right\} \text{ for all } t \]

2. \[ f(T, S_T) = \max\{S_T - K, 0\}. \]

This is called the Black-Scholes partial differential equation (PDE). The second condition is the boundary condition, which is required to solve differential equations. We can in fact solve this PDE in closed form.

Skipping the details, the solution of the PDE is:

\[
f(t, S) = S \Phi \left( \frac{\ln(S/K) + (r - \frac{1}{2} \sigma^2) (T - t)}{\sigma \sqrt{T - t}} \right) - K e^{-r(T-t)} \Phi \left( \frac{\ln(S/K) + (r - \frac{1}{2} \sigma^2) (T - t)}{\sigma \sqrt{T - t}} \right).
\]  

(12.120)

This is the price for a European call option traded in time \(t\), with a current stock price of \(S\), and a strike price of \(K\) that can be exercised at time \(T > t\). This equation is commonly referred to as the Black-Scholes formula.

The function \(\Phi\) is the cumulative density distribution (cdf) for the normal distribution. By definition, if the random variable \(X\) is normally distributed with mean \(0\) and variance \(1\) (i.e., \(X \sim N(0,1)\)), then \(\Phi(z) = \text{prob}(X \leq z)\). Some simple values include \(\Phi(0) = \frac{1}{2}\). There is a closed form for this distribution given by \(\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt\), but practitioners typically use charts or computers to approximate these values. The normal distribution is present in the Black-Scholes formula, because the Brownian motion (the random variable governing the stock price movements) is normally distributed.

One of the remarkable things about this formula is that the price of the option does not depend on \(\mu\), the growth rate of the stock price.
12.7 Exercises

1. No arbitrage pricing

Suppose that there are $S = 4$ states and $J = 2$ assets with payouts

$$R = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 0 & 1 \\ 3 & 2 \end{bmatrix}.$$ 

Which of the following asset prices satisfy no arbitrage: (i) $q = (0, 2)$, (ii) $q = (2, 1.75)$, and (iii) $q = (2, 1.25)$?

2. No arbitrage pricing

Suppose that there are $S = 4$ states and $J = 3$ assets with payouts

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & \frac{1}{3} \\ \frac{1}{2} & \frac{2}{3} & 1 \end{bmatrix}.$$ 

Which of the following asset prices satisfy no arbitrage: (i) $q = (1, 4, 2)$, (ii) $q = (2, 1, 1)$, and (iii) $q = (3, 2, 1)$?

3. Lucas asset pricing model

Solve for the Lucas asset prices when $S = 4$, the dividends are $(d_1,d_2,d_3,d_4) = (4.5, 5, 5.25, 5.4)$, the utility function is $u(c) = \frac{c^{1-\rho}}{1-\rho}$ for relative risk aversion equal to $\rho = 3$, the discount factor $\beta = 0.95$, and the transition matrix is

$$\Gamma = \begin{bmatrix} 0.5 & 0.2 & 0.2 & 0.1 \\ 0.2 & 0.5 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.2 & 0.5 \end{bmatrix}.$$ 

4. Lucas asset pricing model

Consider the following extension of the Lucas asset pricing model.
Each household is initially endowed with one unit each of two trees: one producing apples and one producing bananas. With two goods, the household utility function is \( u : \mathbb{R}_+^2 \to \mathbb{R} \), a function of both apple consumption and banana consumption. The production of each tree follows the same Markov process. Write down the Bellman equation, specify the market clearing conditions, and find equations for the prices of the two trees. Importantly, do no forget to include the relative price of bananas to apples.

5. Lucas asset pricing model

Using the price equations from the previous exercise, write down an expression for the price of a 1-period option to trade a share of the first tree for a share of the second tree.

6. Bubbles

Recall that a bubble exists if and only if

\[
1 > \sum_{\tau=t+1}^{\infty} \left( \prod_{k=t+1}^{\tau-1} \frac{p_k}{p_k + d_k} \right) \frac{d_\tau}{p_\tau + d_\tau}.
\]

Consider a stock price that grows at a constant rate \( r \) (for instance \( r = 2\% \)) and pays out dividends that are a constant fraction \( n \) of the stock price (for instance \( n = 1\% \)). The values for \( r \) and \( n \) can take any values provided that \( r \geq 0 \) and \( n > 0 \). Does a bubble exist? Why or why not?

7. Bubbles

For the economy in the bubbles example, verify that all of the equilibrium conditions (budget constraints, market clearing conditions, Euler equations, and transversality condition) are satisfied for the following variables:

\[
p(t) = p = \frac{3\beta - 2}{3(1 + \beta)} \quad \forall t.
\]

\[
(c^1(t), \theta^1(t)) = \begin{cases} 
(3 - 3p, 2) & \text{for } t \text{ odd} \\
(2 + 3p, -1) & \text{for } t \text{ even}
\end{cases}.
\]

\[
(c^2(t), \theta^2(t)) = \begin{cases} 
(2 + 3p, -1) & \text{for } t \text{ odd} \\
(3 - 3p, 2) & \text{for } t \text{ even}
\end{cases}.
\]
8. *The Black-Scholes model*

The solution to the Black-Scholes model provides a formula for the price of a European call option traded in time \( t \), with a current stock price of \( S \), and a strike price of \( K \) that can be exercised at time \( T > t \). Suppose that the strike price is $95 and the current stock price is $100. Suppose that the risk-free return is equal to \( r = 3\% \) and the standard deviation of the stock price is equal to \( \sigma = 16\% \). Create a graph with \( T - t \) on the x-axis and the price of the European call option on the y-axis.
Bibliography


13

Leverage Cycle

13.1 2-period model: No borrowing

13.1.1 Sneak peek

Summary

This chapter analyzes a model of endogenous leverage. Leverage is a measure of the collateral requirements for each unit of borrowing. The financial setting lacks commitment, meaning that lenders require borrowers to put forth collateral in order to secure a loan. The collateral arrangements are very simple: when it comes time to repay the loan, the borrower may either repay the entire loan or may default and forfeit the collateral. All agents in the economy are aware of these arrangements, meaning lenders face a trade-off: more collateral protects their investment from default, but also discourages borrowing (meaning lenders earn a lower interest rate).

In this model, the leverage, namely the amount of collateral that must be held for each unit of borrowing, is endogenously determined. This means that any borrowing that takes place must be with a collateral requirement that both the lender and the borrower agree to. If either party finds another collateralized borrowing arrangement more profitable, then they will switch to the more profitable arrangement. By allowing leverage to be determined endogenously, the model can account for how changes in leverage (the so-called leverage cycle) can lead to large fluctuations in asset prices and welfare.

We begin with the basics of the 2-period model. In this first section, agents are not allowed to borrow. This will allow us to understand just how important, and volatile, the
introduction of borrowing in the next section will be.

**Notation**

The variables to be introduced in this section are given in the following table:

- $c^h(0)$ consumption in state 0
- $c^h(G)$ consumption in state $G$
- $c^h(B)$ consumption in state $B$
- $\theta^h$ asset holdings (chosen in state 0)
- $p$ asset price
- $s^h$ commodity storage (chosen in state 0)
- $h$ household index, probability of state $G$ for household $h$

**Main takeaways**

After completing this section, you will be able to answer the following questions:

- How do we set up the financial model?
- What is the relation between the asset price and the fraction of households willing to purchase the asset?

### 13.1.2 Heterogeneous beliefs

In this model, there are two time periods, $t = 0$ and $t = 1$. In time period $t = 1$, there are two possibly states of nature: good ($G$) and bad ($B$). There is one commodity and one asset.

The dividends of the asset in state $G$ equals 1, while the dividends of the asset in state $B$ equals 0.2.

There exists a unit mass of households.

Households are characterized by their beliefs. The beliefs are uniformly distributed over the unit interval $[0, 1]$. Household $h$ is a household with beliefs $h$, meaning that the household assigns probability $h$ to state $G$. Since probabilities must sum to 1, household $h$ assigns probability $1 - h$ to state $B$. The timing and the beliefs are illustrated in Figure 13.1.1.

Apart from beliefs, households are identical in every way.
13.1. 2-PERIOD MODEL: NO BORROWING

For household $h$, consumption is $c^h(0)$ in period $t = 0$, $c^h(G)$ in the good state, and $c^h(B)$ in the bad state. The utility function is given by:

$$u^h(c^h(0), c^h(G), c^h(B)) = c^h(0) + hc^h(G) + (1 - h)c^h(B).$$  \hspace{1cm} (13.1)

Households with a linear expected utility function are said to be risk-neutral.

Households are initially endowed with 1 unit of the commodity in time period $t = 0$. The households do not have any endowments in either state in time period $t = 1$.

13.1.3 Financial assets

The asset is in unit net supply. Each household is initially endowed with one unit of the asset and decides how many units of the asset to hold going forward. Denote $\theta^h$ as the asset position chosen by household $h$. The price of the asset is $p$.

It is not possible to short-sell the asset, meaning that $\theta^h \geq 0$. A household with $\theta^h = 0$ has sold their entire initial endowment of 1 unit of the asset, but cannot sell any more. Short-selling is not permitted.

The commodity is durable. This means that the commodity can be stored. Denote $s^h$ as the amount of the commodity that is stored by the household. Obviously, storage is nonnegative: $s^h \geq 0$.

13.1.4 Household budget constraints

The budget constraint for households in period $t = 0$ is:

$$c^h(0) + s^h + p\theta^h \leq 1 + p.$$  \hspace{1cm} (13.2)

The left-hand side is the household expenditures. The household chooses its consumption $c^h(0)$, the amount to store $s^h$, and the asset position $\theta^h$ (with price $p$). The right-hand side is the household income. The initial household endowment is 1 unit of the commodity and 1 unit of the asset (where $p$ is the price of the asset).

The budget constraints for households in each of the two states in period $t = 1$ are:

$$c^h(G) \leq s^h + \theta^h.$$  \hspace{1cm} (13.3)

$$c^h(B) \leq s^h + 0.2\theta^h.$$
Recall that the dividends of the asset is 1 in state G and 0.2 in state B.

### 13.1.5 Market clearing conditions

The market clearing condition for the asset market is \( \int_0^1 \theta^h dh = 1 \). The total initial amount of asset available equals 1, so the total amount after trading must be equal to 1 as well (unit net supply).

The market clearing conditions for the commodity markets are:

\[
\int_0^1 (c^h(0) + s^h) dh = 1,
\]

\[
\int_0^1 c^h(G) dh = 1 + \int_0^1 s^h dh.
\]

\[
\int_0^1 c^h(B) dh = 0.2 + \int_0^1 s^h dh.
\]

The aggregate amount of the commodity in period \( t = 0 \) is 1. In period \( t = 1 \), the aggregate amount of the commodity equals the dividends plus the aggregate amount stored. Dividends are equal to 1 in state G and 0.2 in state B. The total amount stored is \( \int_0^1 s^h dh \).

### 13.1.6 Solving for equilibrium: Equation 1

Define the household \( h^* \) as the cutoff household such that (i) all households \( h \geq h^* \) (the optimists) will buy the asset and (ii) all households \( h \leq h^* \) (the pessimists) will sell the asset. The household \( h^* \) is indifferent between buying and selling the asset. The price of the asset is equal to the expected value of the asset for household \( h^* \):

\[
p = h^* + (1 - h^*) (0.2) .
\]

Recall that the asset has dividends equal to 1 in state G (which occurs with probability \( h^* \) for the cutoff household) and dividends equal to 0.2 in state B (which occurs with probability \( 1 - h^* \) for the cutoff household).

### 13.1.7 Solving for equilibrium: Equation 2

Households \( h \leq h^* \) will sell their entire initial endowment of the asset, meaning that \( \theta^h = 0 \).
Households $h \geq h^*$ will set $c^h(0) = s^h = 0$ in order to buy as many units of the asset as possible. The updated budget constraint for households $h \geq h^*$ is given by:

$$ p \theta^h = 1 + p. $$

(13.6)

Take the integral of the budget constraint over all households $h \geq h^*$:

$$ \int_{h^*}^{1} p \theta^h dh = \int_{h^*}^{1} (1 + p) dh. $$

(13.7)

We can first factor out the terms from each of the integrals that are independent of $h$:

$$ p \int_{h^*}^{1} \theta^h dh = (1 + p) \int_{h^*}^{1} dh. $$

(13.8)

From the asset market clearing condition $\int_{h^*}^{1} \theta^h dh = 1$. The fraction of optimistic households $h \geq h^*$ is equal to:

$$ \int_{h^*}^{1} dh = 1 - h^*. $$

(13.9)

This follows because households are uniformly distributed. The integral of the budget constraint over all households $h \geq h^*$ is given by:

$$ p = (1 - h^*) (1 + p). $$

(13.10)

### 13.1.8 Solving for equilibrium: Two equations and two unknowns

We now have two equations and two unknowns:

$$ p = h^* + (1 - h^*) (0.2). $$

(13.11)

$$ p = (1 - h^*) (1 + p). $$

(13.12)

The first equation can be solved for $h^*$:

$$ h^* = \frac{p - 0.2}{0.8}. $$

(13.13)
Inserting into the second equation results in a quadratic equation for $p$:

$$p = \left( \frac{1 - p}{0.8} \right) (1 + p).$$

(13.14)

Gathering terms, we arrive at the following quadratic equation:

$$p^2 + 0.8p - 1 = 0.$$  

(13.15)

Solving this quadratic equation yields:

$$p = 0.677.$$  

(13.16)

$$h^* = 0.596.$$

### 13.2 2-period model: With borrowing

#### 13.2.1 Sneak peek

**Summary**

This section introduces borrowing. There is one option for borrowing, namely that the collateral requirement for each unit of borrowing is fixed. The collateral is the real asset. Borrowers stake the real asset as collateral in order to borrow and then turn around and use these borrowed funds in order to purchase more units of the real asset. The exogenous collateral requirement specifies that for each 0.2 units of borrowing, 1 unit of the real asset must be put forth as collateral. Investors that wish to buy the asset are simultaneously increasing their collateral when they buy the asset. The effective price of the asset is then equal to the actual price minus the amount that can be borrowed. In this case, the effective price is $p - 0.2$, since each unit of the asset allows for 0.2 units of borrowing.

When the uncertainty is realized, borrowers have to decide whether to repay their loan or to default and forfeit the collateral. The collateral requirement was exogenously specified such that the borrower will always repay the loan. In fact, the borrower is indifferent between default and repayment in the bad state. Such a borrowing contract is called a ‘no-default’ loan.

After investigating the equilibrium outcome if borrowing on one contract is permitted, the following section will allow for the possibility that many borrowing contracts are traded.
13.2. 2-PERIOD MODEL: WITH BORROWING

Notation

The variables to be introduced in this section are given in the following table:

- $b^h$: amount that household $h$ borrows in state 0
- $r$: interest rate at which borrowing takes place
- $h^*$: marginal household, indifferent between buying and selling asset
- $h^*$: marginal household, indifferent between borrowing and lending

Main takeaways

After completing this section, you will be able to answer the following questions:

- When borrowing is permitted, what is the equilibrium interest rate?
- When borrowing is permitted, is the asset price higher or lower compared to when borrowing is not permitted?
- What are three common measures of the levels of borrowing in a market?

13.2.2 Introducing collateralized borrowing

Denote the promised repayment amount for household $h$ as $b^h$. Any household with $b^h < 0$ is a lender, while any household with $b^h > 0$ is a borrower. Any borrowing that occurs takes place at the interest rate $r$. Consequently, the ability to borrow changes the period $t = 0$ budget constraint to:

$$c^h(0) + s^h + p\theta^h \leq 1 + p + \frac{1}{1 + r} b^h.$$  \hspace{1cm} (13.17)

The market clearing condition for borrowing is given by $\int_0^1 b^h \, dh = 0$. The market clearing condition says that the total amount of borrowing $\int_0^1 \max \{b^h, 0\} \, dh$ must be equal to the total amount of lending $\int_0^1 \max \{-b^h, 0\} \, dh$.

Lenders in this new borrowing market are concerned about default. Consequently, they are going to require borrowers to provide collateral in order to secure a loan. The collateral is going to be the household’s asset holding $\theta^h$. The state $G$ and state $B$ amounts that the
borrower has to repay, for each of the 2 contingencies, are given in the table below:

<table>
<thead>
<tr>
<th></th>
<th>repay loan</th>
<th>forfeit collateral</th>
</tr>
</thead>
<tbody>
<tr>
<td>state G</td>
<td>$b^h$</td>
<td>$1 \cdot \theta^h$</td>
</tr>
<tr>
<td>state B</td>
<td>$b^h$</td>
<td>$0.2 \cdot \theta^h$</td>
</tr>
</tbody>
</table>

Notice that by forfeiting collateral ($\theta^h$ units of the asset), the borrower is forgoing the dividends of the asset. The borrower is always going to choose the option that minimizes the repayment amount, so chooses min \(b^h, \theta^h\) in state G and min \(b^h, 0.2\theta^h\) in state B.

Consequently, the amount repaid to the lender is given by \(b^h \min \{1, \Psi\}\) in state G and \(b^h \min \{1, 0.2\Psi\}\) in state B, where \(\Psi\) is the collateral-to-loan ratio \(\frac{\theta^h}{r}\) chosen by the borrowers.

The budget constraints in state G and state B are given by:

\[
\begin{align*}
\epsilon^h (G) & \leq s^h + \theta^h - b^h \min \{1, \Psi\}, \\
\epsilon^h (B) & \leq s^h + 0.2\theta^h - b^h \min \{1, 0.2\Psi\}.
\end{align*}
\] (13.18) (13.19)

For now, suppose that the collateral constraint is set such that \(b^h \leq 0.2\theta^h\). This means that 0.2 units of borrowing is permitted for each one unit of collateral, or 5 units of collateral is required for each 1 unit of borrowing. Since \(b^h \leq 0.2\theta^h\), then \(\Psi \geq 5\), min \(\{b^h, \theta^h\} = b^h\) in state G, and min \(\{b^h, 0.2\theta^h\} = b^h\) in state B, meaning that the borrowers always choose to repay the loan. If the borrower borrows up to the limit \(b^h = 0.2\theta^h\), then \(\Psi = 5\) and the payouts are the same in state B for both default and repayment.

We can now solve for the equilibrium with borrowing. We conjecture that all households \(h \geq h^*\) will borrow up to the limit \(b^h = 0.2\theta^h\) and the interest rate is \(r = 0\). Under these two conjectures, the collateral-to-loan ratio \(\Psi = 5\). We will go back later to confirm these conjectures.

13.2.3 Solving for the equilibrium: Equation 1

Similar to the equilibrium solution without borrowing, there is going to be a cutoff. The marginal household \(h^*\) is indifferent between buying and selling the asset. This means that the price of the asset must be identical to the expected value of the asset for household \(h^*\):

\[
p = h^* + (1 - h^*) (0.2).
\] (13.20)
The optimistic households $h \geq h^*$ will purchase the asset and the pessimistic households $h \leq h^*$ will sell the asset. This equation is the same as found for the no-borrowing economy.

### 13.2.4 Solving for the equilibrium: Equation 2

The households $h \leq h^*$ sell their entire initial endowment of the asset, meaning they choose $\theta^h = 0$.

The households $h \geq h^*$ buy as many units of the asset as possible. They borrow up to the limit $b^h = 0.2\theta^h$ and set $c^h(0) = s^h = 0$ to buy as many units of the asset as possible. Under the assumption that $r = 0$, the updated budget constraint for households $h \geq h^*$ is given by:

$$p \theta^h = 1 + p + 0.2\theta^h.$$  \hspace{1cm} (13.21)

Subtract $0.2\theta^h$ from both sides:

$$(p - 0.2)\theta^h = 1 + p.$$  \hspace{1cm} (13.22)

Take the integral of the budget constraint over all households $h \geq h^*$:

$$\int_{h^*}^1 (p - 0.2)\theta^h dh = \int_{h^*}^1 (1 + p) dh.$$  \hspace{1cm} (13.23)

Using the facts from the previous section, the integrals simplify to:

$$p - 0.2 = (1 - h^*) (1 + p).$$  \hspace{1cm} (13.24)

### 13.2.5 Solving for equilibrium: Two equations and two unknowns

An end-of-chapter exercise asks you to verify that $(h^*, p) = (0.686, 0.749)$ is the solution to the system of equations

$$p = h^* + (1 - h^*) (0.2).$$  \hspace{1cm} (13.25)

$$p - 0.2 = (1 - h^*) (1 + p).$$  \hspace{1cm} (13.26)
13.2.6 Comparing no-borrowing and borrowing outcomes

Consider the following table:

<table>
<thead>
<tr>
<th></th>
<th>No borrowing</th>
<th>Borrowing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset price</td>
<td>$p = 0.677$</td>
<td>$p = 0.749$</td>
</tr>
<tr>
<td>Marginal household</td>
<td>$h^* = 0.596$</td>
<td>$h^* = 0.686$</td>
</tr>
</tbody>
</table>

Borrowing allows for a higher asset price. Borrowing also allows for a smaller group of optimistic households to drive up the asset price.

13.2.7 Leverage definitions

There are several equivalent ways in which we can refer to the amount that households borrow:

1. Leverage

   Leverage is equal to the asset price divided by the effective asset price actually paid accounting for borrowing. The effective price is $p - 0.2$, since the households borrow $b^h = 0.2\theta^h$.

   $$\text{Leverage} = \frac{p}{p - 0.2} = 1.36. \quad (13.27)$$

2. Loan to value

   The loan to value is the ratio of the amount borrowed (for each unit of the asset held) divided by the asset price.

   $$\text{Loan to value} = \frac{0.2}{p} = 27\%. \quad (13.28)$$

3. Margin (or haircut)

   The margin (or haircut) is equal to the effective asset price divided by the actual asset price.

   $$\text{Margin} = \frac{p - 0.2}{p} = 73\%. \quad (13.29)$$

   The margin and the loan to value ratio must sum to 1. The margin is the inverse of the leverage ratio.
13.2. 2-PERIOD MODEL: WITH BORROWING

13.2.8 Inequality

The Gini coefficient (also called the Gini index) measures how unequal a distribution is. When the Gini coefficient equals 1, the distribution is entirely unequal. When the Gini coefficient equals 0, the distribution is perfectly equal. In equilibrium, there are two types of households: optimists with \( h \geq h^* \) and pessimists with \( h < h^* \). The consumption in the final time period are displayed in the following table:

<table>
<thead>
<tr>
<th></th>
<th>state ( G )</th>
<th>state ( B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h \geq h^* )</td>
<td>2.549</td>
<td>0</td>
</tr>
<tr>
<td>( h &lt; h^* )</td>
<td>1.749</td>
<td>1.749</td>
</tr>
</tbody>
</table>

To verify these values, consider first optimistic households with \( h \geq h^* \). The fraction of optimistic households is 31.4% of the population. Since the total asset amount is 1, then each of the optimistic households owns \( \theta^h = \frac{1}{0.314} \approx 3.186 \) units of asset. The asset payout for each household is equal to \((1 - 0.2) \theta^h = 2.549\) units in state \( G \) and \((0.2 - 0.2) \theta^h = 0\) in state \( B \). These are also the consumption values for the optimistic households in both states.

Pessimistic households are indifferent between storing the commodity and consuming in the initial period \( t = 0 \). Suppose that they store the commodity. This means that the initial income for all pessimistic households is being used either to lend or to store. The total income for each pessimistic household equals \( 1 + p = 1.749 \). Both storage and lending have zero interest rate, so each pessimistic household receives a payout (and consumption) equal to 1.749 in both states \( G \) and \( B \).

Total consumption in state \( G \) is equal to

\[
(0.314) (2.549) + (0.686) (1.749) = 2,
\]

which is the total amount of available resources given that all commodity from the initial period is stored. Total consumption in state \( B \) is equal to

\[
(0.314) (0) + (0.686) (1.749) = 1.2,
\]

which is the total amount of available resources given that all commodity from the initial period is stored.

In state \( G \), the high-consumption group is the optimists \( h \geq h^* \). This group makes up 31.4% of the population and consumes \( \frac{(31.4\%) (2.549)}{2} = 0.4 \) of the total consumption. When
the high-consumption group comprise 40% of the total consumption and only 31.4% of the population, the Gini coefficient is found from the simple formula:

\[
Gini(G) = 0.40 - 0.314 = 0.086. \tag{13.33}
\]

In state \( B \), the high-consumption group is the pessimists \( h < h^* \). The group consume 100% of the total consumption and only 68.6% of the total population. The Gini index is found from the simple formula:

\[
Gini(B) = 1 - 0.686 = 0.314. \tag{13.34}
\]

Obviously the Gini index has two very different values depending upon which state is realized. Policymakers that are interested in Gini must come up with a way to assign probabilities to the two different states in order to compute an expected measure of inequality.

### 13.2.9 Verifying borrowers at limit

We can now show that all borrowers will borrow up to the limit \( b^h = 0.2 \theta^h \). By borrowing up to the limit (with an interest rate \( r = 0 \)), the net expenditures on the asset \( \theta^h \) from the period \( t = 0 \) budget constraint are:

\[
p\theta^h - \frac{1}{1 + r} b^h = p\theta^h - 0.2\theta^h = (p - 0.2) \theta^h. \tag{13.35}
\]

The payout of a household that borrows up to the limit is \( \theta^h - \min \{ b^h, \theta^h \} = (1 - 0.2) \theta^h \) in state \( G \) and \( 0.2\theta^h - \min \{ b^h, 0.2\theta^h \} = 0 \) in state \( B \). Household \( h \) will borrow up to the limit provided that the expected payout exceeds the net expenditures:

\[
h \left( (1 - 0.2) \theta^h \right) + (1 - h) (0) \geq (p - 0.2) \theta^h. \tag{13.36}
\]

The inequality is equivalent to:

\[
h \geq \frac{p - 0.2}{0.8} = h^*. \tag{13.37}
\]

By definition, the households \( h \geq h^* \) are the households that are borrowing. So all borrowers will borrow up to the limit \( b^h = 0.2 \theta^h \).
13.2.10 Verifying zero interest rate

We now will verify that the equilibrium interest rate is \( r = 0 \). Each borrower borrows the amount \( b^h \). The total mass of borrowers is \((1 - 0.686) = 0.314\) as all households \( h \geq 0.686 \) are borrowers. This means that the total amount of borrowing is equal to \( 0.314 \cdot b^h \).

Since total lending equals total borrowing, the total amount lent by all households is equal to \( 0.314 \cdot b^h \). What does this mean about how much is lent by each lender? Since the mass of lenders is 0.686, then the total amount that each will lend is equal to \( \frac{0.314 \cdot b^h}{0.686} \).

The amount borrowed by each household \( h \geq 0.686 \) is equal to

\[
b^h = 0.2 \theta^h = 0.2 \left( \frac{1}{0.314} \right) = 0.637.
\]

So the total amount that each household \( h \leq 0.686 \) is lending is

\[
\frac{0.314 \cdot b^h}{0.686} = 0.292.
\]

Each household initially begins with 1 unit of endowment and 1 unit of the asset, so they have more than enough endowment to cover this lending. Competition among the lenders will drive the interest rate down to \( r = 0 \). The lenders are not impatient and have plenty of endowment to lend. If one lender charged an interest rate \( r = 0.01 \), all other lenders could swoop in and charge an interest rate \( r = 0.005 \). These lenders with the lower interest rate are still making profit and all borrowers would choose the lower interest rate. This undercutting continues until the equilibrium interest rate is exactly \( r = 0 \).

13.3 2-period model: Borrowing contracts

13.3.1 Sneak peek

Summary

As with reality, the model will now include an infinite number of borrowing contracts that can, in principle, be traded. Each borrowing contract differs according to the amount of collateral required for each unit of borrowing. Based upon the amount of collateral, the interest rate can be determined. It is possible for borrowers and lenders to meet on the market and trade any one of these infinite number of borrowing contracts. The contracts
actually traded in equilibrium will be those such that both the borrowers and lenders agree to the trade.

What we find is that it is optimal for both the borrowers and the lenders to trade the 'no-default' contract, which was introduced in the previous section. This is a common feature of leverage cycle models for the following reason: borrowing contracts with lower collateral requirements must also have higher interest rates in order to protect lenders against the possibility of default. These higher interest rates punish the borrowers in the good states when they do not default (as they have to pay a higher interest rate) and do not punish them in the bad states in which default occurs (they default and don’t pay any interest). These are the exact opposite incentives that would entice borrowers to use such contracts.

**Notation**

The variables to be introduced in this section are given in the following table:

- $j$: the amount that a borrower promises to repay
- $q_j$: the price for a borrowing contract with promised amount $j$
- $r_j$: the interest rate for borrowing contract $j$, where $q_j = \frac{j}{1+r_j}$
- $b_j^h$: amount that household $h$ borrows using contract $j$

**Main takeaways**

After completing this section, you will be able to answer the following questions:

- If all borrowing contracts are available in the market, which one(s) will actually be traded in equilibrium?
- Can we find prices (interest rates) for those contracts that are not traded in equilibrium?
- How can the model be solved for any initial endowment, any bad state dividend, and any range of household beliefs?

**13.3.2 Borrowing contracts**

If a household borrows 0.5 units and secures it with collateral equal to 1 house, then such a contract is proportional to one in which the household borrows 1 unit and secures it with
2 houses. Given that these collateral contracts can be scaled, we normalize the collateral amount to be 1 unit of the asset.

The borrowing contract $j$ is characterized by an asset price $q_j$ and a promise to repay $j$. The collateral requirement is 1 unit of the asset for each contract. The previous section considered a case with $j = 0.2$ only. This section considers all values $j \geq 0$.

The interest rate is specified for a contract such that the promised repayment is equal to 1. If the promise to repay is $j$ and the asset price for this promise is $q_j$, then the interest rate on this contact is given by:

$$1 + r_j = \frac{j}{q_j}. \quad (13.40)$$

It is equivalent to refer to the incentives for borrowing contracts with either the asset price $q_j$ or the interest rate $r_j$.

Denote the number of units of borrowing contract $j$ held by household $h$ as $b^h_j$ (recall contract $j$ has a promise to repay the amount $j$). As in the previous section, $b^h_j > 0$ means a household is borrowing using this contract, while $b^h_j < 0$ means a household is saving. For each unit of the borrowing contract, the state $G$ and state $B$ amounts that the borrower has to repay, for each of the 2 contingencies, are given in the table below:

<table>
<thead>
<tr>
<th>repay loan</th>
<th>forfeit collateral</th>
</tr>
</thead>
<tbody>
<tr>
<td>state G</td>
<td>$j$</td>
</tr>
<tr>
<td>state B</td>
<td>$j$</td>
</tr>
</tbody>
</table>

The amount repaid by the borrower is therefore equal to $b^h_j \min \{1, j\}$ in state $G$ and $b^h_j \min \{0.2, j\}$ in state $B$.

### 13.3.3 Updating the model

The period $t = 0$ budget constraint for the households is given by:

$$c^h(0) + s^h + p\theta^h \leq 1 + p + \int_{j \geq 0} q_j b^h_j dj. \quad (13.41)$$

Recall that for any contract $j$, each unit borrowed must be supported with one unit of the asset as collateral:

$$\int_{j \geq 0} \max \{b^h_j, 0\} dj \leq \theta^h. \quad (13.42)$$

Given the possibility for default, the budget constraints in the two states in period $t = 1$
are:

\[
\begin{align*}
\ell^h (G) & \leq s^h + \theta^h - \int_{j \geq 0} b^h_j \min \{1, j\} dj. \\
\ell^h (B) & \leq s^h + 0.2 \theta^h - \int_{j \geq 0} b^h_j \min \{0.2, j\} dj.
\end{align*}
\] (13.43)

The market clearing conditions for the commodities and the asset are identical to what we have previously introduced. The market clearing conditions for the borrowing contracts are such that \( \int_0^1 b^h_j dh = 0 \) for all possible values of \( j \).

### 13.3.4 Interest rates for all borrowing contracts

In this setting, there are going to be many potential borrowing contracts that can be traded. In equilibrium, only one such borrowing contract will actually be traded. That borrowing contract is \( j = 0.2 \). The equilibrium is identical to what we found previously in which the asset price is equal to \( p = 0.749 \) and all households \( h \geq h^* = 0.686 \) are purchasing the asset. Even though the borrowing contract for \( j = 0.3 \) isn’t traded in equilibrium, we can still find its equilibrium price (and same for all other \( j \neq 0.2 \)).

For the contract \( j = 0.2 \), we have previously shown that the interest rate is equal to \( r = 0 \). Therefore, the asset price is \( q_{0.2} = \frac{j}{1 + r} = \frac{0.2}{1} = 0.2 \). There is no risk in the ‘no-default’ loan.

For the contract \( j = 0.3 \), the same one unit of collateral now corresponds to \( j = 0.3 \) promised repayment. This is a lower collateral requirement. By requiring lower collateral, the lenders must receive a higher interest rate.

The marginal buyer \( h^* \) is indifferent between purchasing and selling the asset. That same household is also indifferent between borrowing and saving using any of the borrowing contracts. For the contract \( j = 0.3 \), the value assigned by the marginal buyer \( h^* = 0.686 \) is:

\[
\begin{align*}
q_j & = h^* \min \{1, j\} + (1 - h^*) \min \{0.2, j\} \\
& = (0.686) \min \{1, 0.3\} + (1 - 0.686) \min \{0.2, 0.3\} = 0.269.
\end{align*}
\]

The interest rate formula dictates

\[
1 + r_{0.3} = \frac{j}{q_{0.3}} = \frac{0.3}{0.269} = 1.115,
\] (13.44)

so the interest rate is equal to \( r_{0.3} = 11.5\% \).
13.3.5 Only one borrowing contract traded

The asset returns are defined as

\[ \text{return} = \frac{\text{payout}}{\text{price}}. \]

The returns for contract \( j = 0.2 \) and contract \( j = 0.3 \) are given in the following table, where we recall that \( q_{0.3} = 0.269 \):

<table>
<thead>
<tr>
<th>State</th>
<th>( \frac{\min{1,j}}{q_{0.2}} )</th>
<th>( \frac{\min{0.2,j}}{q_{0.2}} )</th>
<th>( \frac{\min{1,j}}{q_{0.3}} )</th>
<th>( \frac{\min{0.2,j}}{q_{0.3}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G )</td>
<td>0.2</td>
<td>0.2</td>
<td>1</td>
<td>0.743</td>
</tr>
<tr>
<td>( B )</td>
<td>0.2</td>
<td>0.2</td>
<td>0.3</td>
<td>1.115</td>
</tr>
</tbody>
</table>

Borrowers are optimistic households with probabilities \( h \geq h^* \). As borrowers, they seek to minimize the returns (what they owe) from borrowing contracts. As optimists, they believe that state \( G \) is very likely to occur. So borrowers want low returns in state \( G \) and don’t mind high returns in state \( B \). This rules out contract \( j = 0.3 \) for borrowers.

Lenders are pessimistic households with probabilities \( h \leq h^* \). As lenders, they seek to maximize the returns (what they receive) from borrowing contracts. As pessimists, they believe that state \( B \) is very likely to occur. So lenders want high returns in state \( B \) and don’t mind low returns in state \( G \). This rules out contract \( j = 0.3 \) for lenders.

Both borrowers and lenders prefer contract \( j = 0.2 \) to all other contracts with \( j > 0.2 \).

For the contract \( j = 0.1 \), the asset price is \( q_{0.1} = 0.1 \). Borrowers will always repay the promise 0.1 as it is lower than the value of collateral. This is a risk-free contract with a very high collateral requirement. The returns for contract \( j = 0.2 \) and contract \( j = 0.1 \) are identically 1 in both states. With contract \( j = 0.1 \), 1 unit of collateral allows for 0.1 units of borrowing. This is half as much borrowing as permitted by contract \( j = 0.2 \), for the same 1 unit of collateral. For this reason, borrowers strictly prefer contract \( j = 0.2 \).

Borrowers prefer contract \( j = 0.2 \) to all other contracts with \( j < 0.2 \).

The only contract that both agree to trade is contract \( j = 0.2 \).

13.3.6 Example: General parameter values

The parameters of the model are the dividend payouts and the initial household endowment. Denote the dividend payouts as \( d(G) \) and \( d(B) \), respectively. As a normalization, we set \( d(G) = 1 \). Denote the initial household endowment as \( E \). We have previously considered the values \( d(B) = 0.2 \) and \( E = 1 \).
Having seen that the only borrowing contract traded in equilibrium is such that the promised repayment is exactly equal to the dividend in state $B$ (the lowest dividend), we can collect the steps to solve for an equilibrium:

1. The first equilibrium equation is the cutoff rule. Equation (13.20) for the general case is given by:

   $$ p = h^* + (1 - h^*)d(B). \quad (13.45) $$

2. The second equilibrium equation is the market clearing condition. Households $h \leq h^*$ have asset holdings $\theta^h = 0$. The budget constraint for households $h \geq h^*$ is

   $$ p\theta^h = E + p + d(B)\theta^h. \quad (13.46) $$

Subtract $d(B)\theta^h$ from both sides:

   $$ (p - d(B))\theta^h = E + p. \quad (13.47) $$

Integrate the budget constraint over all households $h \geq h^*$ and use the market clearing condition $\int_{h^*}^1 \theta^hdh = 1$:

   $$ p - d(B) = (1 - h^*) (E + p). \quad (13.48) $$

The two equations (13.45) and (13.48) can then be solved for the equilibrium variables $(p, h^*)$ as a function of the parameters $(d(B), E)$. Equilibrium leverage, loan to value, and margin are defined by:

   $$ \text{Leverage} = \frac{p}{p - d(B)}. \quad (13.49) $$

   $$ \text{Loan to value} = \frac{d(B)}{p}. \quad (13.50) $$

   $$ \text{Margin} = \frac{p - d(B)}{p}. \quad (13.51) $$

### 13.3.7 Example: Restricted household beliefs

This example generalizes the model by allowing the household beliefs to be uniformly distributed over the interval $[Lo, Hi]$. Since the probabilities must be between 0 and 1, then $0 \leq Lo < Hi \leq 1$. There remains, as above, a unit mass of households. The parameters of the model are the dividend payouts, the initial household endowment, and the range for
13.3. 2-PERIOD MODEL: BORROWING CONTRACTS

household beliefs. The economy previously considered in this chapter specifies \( d(B) = 0.2, \ E = 1, \ Lo = 0, \) and \( Hi = 1. \) The equilibrium steps are repeated:

1. The first equilibrium equation is the cutoff rule. Equation (13.20) for the general case remains unchanged:
   \[
   p = h^* + (1 - h^*)d(B). \tag{13.52}
   \]

2. The second equilibrium equation is the market clearing condition. Households \( h \leq h^* \) have asset holdings \( \theta^h = 0. \) The budget constraint for households \( h \geq h^* \) is
   \[
   p\theta^h = E + p + d(B)\theta^h. \tag{13.53}
   \]
   Subtract \( d(B)\theta^h \) from both sides:
   \[
   (p - d(B))\theta^h = E + p. \tag{13.54}
   \]

Integrate the budget constraint over all households \( h \geq h^* \) and use the market clearing condition \( \int_{h^*}^{Hi} \theta^h dh = 1 : \)

\[
 p - d(B) = \frac{Hi - h^*}{Hi - Lo} (E + p). \tag{13.55}
\]

Recall that all households \( h \geq h^* \) have identical income \( E + p, \) so the total income for all households \( h \geq h^* \) is equal to the income \( E + p \) multiplied by the fraction of households \( h \geq h^*. \) The fraction of households \( h \geq h^* \) is equal to \( \frac{Hi - h^*}{Hi - Lo}. \) Consider that when \( Lo = 0 \) and \( Hi = 1, \) the fraction of households is \( 1 - h^*. \)

To be completely thorough, the integral of equation (13.54) for \( h \geq h^* \) is:

\[
\int_{h^*}^{Hi} (p - d(B))\theta^h f(h) dh = \int_{h^*}^{Hi} (E + p) f(h) dh, \tag{13.56}
\]

where \( f(h) \) is the probability density function (pdf) for the uniform distribution \( Unif[Lo, Hi]. \)

The asset market clearing condition is \( \int_{h^*}^{Hi} \theta^h f(h) dh = 1. \) For the uniform distribution \( Unif[Lo, Hi], \) the pdf is \( f(h) = \frac{1}{Hi - Lo}. \) When \( Lo = 0 \) and \( Hi = 1, \) the pdf is simply
\[ f(h) = 1. \] The right-hand side of (13.56) is equal to:

\[
\int_{h^*}^{H_i} (E + p) \frac{1}{H_i - Lo} \, dh = \frac{E + p}{H_i - Lo} \int_{h^*}^{H_i} \, dh = \frac{E + p}{H_i - Lo} (H_i - h^*). \tag{13.57}
\]

This is exactly what we had found above using the "fraction of households" approach.

The two equations (13.52) and (13.55) can then be solved for the equilibrium variables \((p, h^*)\) as a function of the parameters \((d(B), E, Lo, Hi)\). From (13.52),

\[
p - d(B) = h^* (1 - d(B)) \tag{13.58}
\]

\[
p = d(B) + h^* (1 - d(B)). \tag{13.59}
\]

Inserting these into (13.55):

\[
h^* (1 - d(B)) = \frac{H_i - h^*}{H_i - Lo} (E + d(B) + h^* (1 - d(B))). \tag{13.60}
\]

This can be simplified to a quadratic equation in terms of the variable \(h^*\):

\[(h^*)^2 (1 - d(B)) + h^* (E + d(B) - Lo (1 - d(B))) - Hi (E + d(B)) = 0. \tag{13.61}
\]

From there, the asset price \(p\) can be found.

There are 2 real solutions to the quadratic equation, but we only take the larger value as we require that \(0 \leq h^* \leq 1\).

It is easy to verify that both \(h^*\) and \(p\) are strictly increasing in all 4 parameters: \(d(B), E, Lo,\) and \(Hi\). An increase in the \(d(B)\) means that households can borrow more as lenders are guaranteed to get back at least \(d(B)\) in all states. An increase in \(E\) means that borrowers have more initial income to invest. An increase in either \(Lo\) or \(Hi\) means that the pool of investors is more optimistic, on average, about the asset dividend. Any of these 4 increases will lead to an increase in both \(h^*\) and \(p\).
13.4 3-period model: The leverage cycle

13.4.1 Sneak peek

Summary

The 3-period model is an extension of the 2-period model and now agents can invest and borrow in both periods $t = 0$ and $t = 1$. This allows agents to adjust their portfolio in the middle period $t = 1$ after a partial realization of uncertainty. In this middle period $t = 1$, investors receive a signal about the dividend value in the final period $t = 2$. Since the future dividend value determines the current price of the asset (and thus its value as collateral), this signal is very important. Based upon this signal, investors can adjust their portfolios and enter into new borrowing arrangements.

When the signal is good, meaning that a high dividend is more likely, then borrowing will become looser and less collateral is required for every unit of borrowing. When the signal is bad, meaning that a low dividend is more likely, then borrowing will become tighter and more collateral for every unit of borrowing.

The fundamentals in this model are the probabilities of a high and low dividend. It is these beliefs that drive the asset trade and determine the asset price. The resulting financial impact of a signal can be larger than the change in the value of the expected dividend caused by the probability change. The financial impact is usually measured in terms of the change in asset price. The reason that the change in probabilities (fundamentals) is amplified into a financial crisis is the leverage cycle.

The leverage cycle begins with the economy in a vulnerable state due to excess leverage (or excess borrowing) by the optimistic households. A change in the fundamentals means that a bad signal is realized and the probability of a low dividend has increased. Households respond, given the initial state of excess leverage, by over-correcting with excess deleveraging. They will abandon their borrowing contracts, often at a loss (bankruptcy). This leads to a drop in the asset price, whose scale is much larger than the change in the probabilities.
Notation

The variables to be introduced in this section are given in the following table:

- $c^h(\sigma)$ consumption in state $\sigma \in \{0, G, B, GG, GB, BG, BB\}$
- $e^h(\sigma)$ endowment in state $\sigma$, the only nonzero endowment is $e^h(0) = 1$
- $s^h(\sigma)$ storage chosen in state $\sigma \in \{0, G, B\}$
- $\theta^h(\sigma)$ asset chosen in state $\sigma \in \{0, G, B\}$
- $p(\sigma)$ asset price in state $\sigma \in \{0, G, B\}$
- $b_j^h(\sigma)$ borrowing contract $j$ chosen in state $\sigma \in \{0, G, B\}$
- $q_j(\sigma)$ price for borrowing contract $j$ in state $\sigma \in \{0, G, B\}$
- $r_j(\sigma)$ interest rate for borrowing contract $j$ in state $\sigma \in \{0, G, B\}$

Main takeaways

After completing this section, you will be able to answer the following questions:

- What is the value of an asset in the periods before it pays out a dividend?
- How is the system of equations set up to solve for the leverage cycle?
- How is the asset price crash decomposed into fundamental and non-fundamental components?

13.4.2 Model setup

The time periods are $t = 0$, $t = 1$, and $t = 2$. In both periods $t = 1$ and $t = 2$, a shock occurs, either good ($G$) or bad ($B$). In period $t = 1$, there are 2 possible histories of shocks: $G$ and $B$. In period $t = 2$, there are 4 possible histories of shocks (where the shock in period $t = 1$ is listed first and the shock in period $t = 2$ is listed second): $GG$, $GB$, $BG$, and $BB$. For each of the histories in period $t = 2$, there is a dividend value for the one asset in the economy. Denote the dividend values as $d(GG)$, $d(GB)$, $d(BG)$, and $d(BB)$.

A standard economy will have dividend values given by $(d(GG), d(GB), d(BG), d(BB)) = (1, 1, 1, 0.2)$. As a normalization, we always set $d(GG) = 1$. As a matter of consistency, we impose that $d(GG) \geq d(GB) \geq d(BB)$ and $d(GG) \geq d(BG) \geq d(BB)$.
13.4.3 Heterogeneous beliefs

Household $h$ believes that each shock $G$ will occur with probability $h$ and that the shock realizations are independent and identically distributed. This means that household $h$ assigns the following probabilities to dividend payouts:

- Dividend $d(GG)$ with probability $h^2$(13.62)
- Dividend $d(GB)$ with probability $h(1-h)$
- Dividend $d(BG)$ with probability $h(1-h)$
- Dividend $d(BB)$ with probability $(1-h)^2$

Households have heterogeneous beliefs as they disagree about what probability to assign to that distribution. The probability assigned to each shock and the dividends are illustrated in Figure 13.4.1.

The period $t = 0$ state is labeled 0, the period $t = 1$ states are labeled $G$ and $B$, and the period $t = 2$ states are labeled $GG$, $GB$, $BG$, and $BB$. The consumptions are given in the following table:

<table>
<thead>
<tr>
<th>state</th>
<th>0</th>
<th>G</th>
<th>B</th>
<th>GG</th>
<th>GB</th>
<th>BG</th>
<th>BB</th>
</tr>
</thead>
<tbody>
<tr>
<td>consumption</td>
<td>$c^h(0)$</td>
<td>$c^h(G)$</td>
<td>$c^h(B)$</td>
<td>$c^h(GG)$</td>
<td>$c^h(GB)$</td>
<td>$c^h(BG)$</td>
<td>$c^h(BB)$</td>
</tr>
</tbody>
</table>

Households are risk-neutral, meaning that their utility function is the expected value of consumption. The probabilities are different and governed by the heterogeneous beliefs previously described. For household $h$, the utility function is given by:

\[
u^h(c^h(0), c^h(G), c^h(B), c^h(GG), c^h(GB), c^h(BG), c^h(BB)) = c^h(0) + h c^h(G) + (1 - h) c^h(B) + h^2 c^h(GG) + h(1 - h) c^h(GB) \\
+ h(1 - h) c^h(BG) + (1 - h)^2 c^h(BB).
\]

As in the 2-period model, households have an endowment in state 0 of $E$ units of the commodity and 1 unit of the asset. Households have no endowments (of either commodity or asset) in future states. In our standard economy, $E = 1$. 
13.4.4 Assets and storage

There is a unit mass of households, with each initially holding 1 unit of the asset. This means that the total initial amount of the asset equals 1. The asset is traded in states 0, G, and B, and in all of these states, the total amount of the asset after trading must be equal to 1.

The asset is traded in 3 different states: 0, G, and B. The asset holdings for household \( h \) in each of these 3 states is denoted by \( \theta^h(0) \geq 0 \), \( \theta^h(G) \geq 0 \), and \( \theta^h(B) \geq 0 \). We require that the asset positions cannot be negative, meaning that it is not possible to short-sell the asset. The asset holdings in each of those 3 states is influenced by the asset prices in each of those 3 states and these asset prices are denoted by \( p(0) \), \( p(G) \), and \( p(B) \).

In addition to asset holdings, households have one additional financial instrument and one storage opportunity. In the 3 states 0, G, and B, households can choose to store the commodity into the subsequent period. Storage in state 0 is denoted \( s^h(0) \geq 0 \) and allows a transfers of \( s^h(0) \) units of the commodity from state 0 into both states G and B (these are the states that occur in the period immediately following state 0). Notice that storage must be nonnegative. Storage in state G is denoted \( s^h(G) \geq 0 \) and allows a transfers of \( s^h(G) \) units of the commodity from state G into both states GG and GB (these are the states that occur in period \( t = 2 \) that immediately follow state G). Similarly for state B, storage transfers \( s^h(B) \) units of the commodity from state B into both states BG and BB.

The other option is the financial instrument, which is collateralized borrowing. The asset will continue to serve as collateral for all borrowing. Collateralized borrowing takes place in states 0, G, and B. Borrowing contracts are indexed by \( j \) (the promised amount) and the collateral requirement is one unit of the asset. The contract \( j \) borrowing is denoted by \( b^h_j(0) \) in state 0, by \( b^h_j(G) \) in state G, and by \( b^h_j(B) \) in state B. The prices for the borrowing contract \( j \) are denoted by \( q_j(0) \) in state 0, by \( q_j(G) \) in state G, and by \( q_j(B) \) in state B.

13.4.5 Household budget constraint and collateral constraints

Budget constraints in the model are always of the form:

\[
\text{expenditures} \leq \text{income},
\]

where the outflows for expenditures are consumption, storage, and asset holding and the sources of income are endowment, storage, asset value, asset dividend, and net borrowing.
In state 0, there is no storage from the previous period, no borrowing from the previous period, and no asset dividend (households are endowed with 1 unit of the asset to begin with, so their income includes the asset value). The state 0 budget constraint for the households is given by:

\[ c^h(0) + s^h(0) + p(0)\theta^h(0) \leq E + p(0) + \int_{j \geq 0} q_j(0)b^h_j(0)dj. \]  

(13.65)

The income from net borrowing is given by \( \int_{j \geq 0} q_j(0)b^h_j(0)dj \). The collateral constraint for state 0 is given by:

\[ \int_{j \geq 0} \max \{ b^h_j(0), 0 \} \, dj \leq \theta^h(0). \]  

(13.66)

This states that each unit of borrowing contract is supported by 1 unit of collateral (1 unit of the asset).

In state \( G \), there is no endowment and no asset dividend. Net borrowing is equal to

\[ -\int_{j \geq 0} b^h_j(0) \min \{ p(G), j \} \, dj + \int_{j \geq 0} q_j(G)b^h_j(G)\, dj. \]  

(13.67)

The first term is the net repayment from past borrowing and the second term is the net income from current borrowing. Consider the past borrowing term. Borrowers with \( b^h_j(0) > 0 \) have a binary decision in state \( G \): either repay the promised amount \( j \) or default and forfeit 1 unit of the asset, which is valued at \( p(G) \). Borrowers will choose the minimum and receive payout \( \min \{ p(G), j \} \). Lenders know just as well as borrowers the repayment decision, so lenders \( b^h_j(0) < 0 \) receive the same payout for each unit lent. The budget constraint in state \( G \) is:

\[ c^h(G) + s^h(G) + p(G)\theta^h(G) \leq s^h(0) + p(G)\theta^h(0) - \int_{j \geq 0} b^h_j(0) \min \{ p(G), j \} \, dj + \int_{j \geq 0} q_j(G)b^h_j(G)dj. \]  

(13.68)

The collateral constraint in state \( G \) is given by:

\[ \int_{j \geq 0} \max \{ b^h_j(G), 0 \} \, dj \leq \theta^h(G). \]  

(13.69)

This takes the same form as the state 0 collateral constraint and requires that each unit of borrowing contract is supported by 1 unit the asset.
In similar fashion, the budget constraint for state $B$ can be written:

$$c^h(B) + s^h(B) + p(B)\theta^h(B) \leq s^h(0) + p(B)\theta^h(0) - \int_{j \geq 0} b^h_j(G) \min \{p(B), j\} \, dj + \int_{j \geq 0} q_j(B) b^h_j(B) \, dj.$$  (13.70)

The collateral constraint in state $B$ is given by:

$$\int_{j \geq 0} \max \{b^h_j(B), 0\} \, dj \leq \theta^h(B).$$

The final 4 states are states $GG$, $GB$, $BG$, and $BB$ and occur in period $t = 2$. Since period $t = 2$ is the final period, asset markets and borrowing contract markets do not open. Thus, there are no collateral constraints in these states, but there are budget constraints. The budget constraints in each of these states do not contain storage or asset expenditures, there is no endowment, and assets have no value (the lack of a future period means that the asset price is zero). Assets do have dividend payouts and net borrowing is equal to the net repayment from previous period borrowing.

In state $GG$, the dividend payout is equal to $d(GG)$ (which we normalized to $d(GG) = 1$). Net borrowing is equal to

$$-\int_{j \geq 0} b^h_j(G) \min \{d(GG), j\} \, dj.$$  (13.71)

This term is the net repayment from past borrowing. Borrowers with $b^h_j(G) > 0$ have a binary decision in state $GG$: either repay the promised amount $j$ or default and forfeit 1 unit of the asset, which is valued at $d(GG)$ (assets in period $t = 2$ only have dividend value). Borrowers will choose the minimum $\min \{d(GG), j\}$. Lenders know just as well as borrowers that this is the decision that will be made, so lenders $b^h_j(G) < 0$ receive the same payout for each unit lent. The budget constraint in state $GG$ is:

$$c^h(GG) \leq s^h(G) + d(GG)\theta^h(G) - \int_{j \geq 0} b^h_j(G) \min \{d(GG), j\} \, dj.$$  (13.72)

In similar fashion, the budget constraints for states $GB$, $BG$, and $BB$ are given by:

$$c^h(GB) \leq s^h(G) + d(GB)\theta^h(G) - \int_{j \geq 0} b^h_j(G) \min \{d(GB), j\} \, dj.$$  (13.73)
$$c^h(BG) \leq s^h(B) + d(BG)\theta^h(B) - \int_{j \geq 0} b^h_j(B) \min \{d(BG), j\} \, dj.$$  
$$c^h(BB) \leq s^h(B) + d(BB)\theta^h(B) - \int_{j \geq 0} b^h_j(B) \min \{d(BB), j\} \, dj.$$
13.4. 3-PERIOD MODEL: THE LEVERAGE CYCLE

13.4.6 Market clearing conditions

The market clearing conditions are less important for our analysis, but we need to write them down in order to properly close the model. The possible markets that can occur in any state are the commodity market (consumption, storage, and endowment of commodity), the asset market, and the market for each collateralized borrowing contract $j$. State 0 contains all of these markets and the market clearing conditions in state 0 are given by:

$$\int_0^1 (c^h(0) + s^h(0)) \, dh = E. \tag{13.74}$$

$$\int_0^1 \theta^h(0) \, dh = 1.$$  

$$\int_0^1 b^h_j(0) \, dh = 0 \text{ for all } j \geq 0.$$  

Since each household has $E$ units of the commodity as endowment and there are a unit mass of households, then the total amount of the commodity is $E$ units. This total must be split between total household consumption and storage. The total amount of asset equals 1 unit. Finally, the borrowing contracts are zero net sum, meaning that the total amount of each borrowing contract $j$ must be equal to 0. This states that each unit of borrowing must be offset by an equal unit of lending.

State $G$ contains all of the markets and the market clearing conditions in state $G$ are given by:

$$\int_0^1 (c^h(G) + s^h(G)) \, dh = \int_0^1 s^h(0) \, dh. \tag{13.75}$$

$$\int_0^1 \theta^h(G) \, dh = 1.$$  

$$\int_0^1 b^h_j(G) \, dh = 0 \text{ for all } j \geq 0.$$  

Here, since there is no commodity endowment in state $G$, the total amount of commodity equals the total amount stored from the previous period $\int_0^1 s^h(0) \, dh$. This total must be split between total household consumption and storage. The total amount of asset equals 1 unit. Finally, the borrowing contracts are zero net sum, meaning that the total amount of each borrowing contract $j$ must be equal to 0. In similar fashion, the market clearing conditions
in state $B$ are given by:

\[
\int_0^1 (c^h(B) + s^h(B)) \, dh = \int_0^1 s^h(0) \, dh. \tag{13.76}
\]

\[
\int_0^1 \theta^h(B) \, dh = 1.
\]

\[
\int_0^1 b_j^h(B) \, dh = 0 \text{ for all } j \geq 0.
\]

States $GG$, $GB$, $BG$, and $BB$ do not contain financial markets (a market for the asset or markets for the collateralized borrowing contracts). The only markets are the commodity markets. The market clearing condition for the commodity market in state $GG$ is given by:

\[
\int_0^1 c^h(GG) \, dh = d(GG) + \int_0^1 s^h(G) \, dh. \tag{13.77}
\]

The total amount of storage from the previous period is equal to $\int_0^1 s^h(G) \, dh$. The asset is a real asset, meaning that the dividend payout is in units of the commodity and adds to the total amount of commodity. Since the total asset holdings are $\int_0^1 \theta^h(G) \, dh = 1$, then the total dividend payouts are $\int_0^1 d(GG) \theta^h(G) \, dh = d(GG)$. The total amount of commodity is then equal to $d(GG) + \int_0^1 s^h(G) \, dh$, and this must be equal to the total household consumption (with no future periods, it is pointless for households to engage in further storage).

In similar fashion, the market clearing conditions for the commodity markets in states $GB$, $BG$, and $BB$ are given by:

\[
\int_0^1 c^h(GB) \, dh = d(GB) + \int_0^1 s^h(G) \, dh. \tag{13.78}
\]

\[
\int_0^1 c^h(BG) \, dh = d(BG) + \int_0^1 s^h(B) \, dh.
\]

\[
\int_0^1 c^h(BB) \, dh = d(BB) + \int_0^1 s^h(B) \, dh.
\]
13.4. 3-PERIOD MODEL: THE LEVERAGE CYCLE

13.4.7 Solving the model: Initial facts

We will solve the model for an economy in which the belief variable $h \sim \text{Unif}[0, 1]$. Eventually, when we plug values in for a standard economy, we will consider dividend values $(d(GG), d(GB), d(BG), d(BB)) = (1, 1, 1, 0.2)$ and endowment $E = 1$. In an example in the final subsection, we will consider a procedure to solve for economies with more general belief structures.

Since $d(GG) \geq d(GB)$ and $d(GB) \geq d(BB)$, the price of the asset in state $G$ must be higher than the price of the asset in state $B$:

$$p(G) \geq p(B). \quad (13.79)$$

State $G$ is our good signal state, signifying that there is an increased chance of a high dividend in the following period, while state $B$ is our bad signal state, signifying that there is an increased chance of a low dividend in the following period.

In state 0, the borrowing conditions are determined by the anticipated payout of borrowing contract $j$, which is $\min \{p(G), j\}$ in state $G$ and $\min \{p(B), j\}$ in state $B$. Just as in the 2-period model, the contract that is actually traded will be such that:

$$j = \min \{p(G), p(B)\}.$$  

Recall that $p(G) \geq p(B)$. Therefore, the contract $j = p(B)$ is the one traded. This is a no-default loan, which is the largest amount of borrowing (the largest value for $j$) that can take place such that the loan is always repaid in both state $G$ ($j \leq p(G)$) and state $B$ ($j \leq p(B)$). The interest rate for this contract is $r_j = 0$. Using the relation $1 + r_j = \frac{j}{q_j}$, the price of this contract is $q_j = p(B)$.

In state $G$, the borrowing conditions are determined by the anticipated payout of borrowing contract $j$, which is $\min \{d(GG), j\}$ in state $GG$ and $\min \{d(GB), j\}$ in state $GB$. As above, the contract that is actually traded will be such that:

$$j = \min \{d(GG), d(GB)\}.$$  

Recall that $d(GG) \geq d(GB)$ (where $d(GG) = 1$ is the normalization). Therefore, the contract $j = d(GB)$ is the one traded. This is a no-default loan, which is the largest amount of borrowing (the largest value for $j$) that can take place such that the loan is always repaid in both state $GG$ ($j \leq d(GG)$) and state $GB$ ($j \leq d(GB)$). The interest rate for this contract
is \( r_j = 0 \). Using the relation \( 1 + r_j = \frac{j}{q_j} \), then the price of this contract is \( q_j = d(GB) \).

In state \( B \), using the exact same logic, the traded contract is \( j = d(BB) \) and the price of the contract is \( q_j = d(BB) \).

### 13.4.8 Solving the model: Cutoff rules

If you recall in the 2-period model, we talked about the marginal buyer \( h^* \). We will adopt the same approach here, but we now need to talk about the marginal buyer in states 0, \( G \), and \( B \). The marginal buyer in state 0 is denoted \( h^*(0) \), the marginal buyer in state \( G \) is denoted \( h^*(G) \), and the marginal buyer in state \( B \) is denoted \( h^*(B) \).

In state 0, the marginal buyer \( h^*(0) \) is indifferent between buying and selling the asset. From state 0, the two states in the subsequent period are \( G \) and \( B \). The asset price \( p(0) \) is equal to the valuation of the asset by the marginal buyer \( h^*(0) \):

\[
p(0) = h^*(0)p(G) + (1 - h^*(0))p(B). \tag{13.80}
\]

The value \( p(G) \) is the payout of the asset in state \( G \) and \( p(B) \) is the payout of the asset in state \( B \).

In state \( G \), the marginal buyer \( h^*(G) \) is indifferent between buying and selling the asset. From state \( G \), the two states in the subsequent period are \( GG \) and \( GB \). The asset price \( p(G) \) is equal to the valuation of the asset by the marginal buyer \( h^*(G) \):

\[
p(G) = h^*(G)d(GG) + (1 - h^*(G))d(GB). \tag{13.81}
\]

The value \( d(GG) \) is the payout of the asset in state \( GG \) and \( d(GB) \) is the payout of the asset in state \( GB \).

In state \( B \), the marginal buyer \( h^*(B) \) is indifferent between buying and selling the asset. The asset price \( p(B) \) is equal to the valuation of the asset by the marginal buyer \( h^*(B) \):

\[
p(B) = h^*(B)d(BG) + (1 - h^*(B))d(BB). \tag{13.82}
\]

These three equations are very similar to the equation (13.20) from the 2-period model.
13.4.9 Solving the model: Budget constraints

State 0

Recall the state 0 budget constraint for households:

\[ c_h(0) + s_h(0) + p(0)\theta_h(0) \leq E + p(0) + \int_{j \geq 0} q_j(0)b_j^h(0) dj. \]  \hspace{1cm} (13.83)

From above, \( j = p(B) \) and \( q_j(0) = p(B) \). The budget constraint is binding as households maximize their utility. The binding budget constraint (given \( j = p(B) \) and \( q_j(0) = p(B) \)) is given by:

\[ c_h(0) + s_h(0) + p(0)\theta_h(0) = E + p(0) + p(B)b_{p(B)}^h(0). \]  \hspace{1cm} (13.84)

In state 0, the households \( h \leq h^*(0) \) sell the asset and set \( \theta_h(0) = 0 \).

In state 0, the households \( h \geq h^*(0) \) buy the asset. These households can borrow up to the limit \( b_{p(B)}^h(0) = \theta_h(0) \). The households set \( c_h(0) = s_h(0) = 0 \) in order to buy as many units of the asset as possible. The budget constraint is updated as:

\[ p(0)\theta_h(0) = E + p(0) + p(B)\theta_h(0). \]  \hspace{1cm} (13.85)

Subtract \( p(B)\theta_h(0) \) from both sides:

\[ (p(0) - p(B))\theta_h(0) = E + p(0). \]  \hspace{1cm} (13.86)

Integrate over all households \( h \geq h^*(0) \):

\[ \int_{h^*(0)}^{1} (p(0) - p(B))\theta_h(0)dh = \int_{h^*(0)}^{1} (E + p(0)) dh. \]  \hspace{1cm} (13.87)

We now use two facts. First, \( \int_{h^*(0)}^{1} \theta^h(0)dh = 1 \) as households \( h \geq h^*(0) \) are the only ones to buy the asset. Second, since \( h \sim Unif [0, 1] \), then \( \int_{h^*(0)}^{1} dh = 1 - h^*(0) \) (further details are found in the 2-period model). Thus, the final version of the equation is:

\[ p(0) - p(B) = (E + p(0))(1 - h^*(0)). \]  \hspace{1cm} (13.88)
State G

Recall the state $G$ budget constraint for households:

$$c^h(G) + s^h(G) + p(G)\theta^h(G)$$

$$\leq s^h(0) + p(G)\theta^h(0) - \int_{j \geq 0} b^h_j(0) \min\{p(G), j\} dj + \int_{j \geq 0} q_j(G)b^h_j(G) dj. \tag{13.89}$$

With the good signal $G$, the marginal buyer must satisfy: $h^*(G) \geq h^*(0)$. The buyers are now even more optimistic than before. Why? Given their investments from state 0, they now have more available income in state $G$ to invest.

In state $G$, the households $h \leq h^*(G)$ sell the asset and set $\theta^h(G) = 0$. Let’s focus on households $h \geq h^*(G)$ as these are the households that buy the asset. Since $h^*(G) \geq h^*(0)$, then these households had previously bought the asset in state 0 and borrowed using collateral to do so. Since $\int_{h^*(0)}^1 \theta^h(0) dh = 1$, then the asset holding for each household $h \geq h^*(G)$ is equal to $\theta^h(0) = \frac{1}{1-h^*(0)}$. The borrowed amount is $b^h_{pl(B)}(0) = \theta^h(0) = \frac{1}{1-h^*(0)}$.

These households set $s^h(0) = 0$. This means that their income is given by:

$$s^h(0) + p(G)\theta^h(0) - \int_{j \geq 0} b^h_j(0) \min\{p(G), j\} dj$$

$$= 0 + p(G) \left( \frac{1}{1-h^*(0)} \right) - p(B) \left( \frac{1}{1-h^*(0)} \right)$$

$$= \frac{p(G) - p(B)}{1-h^*(0)}. \tag{13.90}$$

From above, $j = d(GB)$ and $q_j(G) = d(GB)$. The budget constraint is binding as households maximize their utility. The binding budget constraint (given $j = d(GB)$ and $q_j(G) = d(GB)$) is given by:

$$c^h(G) + s^h(G) + p(G)\theta^h(G) = \frac{p(G) - p(B)}{1-h^*(0)} + d(GB)b^h_{d(GB)}(G). \tag{13.91}$$

In state $G$, the households $h \geq h^*(G)$ buy the asset. These households can borrow up to the limit $b^h_{d(GB)}(G) = \theta^h(G)$. The households set $c^h(G) = s^h(G) = 0$ in order to buy as many units of the asset as possible. The budget constraint is updated as:

$$p(G)\theta^h(G) = \frac{p(G) - p(B)}{1-h^*(0)} + d(GB)\theta^h(G). \tag{13.92}$$
Subtract \( d(GB) \) from both sides:

\[
(p(G) - d(GB)) \theta^h(G) = \frac{p(G) - p(B)}{1 - h^*(0)}.
\]  

(13.93)

Integrate over all households \( h \geq h^*(G) \):

\[
\int_{h^*(G)}^1 (p(G) - d(GB)) \theta^h(G) dh = \int_{h^*(G)}^1 \left( \frac{p(G) - p(B)}{1 - h^*(0)} \right) dh.
\]  

(13.94)

We now use two facts. First, \( \int_{h^*(G)}^1 \theta^h(G) dh = 1 \) as households \( h \geq h^*(G) \) are the only ones to buy the asset. Second, since \( h \sim Unif [0,1] \), then \( \int_{h^*(G)}^1 dh = 1 - h^*(G) \). Thus, the final version of the equation is:

\[
p(G) - d(GB) = (p(G) - p(B)) \frac{1 - h^*(G)}{1 - h^*(0)}.
\]  

(13.95)

State B

Recall the state B budget constraint for households:

\[
c^h(B) + s^h(B) + p(B)\theta^h(B) \\
\leq s^h(0) + p(B)\theta^h(0) - \int_{j \geq 0} b^h_j(0) \min \{p(B), j\} dj + \int_{j \geq 0} q_j(B)b^h_{j}(B) dj.
\]  

(13.96)

With the bad signal \( B \), the marginal buyer must satisfy: \( h^*(0) \geq h^*(B) \). However, not all households \( h \geq h^*(B) \) have income available to invest. Recall in the previous subsection that the households \( h \geq h^*(0) \) set storage \( s^h(0) = 0 \), purchase the asset, and borrow \( b^h_j(0) = \theta^h(0) \) on contract \( j = p(B) \). Consequently, the households \( h \geq h^*(0) \) have income equal to:

\[
s^h(0) + p(G)\theta^h(0) - \int_{j \geq 0} b^h_j(0) \min \{p(G), j\} dj
\]  

(13.97)

\[= 0 + p(B) \left( \frac{1}{1 - h^*(0)} \right) - p(B) \left( \frac{1}{1 - h^*(0)} \right) = 0.
\]

These household \( h \geq h^*(0) \) are bankrupt. Even though they would like to purchase the asset, they have no available income with which to do so. The only way they can borrow is if they have asset to post as collateral, but they cannot buy any asset (zero income).
The new optimistic \( h^*(B) \leq h \leq h^*(0) \) were lenders in state 0. Using the following logic, we can calculate the income for any lender \( h \leq h^*(0) \) in state \( B \). The lenders choose \( \theta^h(0) = 0 \). Total lending must be equal to total borrowing, which was \( \int_{h^*(0)}^1 \theta^h(0)dh = 1 \). Thus, the payout on total lending equal to 1 is given by \( \min \{ p(B), 1 \} = p(B) \). Finally, the lenders choose to store the entire available amount of commodity in state 0 (as they can use this storage to increase income and therefore borrowing in future states). Total storage equals \( E \), which is the total initial endowment of the commodity. The total income across all lenders \( h \leq h^*(0) \) is equal to \( E + p(B) \). Since \( \int_{h^*(0)}^0 dh = h^*(0) \), then the state \( B \) income held by each lender \( h \leq h^*(0) \) is equal to:

\[
s^h(0) + p(B)\theta^h(0) - \int_{j \geq 0} b^h_j(0) \min \{ p(B), j \} dj = \frac{E + p(B)}{h^*(0)}.
\]  

(13.98)

From above, \( j = d(BB) \) and \( q_j(B) = d(BB) \). The budget constraint is binding as households maximize their utility. The binding budget constraint (given \( j = d(BB) \) and \( q_j(B) = d(BB) \)) is given by:

\[
c^h(B) + s^h(B) + p(B)\theta^h(B) = \frac{E + p(B)}{h^*(0)} + d(BB) b^h_{d(BB)}(B).
\]  

(13.99)

In state \( B \), the households \( h^*(B) \leq h \leq h^*(0) \) buy the asset. These households can borrow up to the limit \( b^h_{d(BB)}(B) = \theta^h(B) \). The households set \( c^h(B) = s^h(B) = 0 \) in order to buy as many units of the asset as possible. The budget constraint is updated as:

\[
p(B)\theta^h(B) = \frac{E + p(B)}{h^*(0)} + d(BB) \theta^h(B).
\]  

(13.100)

Subtract \( d(BB) \theta^h(B) \) from both sides:

\[
(p(B) - d(BB)) \theta^h(B) = \frac{E + p(B)}{h^*(0)}.
\]  

(13.101)

Integrate over all households \( h^*(B) \leq h \leq h^*(0) \):

\[
\int_{h^*(B)}^{h^*(0)} (p(B) - d(BB)) \theta^h(B)dh = \int_{h^*(B)}^{h^*(0)} \left( \frac{E + p(B)}{h^*(0)} \right) dh.
\]  

(13.102)
We now use two facts. First, \( \int_{h^*(B)}^{h^*(0)} \theta h(B) dh = 1 \) as households \( h^*(B) \leq h \leq h^*(0) \) are the only ones to buy the asset. Second, since \( h^{-\text{Unif} [0, 1]} \), then \( \int_{h^*(B)}^{h^*(0)} dh = h^*(0) - h^*(B) \). Thus, the final version of the equation is:

\[
p(B) - d(BB) = (E + p(B)) \frac{h^*(0) - h^*(B)}{h^*(0)}. \tag{13.103}
\]

### 13.4.10 Solving the model: 6 equations and 6 unknowns

We have 6 unknowns \((p(0), p(G), p(B), h^*(0), h^*(G), h^*(B))\) and 6 equations: 3 cutoff rules and 3 budget constraint equations.

The following 4-step procedure, together with the Solver application in Excel, is used to solve this system of 6 equations and 6 unknowns.

1. **Choose values for the parameters:** dividend values \((d(GG), d(GB), d(BG), d(BB))\) and endowment \(E\). We will run through an example for a standard economy with dividend values \((d(GG), d(GB), d(BG), d(BB)) = (1, 1, 0.2)\) and endowment \(E = 1\).

2. **It is helpful to understand how the variable are related for this next step.** Since the marginal buyer must be a value for \( h \) somewhere in the set \([0, 1]\), then \( h^*(0) \in [0, 1] \), \( h^*(G) \in [0, 1] \), and \( h^*(B) \in [0, 1] \). Given the normalization \( d(GG) = 1 \), then all prices must satisfy \( p(0) \in [0, 1] \), \( p(G) \in [0, 1] \), and \( p(B) \in [0, 1] \). Given that state \( G \) is the good signal and state \( B \) is the bad signal, then \( h^*(G) \geq h^*(0) \geq h^*(B) \) and \( p(G) \geq p(0) \geq p(B) \).

Make a guess for the values \( h^*(0) \in [0, 1] \), \( h^*(G) \in [0, 1] \), and \( h^*(B) \in [0, 1] \) such that \( h^*(G) \geq h^*(0) \geq h^*(B) \). Given values for these guesses, use the cutoff rule equations (13.80), (13.81), and (13.82) to determine the values for \((p(0), p(G), p(B))\):

\[
p(0) = h^*(0)p(G) + (1 - h^*(0))p(B). \tag{13.104}
\]
\[
p(G) = h^*(G)d(GG) + (1 - h^*(G))d(GB).
\]
\[
p(B) = h^*(B)d(BG) + (1 - h^*(B))d(BB).
\]

3. **We will now use the Solver application in Excel in order to find the appropriate values for \((h^*(0), h^*(G), h^*(B))\) that solve the budget constraint equations (13.88), (13.95),**
and (13.103):

\[ p(0) - p(B) = (E + p(0)) (1 - h^*(0)). \quad (13.105) \]

\[ p(G) - d(GB) = (p(G) - p(B)) \frac{1 - h^*(G)}{1 - h^*(0)}. \]

\[ p(B) - d(BB) = (E + p(B)) \frac{h^*(0) - h^*(B)}{h^*(0)}. \]

In Solver, you will have one function for Set Objective and two constraints for Subject to the Constraints. Both the objective function and the constraints are set equal to the values required so that the above equations (13.88), (13.95), and (13.103) are satisfied. With 3 targets, there must be 3 variables for By Changing Variable Cells. These 3 variables are the guesses for the values \( (h^*(0), h^*(G), h^*(B)) \). Using the default Solving Method, you select Solve and Excel will find the values for \( (h^*(0), h^*(G), h^*(B)) \) such that equations (13.88), (13.95), and (13.103) are satisfied.

4. As a check of the procedure, make sure that (i) \( h^*(0) \in [0, 1] \), \( h^*(G) \in [0, 1] \), and \( h^*(B) \in [0, 1] \), (ii) \( p(0) \in [0, 1] \), \( p(G) \in [0, 1] \), and \( p(B) \in [0, 1] \), (iii) \( h^*(G) \geq h^*(0) \geq h^*(B) \), and (iv) \( p(G) \geq p(0) \geq p(B) \).

An end-of-chapter exercise asks you to solve for the values of

\[(p(0), p(G), p(B), h^*(0), h^*(G), h^*(B))\]  

that satisfy the 6 equations. For the standard economy with dividend values

\[(d(GG), d(GB), d(BG), d(BB)) = (1, 1, 1, 0.2)\]  

and endowment \( E = 1 \), you should find that the asset prices are equal to \( (p(0), p(G), p(B)) = (0.958, 1, 0.690) \).

### 13.4.11 Decomposing the crash

We have found the wreckage from the crash and located the black box. We can identify the 3 principal causes of the crash, and assign weights to each of the causes.


3 principal causes

For the standard economy, the equilibrium asset prices are $p(0) = 0.958$ and $p(B) = 0.690$. When I refer to a crash, I am referring to the drop in the asset price from $p(0) = 0.958$ to $p(B) = 0.690$. This crash is caused by three factors:

1. Bad news

   The bad news comes from a change in fundamentals, since the probability of a low dividend is now higher.

   In state 0, the expected dividend payout of the asset is equal to

   \[ h^2 d(GG) + h(1-h) d(GB) + h(1-h) d(BG) + (1-h)^2 d(BB) \]  
   \[(13.108)\]

   for any household $h$.

   In state $B$, the expected dividend payout of the asset is equal to

   \[ hd(BG) + (1-h) d(BB) \]
   \[(13.109)\]

   for any household $h$. The drop in expected payout is equal to

   \[ h^2 (d(GG) - d(BG)) + h (1-h) (d(GB) - d(BB)) \]  
   \[(13.110)\]

   By assumption, $d(GG) \geq d(BG)$ and $d(GB) \geq d(BB)$, so the drop in expected payout is positive. The drop is largest for household $h = 0.5$, and is equal to $\frac{1}{4} (d(GG) - d(BG) + d(GB))$ for this household.

   And yet the asset price has fallen by the amount $p(0) - p(B)$, which can be evaluated analytically from equations (13.80), (13.81), and (13.82) and is given by the expression:

   \[ h^* (0) h^* (G) d(GG) - h^* (0) h^* (B) d(BG) \]
   \[ + h^* (0) (1-h^* (G)) d(GB) - h^* (0) (1-h^* (B)) d(BB) \]  
   \[(13.111)\]

   The fall in the asset price is larger (and can sometimes be much larger) than the maximum drop in valuation. For the standard economy, the maximum drop in valuation is equal to $\frac{1}{4} (d(GG) - d(BG) + d(GB) - d(BB)) = 0.2$ and yet the financial crash is of size $p(0) - p(B) = 0.268$. This indicates that the crash must be caused by factors other than fundamentals (bad news).
2. Bankruptcy

The buyers $h \geq h^*(0)$ that are leveraged in state 0 go bankrupt in state $B$. These are the optimistic buyers who would otherwise be purchasing the asset and maintaining the high asset price (if they had income with which to purchase the asset).

3. Deleveraging

Leveraging plummets (households deleverage), meaning that less borrowing takes place. The following table displays the leverage measures in states 0 and $B$ for the standard economy:

<table>
<thead>
<tr>
<th>Leverage</th>
<th>State 0</th>
<th>State $B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{p(0)}{p(0)-p(B)}$</td>
<td>$= 3.58$</td>
<td>$\frac{p(B)}{p(B)-d(BB)}$</td>
</tr>
<tr>
<td>Loan to value</td>
<td>$\frac{p(B)}{p(0)}$</td>
<td>$= 72%$</td>
</tr>
<tr>
<td>Margin (haircut)</td>
<td>$\frac{p(0)-p(B)}{p(0)}$</td>
<td>$= 28%$</td>
</tr>
</tbody>
</table>

By all measures, the amount of borrowing has decreased as the amount of collateral required for each unit of borrowing has increased.

Let us determine the relative contribution played by each of these three causes.

**Bad news contribution**

First, we focus on the change in fundamentals. To find the effects of a change in fundamentals, we will find a way to hold the fundamentals fixed and then measure the change.

To hold the fundamentals fixed, we will look at the difference in the price $p(0)$ and the price $p(B)$ when both states 0 and $B$ have identical expected dividend payouts. In the current model, the expected dividend payout in state $B$ is equal to $hd(BG) + (1 - h) d(BB)$. What we need to introduce is a new model in which the expected dividend payout in state 0 is also equal to $hd(BG) + (1 - h) d(BB)$. This means that we need to find new values for the probability of a good shock $G$. We introduce the variable $k$ as the probability of a good shock $G$ for household $h$. With these new probabilities, the expected dividend payout in state 0 is given by:

$$k^2 d(GG) + k (1 - k) d(GB) + k (1 - k) d(BG) + (1 - k)^2 d(BB).$$ (13.112)
For each household $h$, we must find the appropriate value for $k$ such that:

$$k^2 d(GG) + k (1 - k) d(GB) + k (1 - k) d(BG) + (1 - k)^2 d(BB) = hd(BG) + (1 - h) d(BB).$$

(13.113)

For the standard economy with dividend values $(d(GG), d(GB), d(BG), d(BB)) = (1, 1, 1, 0.2)$, this is pretty straightforward. Since $d(GG) = d(GB) = d(BG)$, then the probability assigned to $d(BB)$ in the top line must be equal to the probability assigned to $d(BB)$ in the bottom line:

$$(1 - k)^2 = (1 - h).$$

(13.114)

Solving for $k$ yields $k = 1 - \sqrt{1 - h}$. Recall that $k$ is the probability of a good shock $G$ for household $h$. With these updated probabilities, the new model has the same 3 budget constraint equations (13.88), (13.95), and (13.103), but must have 3 new cutoff rules:

$$p(0) = \left(1 - \sqrt{1 - h^*(0)}\right) p(G) + \sqrt{1 - h^*(0)} p(B).$$

(13.115)

$$p(G) = \left(1 - \sqrt{1 - h^*(G)}\right) d(GG) + \sqrt{1 - h^*(G)} d(GB).$$

$$p(B) = \left(1 - \sqrt{1 - h^*(B)}\right) d(BG) + \sqrt{1 - h^*(B)} d(BB).$$

An end-of-chapter exercise asks you to solve for the values of

$$(p(0), p(G), p(B), h^*(0), h^*(G), h^*(B))$$

that satisfy the 6 equations of the new model. You should find that the new state 0 price is $p(0) = 0.816$. This value is denoted ‘New $p(0)$.’

Holding fundamentals fixed (expected dividend at state 0 in new model is equal to the expected dividend at state $B$ in the original model), the size of the crash is equal to

$$\text{New } p(0) - \text{Original } p(B) = 0.816 - 0.690 = 0.126.$$ 

(13.117)

The total size of the financial crash was found to be:

$$\text{Original } p(0) - \text{Original } p(B) = 0.958 - 0.690 = 0.268,$$

(13.118)

so the fraction $\frac{0.126}{0.268} = 47\%$ is caused by non-fundamentals. The remaining fraction must be
caused by fundamentals (i.e., bad news). Therefore, the component of the financial crash that is attributed to the bad news is equal to:

\[
\text{Bad News Fraction} = 1 - \frac{\text{New } p(0) - \text{Original } p(B)}{\text{Original } p(0) - \text{Original } p(B)} = 1 - \frac{0.126}{0.268} = 53\%.
\]  

**Bankruptcy contribution**

Second, we focus on the effect of bankruptcy. The price in state B is lower than it should be due to the effects of bankruptcy. The bankrupt households \( h \geq h^*(0) \) are those that are leveraged in state 0. Because of bankruptcy, they are not able to purchase the asset in state B.

Compare the equilibrium in this 3-period model to the equilibrium from the 2-period model. We can view the 2-period model as starting the economy in state B. In both cases, the probability of the low dividend is equal to \( 1 - h \). The difference is that in the 2-period model, the optimistic households are not bankrupt (there is no prior period for borrowing). The asset price (from the 2-period model) for \((d(G), d(B)) = (1, 0.2)\) and \( E = 1 \) (the comparable form of the standard economy) was found to be \( p = 0.749 \). The asset price in state B of the 3-period model is equal to \( p(B) = 0.690 \). Thus, the fraction of the total change resulting from bankruptcy is equal to

\[
\text{Bankruptcy Fraction} = \frac{0.749 - 0.690}{0.268} = 22\%.
\]  

**Deleveraging contribution**

The third and final cause is deleveraging. We have already seen how the leverage changes between state 0 and state B. This effect must contribute the remaining fraction of the total change:

\[
\text{Deleveraging Fraction} = 100\% - 53\% - 22\% = 25\%.
\]  

13.4.12 **Example: Restricted household beliefs**

This example generalizes the model by allowing the household beliefs to be uniformly distributed over the interval \([Lo, Hi]\). Since the probabilities must be between 0 and 1, then
0 \leq Lo < Hi \leq 1. For the uniform distribution $Unif[Lo, Hi]$, the pdf is $f(h) = \frac{1}{Hi - Lo}$. There remains, as above, a unit mass of households.

The first order of business is to find the 6 equations for the original model. Three of those equations are the cutoff rules, which remain unchanged:

\[ p(0) = h^*(0)p(G) + (1 - h^*(0))p(B). \quad (13.122) \]
\[ p(G) = h^*(G)d(GG) + (1 - h^*(G))d(GB). \]
\[ p(B) = h^*(B)d(BG) + (1 - h^*(B))d(BB). \]

The final 3 equations are from the budget constraint equations. In state 0, the budget constraint for any $h \geq h^*(0)$ simplifies to (see above):

\[ (p(0) - p(B)) \theta^h(0) = E + p(0). \quad (13.123) \]

Integrate over all households $h \geq h^*(0)$:

\[ \int_{h^*(0)}^{Hi} (p(0) - p(B)) \theta^h(0)f(h) \, dh = \int_{h^*(0)}^{Hi} (E + p(0)) f(h) \, dh. \quad (13.124) \]

We now use two facts. First, $\int_{h^*(0)}^{Hi} \theta^h(0)f(h) \, dh = 1$ as households $h \geq h^*(0)$ are the only ones to buy the asset. Second, since $h^* \sim Unif[Lo, Hi]$, then $\int_{h^*(0)}^{Hi} f(h) \, dh = \frac{Hi - h^*(0)}{Hi - Lo}$ (further details are found in the 2-period model). Thus, the final version of the equation is:

\[ p(0) - p(B) = (E + p(0)) \left( \frac{Hi - h^*(0)}{Hi - Lo} \right). \quad (13.125) \]

In state $G$, the budget constraint for any $h \geq h^*(G)$ simplifies to (see above):

\[ (p(G) - d(GB)) \theta^h(G) = \frac{p(G) - p(B)}{1 - h^*(0)}. \quad (13.126) \]

Integrate over all households $h \geq h^*(G)$:

\[ \int_{h^*(G)}^{Hi} (p(G) - d(GB)) \theta^h(G)f(h) \, dh = \int_{h^*(G)}^{Hi} \left( \frac{p(G) - p(B)}{1 - h^*(0)} \right) f(h) \, dh. \quad (13.127) \]
We now use two facts. First, \( \int_{h^*(G)}^{H_i} \theta^h(G) f(h) \, dh = 1 \) as households \( h \geq h^*(G) \) are the only ones to buy the asset. Second, since \( h \sim \text{Unif}[Lo, Hi] \), then \( \int_{h^*(G)}^{H_i} f(h) \, dh = \frac{Hi-h^*(G)}{Hi-Lo} \). Thus, the final version of the equation is:

\[
p(G) - d(GB) = \left( \frac{p(G) - p(B)}{1 - h^*(0)} \right) \left( \frac{Hi-h^*(G)}{Hi-Lo} \right) .
\]  

(13.128)

Finally, in state \( B \), the budget constraint for any \( h^*(B) \leq h \leq h^*(0) \) simplifies to (see above):

\[
(p(B) - d(BB)) \, \theta^h(B) = \frac{E + p(B)}{h^*(0)} .
\]  

(13.129)

Integrate over all households \( h^*(B) \leq h \leq h^*(0) \):

\[
\int_{h^*(B)}^{h^*(0)} (p(B) - d(BB)) \, \theta^h(B) f(h) \, dh = \int_{h^*(B)}^{h^*(0)} \left( \frac{E + p(B)}{h^*(0)} \right) f(h) \, dh .
\]  

(13.130)

We now use two facts. First, \( \int_{h^*(B)}^{h^*(0)} \theta^h(B) f(h) \, dh = 1 \) as households \( h^*(B) \leq h \leq h^*(0) \) are the only ones to buy the asset. Second, since \( h \sim \text{Unif}[Lo, Hi] \), then \( \int_{h^*(B)}^{h^*(0)} f(h) \, dh = \frac{h^*(0)-h^*(B)}{Hi-Lo} \). Thus, the final version of the equation is:

\[
p(B) - d(BB) = \left( \frac{E + p(B)}{h^*(0)} \right) \left( \frac{h^*(0) - h^*(B)}{Hi-Lo} \right) .
\]  

(13.131)

Gathering all 3 budget constraints together:

\[
p(0) - p(B) = (E + p(0)) \left( \frac{Hi-h^*(0)}{Hi-Lo} \right) .
\]  

(13.132)

\[
p(G) - d(GB) = \left( \frac{p(G) - p(B)}{1 - h^*(0)} \right) \left( \frac{Hi-h^*(G)}{Hi-Lo} \right) .
\]  

\[
p(B) - d(BB) = \left( \frac{E + p(B)}{h^*(0)} \right) \left( \frac{h^*(0) - h^*(B)}{Hi-Lo} \right) .
\]

An end-of-chapter exercises asks you to solve for the values of

\[
(p(0), p(G), p(B), h^*(0), h^*(G), h^*(B))
\]  

(13.133)
that satisfy the 6 equations of the original model.

We now wish to decompose the financial crash into its three causes. We know that deleveraging is simply the leftovers, so we focus on bad news and bankruptcy.

For bad news, we consider a new model in which the expected dividend payout from state 0 is equal to the expected dividend payout in the original model from state $B$. When $d(GG) = d(GB) = d(BG)$, the probability of a good shock in the new model (the variable $k$ from above) is given by:

$$ k = 1 - \sqrt{1 - h}. \quad (13.134) $$

The equations that need to change are now the cutoff rule equations (the budget constraint equations remain unchanged). The cutoff rule equations update to:

$$ p(0) = \left(1 - \sqrt{1 - h^*(0)}\right) p(G) + \sqrt{1 - h^*(0)} p(B). \quad (13.135) $$

$$ p(G) = \left(1 - \sqrt{1 - h^*(G)}\right) d(GG) + \sqrt{1 - h^*(G)} d(GB). $$

$$ p(B) = \left(1 - \sqrt{1 - h^*(B)}\right) d(BG) + \sqrt{1 - h^*(B)} d(BB). $$

The budget constraint equations remain:

$$ p(0) - p(B) = (E + p(0)) \left(\frac{Hi - h^*(0)}{Hi - Lo}\right). \quad (13.136) $$

$$ p(G) - d(GB) = \left(\frac{p(G) - p(B)}{1 - h^*(0)}\right) \left(\frac{Hi - h^*(G)}{Hi - Lo}\right). $$

$$ p(B) - d(BB) = \left(\frac{E + p(B)}{h^*(0)}\right) \left(\frac{h^*(0) - h^*(B)}{Hi - Lo}\right). $$

The solution to these 6 equations gives the value $p(0)$ in this new model. The component of the financial crash that is attributed to the bad news is equal to:

$$ \text{Bad News Fraction} = 1 - \frac{\text{New } p(0) - \text{Original } p(B)}{\text{Original } p(0) - \text{Original } p(B)}. \quad (13.137) $$

For bankruptcy, we need to solve for the solution of a 2-period model. Recalling the previous section, if a 2-period model has endowment $E$, dividends $(d(G), d(B))$ (where the normalization $d(G) = 1$ is used), and household beliefs $h \sim \text{Unif}[Lo, Hi]$, then the two
equilibrium equations are:

\[ p = h^* + (1 - h^*)d(B). \]  

\[ p - d(B) = \frac{H_i - h^*}{H_i - L_0} (E + p). \]  

(13.138)

The unknowns are \((p, h^*)\). The value for \(p\) that solves this system of equations is the equilibrium price in the 2-period model, denoted '2-period \(p\)'. The component of the financial crash that is attributed to bankruptcy is equal to:

\[ \text{Bankruptcy Fraction} = \frac{\text{2-period } p - \text{Original } p(B)}{\text{Original } p(0) - \text{Original } p(B)}. \]  

(13.139)

Finally, the deleveraging component is equal to:

\[ \text{Deleveraging Fraction} = 1 - \text{Bad News Fraction} - \text{Bankruptcy Fraction}. \]  

(13.140)

### 13.5 Exercises

1. **2-period model: No borrowing**

   Consider a 2-period leverage cycle model with one asset and one commodity. In the second period, two possible states can occur (Good and Bad). The dividend of the asset in the good state is 1 and the dividend of the asset in the bad state is 0.25. The initial endowment of the commodity equals 1 for all households. The initial endowment of the asset equals 1 for all households. There exists a unit mass of households in the economy. The household beliefs are uniformly distributed in the set \([0, 1]\), where household \(h\) believes the good state occurs with probability \(h\) and the bad state occurs with probability \(1 - h\).

   Do not allow the households to borrow. Solve for the equilibrium price of the asset in the initial period. Solve for the fraction of households that are lenders in the initial period.

2. **2-period model: No borrowing**

   Consider a 2-period leverage cycle model with one asset and one commodity. In the second period, two possible states can occur (Good and Bad). The dividend of the asset in the good state is 1 and the dividend of the asset in the bad state is 0.2. The
initial endowment of the commodity equals 1 for all households. The initial endowment of the asset equals 1 for all households. There exists a unit mass of households in the economy. The household beliefs are uniformly distributed in the set \([0.25, 0.75]\), where household \(h\) believes the good state occurs with probability \(h\) and the bad state occurs with probability \(1 - h\).

Do not allow the households to borrow. Solve for the equilibrium price of the asset in the initial period. Solve for the fraction of households that are lenders in the initial period.

3. **2-period model: With borrowing**

Use the equations

\[
h^* = \frac{p - 0.2}{0.8}
\]

\[
p = (1 - h^*) (1 + p) + 0.2
\]

to solve for the equilibrium variables \((h^*, p)\). You should conclude that \((h^*, p) = (0.686, 0.749)\).

4. **2-period model: With borrowing**

The Gini coefficient is defined in the following way. Choose any consumption level \(c\). Suppose consumption is distributed with cumulative distribution function \(F\) and probability density function \(f\). By definition, \(F(c)\) is the fraction of households with consumption less than or equal to \(c\). The fraction of total consumption allocated to this lower subset of households equals \(\frac{\int_0^c f(x) \, dx}{\int_0^1 f(x) \, dx}\). The Lorenz curve is a plot of \(F(c)\) on the x-axis and \(\frac{\int_0^c f(x) \, dx}{\int_0^1 f(x) \, dx}\) on the y-axis.

Under perfect equality, \(F(c) = \frac{\int_0^c f(x) \, dx}{\int_0^1 f(x) \, dx}\) for all consumption values as the fraction of households \(F(c)\) hold \(F(c)\) fraction of the total consumption. The Gini coefficient is defined as the twice the area between the perfect equality case and the Lorenz curve.

Suppose the fraction \(\theta\) of households consume \(c_1\) and the remaining fraction \(1 - \theta\) consume \(c_2\), where \(c_1 < c_2\). The Lorenz curve starts at \((0, 0)\) and is a straight line up until \((\theta, \frac{\theta c_1}{\theta c_1 + (1 - \theta) c_2})\). This is because the fraction of households with the lowest
consumption (fraction $\theta$) consume the fraction $\frac{\theta c_1}{\theta c_1 + (1 - \theta) c_2}$ of total consumption. The Lorenz curve proceeds from $\left(\theta, \frac{\theta c_1}{\theta c_1 + (1 - \theta) c_2}\right)$ in a straight line to $(1, 1)$.

The Gini coefficient is equal to two times the area between the 45-degree line and the Lorenz curve.

In the notes, the Gini coefficient was determined as the difference between the fraction of total consumption for the high consumers and the fraction of high consumers, which is equal to $\frac{(1-\theta)c_2}{\theta c_1 + (1 - \theta) c_2} - (1 - \theta)$ in this context. Show that the Gini coefficient defined by the Lorenz curve equals the Gini coefficient defined by this function.

5. **2-period model: Borrowing contracts**

   Solve for the interest rate $r_j$ as a function of $j$ for all values $j \in [0, 1]$. Show that this function is strictly increasing.

6. **2-period model: Borrowing contracts**

   Consider a 2-period leverage cycle model with one asset and one commodity. In the second period, two possible states can occur (Good and Bad). The dividend of the asset in the good state is 1 and the dividend of the asset in the bad state is 0.25. The initial endowment of the commodity equals 1 for all households. The initial endowment of the asset equals 1 for all households. There exists a unit mass of households in the economy. The household beliefs are uniformly distributed in the set $[0, 1]$, where household $h$ believes the good state occurs with probability $h$ and the bad state occurs with probability $1 - h$.

   Allow the households to borrow, where borrowing is secured with collateral. Solve for the equilibrium price of the asset in the initial period. Solve for the fraction of households that are lenders in the initial period.

7. **2-period model: Borrowing contracts**

   Consider a 2-period leverage cycle model with one asset and one commodity. In the second period, two possible states can occur (Good and Bad). The dividend of the asset in the good state is 1 and the dividend of the asset in the bad state is 0.2. The initial endowment of the commodity equals 1 for all households. The initial endowment of the asset equals 1 for all households. There exists a unit mass of households in the economy. The household beliefs are uniformly distributed in the set $[0.25, 0.75]$, where
household $h$ believes the good state occurs with probability $h$ and the bad state occurs with probability $1 - h$.

Allow the households to borrow, where borrowing is secured with collateral. Solve for the equilibrium price of the asset in the initial period. Solve for the fraction of households that are lenders in the initial period.

8. 3-period model: The leverage cycle

Consider the standard economy with $h \sim \text{Unif}[0, 1]$, dividend values

$$(d(GG), d(GB), d(BG), d(BB)) = (1, 1, 1, 0.2)$$

and endowment $E = 1$. Solve for the values of

$$(p(0), p(G), p(B), h^*(0), h^*(G), h^*(B))$$

that satisfy the 6 equations of the original model (a good shock $G$ is realized with probability $h$).

9. 3-period model: The leverage cycle

When the household variable $h \sim \text{Unif}[0, 1]$, find an expression for the average drop in the expected value of the asset between states 0 and $B$ (the average across all households)?

10. 3-period model: The leverage cycle

Consider the standard economy with $h \sim \text{Unif}[0, 1]$, dividend values

$$(d(GG), d(GB), d(BG), d(BB)) = (1, 1, 1, 0.2)$$

and endowment $E = 1$. Solve for the values of

$$(p(0), p(G), p(B), h^*(0), h^*(G), h^*(B))$$

that satisfy the 6 equations of the new model (a good shock $G$ is realized with probability $1 - \sqrt{1 - h}$). For this standard economy, you will see that $h^*(G) = 1$ and $p(G) = 1$ must always hold. Run the Solver application in Excel with only one objective function and one constraint, where the two variables are the guesses $h^*(0)$ and $h^*(B)$. 
11. 3-period model: The leverage cycle

Consider the economy with $h \sim \text{Unif}[0.25, 0.75]$, dividend values

$$(d(GG), d(GB), d(BG), d(BB)) = (1, 1, 1, 0.2)$$

and endowment $E = 1$. Solve for the values of

$$(p(0), p(G), p(B), h^*(0), h^*(G), h^*(B))$$

that satisfy the 6 equations of the original model (a good shock $G$ is realized with probability $h$).

12. 3-period model: The leverage cycle

Consider the economy with $h \sim \text{Unif}[0.25, 0.75]$, dividend values

$$(d(GG), d(GB), d(BG), d(BB)) = (1, 1, 1, 0.2)$$

and endowment $E = 1$. Solve for the values of

$$(p(0), p(G), p(B), h^*(0), h^*(G), h^*(B))$$

that satisfy the 6 equations of the new model (a good shock $G$ is realized with probability $1 - \sqrt{1 - h}$). For this standard economy, you will see that $h^*(G) = Hi$ and $p(G) = 1$ must always hold. Run the Solver application in Excel with only one objective function and one constraint, where the two variables are the guesses $h^*(0)$ and $h^*(B)$.

For this economy, what is the size of the Bad News Fraction, the Bankruptcy Fraction, and the Deleveraging Fraction?
Bibliography


Part VII

Search Theory
14

Monetary Search Theory

14.1 Pure exchange model

14.1.1 Sneak peek

Summary

In search and matching models, the demand for money and the demand for labor (our two variables of interest in the following two chapters) are endogenously determined from first principles. The first principles rely upon the observation that economic goods are often indivisible. This can certainly be true for commodities, in which the consumption may only take integer values (it is not possible to consume 1/2 of a car), and is unmistakably true for labor demand. Given that demand must be indivisible, search models make use of an additional assumption inspired by observation: economic exchange takes place in decentralized matches involving pairs of agents and not via a centralized market. Agents are pairwise matched and must decide whether or not to agree to an economic exchange. In any of these economic meetings, there is a trade-off between the costs of search or failing to exchange and the benefits from agreeing to exchange.

We are dealing with a model quite different than those previously introduced. Agents are subject to a search and matching friction. One of the useful features of the model is that it can provide a reason why agents would find it optimal to hold fiat currency. This will be the focus of this chapter. In order to implement effective monetary policy, it is essential that policymakers understand how the demand for money changes, which begins with understanding why and under what conditions agents hold money in the first place.
In the initial model, the setting is as simple as possible. There is an indivisible good and an indivisible unit of currency. Some agents hold the commodity, but cannot consume it, while other agents hold the currency. An economic exchange would consist of two agents agreeing to exchange currency for commodity. The agent who receives the commodity after the exchange is able to consume the commodity and increase utility. The agent who receives the currency is of particular interest as we seek to understand why an agent would trade for seemingly worthless fiat currency.

**Notation**

The variables/parameters to be introduced in this section are given in the following table:

- $M$: fraction of agents that are buyers (with 1 unit of currency)
- $U$: utility gain from consumption
- $C$: cost of production
- $\gamma$: probability for double coincidence of wants
- $\pi$: probability that an individual seller accepts currency
- $\Pi$: fraction of sellers that accept currency
- $V_b$: value function for an unmatched buyer
- $V_s$: value function for an unmatched seller
- $D(\Pi)$: net value of being a buyer, defined as $D(\Pi) = V_b - V_s$

**Main takeaways**

After completing this section, you will be able to answer the following questions:

- Does a nonmonetary equilibrium exist? Under what conditions?
- Does a monetary equilibrium exist? Under what conditions?

### 14.1.2 Model basics

In the pure exchange model, agents may hold 1 unit of fiat currency. Agents are randomly matched each period and must decide if they want to accept a unit of fiat currency in exchange for a good. As we will see, agents only accept fiat currency if they believe that the fiat currency can be used in future transactions to obtain goods.
The model contains an infinite number of discrete time periods. In the model, there are two types of agents:

1. A fraction $M \in (0, 1)$ of all agents are buyers that are currently endowed with one unit each of the fiat currency.

2. The remaining fraction $1 - M$ of all agents are sellers with the ability to produce 1 unit of a perishable consumption good.

The first assumption is that currency is indivisible. This means that each agent can hold either 0 or 1 unit of the currency. The second assumption is that agents cannot consume their own output (it is taboo). This second assumption ensures that agents trade.

### 14.1.3 Random matching

How does the matching process in the market work? In each period, agents are randomly matched. The probability that an agent meets a buyer is $M$ (after all, this is the total fraction of buyers) and the probability that an agent meets a seller is $1 - M$. The fractions $(M, 1 - M)$ remain stationary across all periods.

In any matched pair involving a buyer and a seller, the seller is always able to produce what the buyer consumes. The cost of production for the seller is $C$ and the utility gain for the buyer is $U$. We assume that $U > C$.

In any matched pair involving two sellers, with probability $\gamma \in (0, 1)$ each seller is able to produce what the other consumes (this is a double coincidence of wants).

If a buyer trades with a seller, then their roles switch: the buyer gives up the one unit of currency and exits as a seller, while the seller obtains the one unit of currency and exits as a buyer.

### 14.1.4 Equilibrium equations

In any buyer-seller match, the seller chooses the probability $\pi$ that he/she will accept the 1 unit of fiat currency from the buyer. The overall probability across all exchanges is $\Pi$, which is the fraction of the total buyer-seller matches in which the seller accepts the fiat currency. Each individual seller is arbitrarily small, so they take the overall probability $\Pi$ as given when choosing the optimal $\pi$.

Define the following value functions for the buyer and the seller:
• $V_b$ – value function for an unmatched buyer

• $V_s$ – value function for an unmatched seller

The discount factor from one period to the next is $\beta \in (0, 1)$ for all agents.

### 14.1.5 Buyer’s value function

The value function for the buyer is given by:

$$V_b = \beta (1 - M) \Pi [U + V_s] + \beta (1 - M) (1 - \Pi) V_b + \beta MV_b. \quad (14.1)$$

With probability $(1 - M)\Pi$, the buyer meets a seller and the seller is willing to accept the one unit of fiat currency from the buyer. This means that the seller instantaneously produces the good and then trades it to the buyer. The buyer consumes the good earning utility $U$ and then exits the meeting as a seller (0 currency holding). With probability $(1 - M) (1 - \Pi)$, the buyer is matched with a seller, but the seller is not willing to make the trade, meaning that the buyer remains a buyer. With probability $M$, the buyer meets another buyer and no transaction is possible (each has 1 unit of currency), meaning that the buyer remains a buyer.

Adding and subtracting $V_b$ to the first term on the right-hand side of (14.1) and solving for $V_b$ yields:

$$V_b = \frac{\beta (1 - M) \Pi [U - (V_b - V_s)]}{1 - \beta}. \quad (14.2)$$

### 14.1.6 Seller’s value function

The value function for the seller is given by:

$$V_s = \beta (1 - M) \gamma [U - C + V_s] + \beta (1 - M) (1 - \gamma) V_s + \beta M \pi [V_b - C] + \beta M (1 - \pi) V_s. \quad (14.3)$$

With probability $(1 - M)\gamma$, the seller meets another seller and double coincidence of wants occurs, meaning that each seller can produce what the other one wants. Each seller instantaneously produces a good and then exchanges goods with the other seller. The exchange of goods without money is called barter. The seller obtains utility $U$, incurs cost $C$, and exits the match still without any currency (i.e., remains a seller). With probability $(1 - M) (1 - \gamma)$, the seller is matched with a seller, but they are not able to barter because the double coincidence of wants fails. This means that the seller remains a seller. With probability $M$,
the seller is matched with a buyer. The seller makes the optimal choice for \( \pi \), which is the probability of accepting the 1 unit of fiat currency from the buyer. With probability \( M \pi \), the seller produces the good at cost \( C \) and exits the match with a unit of the fiat currency (i.e., exits as a buyer). With the remaining probability \( M(1 - \pi) \), the seller does not accept the trade and remains a seller.

Adding and subtracting \( V_s \) to the third term on the right-hand side of (14.3) and solving for \( V_s \) yields:

\[
V_s = \frac{\beta(1 - M)\gamma [U - C] + \beta M \pi [V_b - V_s - C]}{1 - \beta}.
\]

(14.4)

### 14.1.7 Incentive to accept money

Notice from the expression for \( V_s \) that the optimal choice for a seller is to choose:

\[
\begin{align*}
\pi &= 0 & \text{if } V_b - V_s - C < 0. \\
\pi &\in [0, 1] & \text{if } V_b - V_s - C = 0. \\
\pi &= 1 & \text{if } V_b - V_s - C > 0.
\end{align*}
\]

(14.5)

The first and third cases are pure strategies: \( \pi = 0 \) means a seller never accepts the currency and \( \pi = 1 \) means that a seller always accepts the currency. If the cost and benefits are exactly equal, \( V_b - V_s - C = 0 \), then a seller is indifferent between accepting and not accepting, so the optimal choice can be a mixed strategy \( \pi \in [0, 1] \). Though mixed strategies can be interesting, we will focus on pure strategies in this text.

Define the difference \( D(\Pi) = V_b - V_s \). We can evaluate this difference using the value function expressions (14.2) and (14.4):

\[
D(\Pi) = \frac{\beta(1 - M)\Pi [U - D(\Pi)] - \beta(1 - M)\gamma [U - C] - \beta M \pi [D(\Pi) - C]}{1 - \beta}.
\]

(14.6)

Notice that we have replaced \( D(\Pi) = V_b - V_s \) from each of the value function expressions. Solving for \( D(\Pi) \) yields:

\[
D(\Pi) \left(1 + \frac{\beta(1 - M)\Pi + \beta M \pi}{1 - \beta}\right) = \frac{\beta(1 - M)\Pi U - \beta(1 - M)\gamma [U - C] + \beta M \pi C}{1 - \beta}, \tag{14.7}
\]

where the factor multiplying \( D(\Pi) \) can be expressed as

\[
\left(1 + \frac{\beta(1 - M)\Pi + \beta M \pi}{1 - \beta}\right) = \frac{1 - \beta + \beta \Pi + \beta M(\pi - \Pi)}{1 - \beta}. \tag{14.8}
\]
The difference \( D(\Pi) \) is then expressed as:

\[
D(\Pi) = \frac{\beta(1 - M)\Pi U - \beta(1 - M)\gamma [U - C] + \beta M\pi C}{1 - \beta + \beta\Pi + \beta M(\pi - \Pi)}. \tag{14.9}
\]

In equilibrium, it must be the case that \( \pi = \Pi \). All agents are homogeneous, meaning that out of all buyer-seller matches, the fraction of all consummated trades is equal to the probability that a trade is consummated. This means that the difference \( D(\Pi) \) is:

\[
D(\Pi) = \frac{\beta\Pi U - \beta M\Pi [U - C] - \beta(1 - M)\gamma [U - C]}{1 - \beta + \beta\Pi}. \tag{14.10}
\]

Remember that \( D(\Pi) = V_b - V_s \) and sellers make the following optimal decisions about accepting currency:

if \( D(\Pi) = V_b - V_s > C \) accept currency \( \pi = \Pi = 1 \)
if \( D(\Pi) = V_b - V_s < C \) reject currency \( \pi = \Pi = 0 \). \tag{14.11}

A monetary equilibrium has \( \Pi = 1 \) and a nonmonetary equilibrium has \( \Pi = 0 \).

### 14.1.8 Existence of a monetary equilibrium

A monetary equilibrium with \( \Pi = 1 \) exists provided that \( D(\Pi) > C \), where the difference equation \( D(\Pi) \) is evaluated at the value \( \Pi = 1 \).

Let’s consider the inequality \( D(\Pi) > C \):

\[
\frac{\beta\Pi U - \beta M\Pi [U - C] - \beta(1 - M)\gamma [U - C]}{1 - \beta + \beta\Pi} > C. \tag{14.12}
\]

Cross-multiplying and combining terms:

\[
\beta\Pi [U - C] - \beta M\Pi [U - C] - \beta(1 - M)\gamma [U - C] > C(1 - \beta). \tag{14.13}
\]

Each term on the left-hand side contains the difference \( U - C \):

\[
[U - C] \beta(1 - M) (\Pi - \gamma) > C(1 - \beta). \tag{14.14}
\]
The monetary equilibrium $\Pi = 1$ is possible provided that:

$$[U - C] \beta(1 - M) (1 - \gamma) > C(1 - \beta), \quad (14.15)$$

which can be expressed as the ratio of utility to production cost:

$$U \beta(1 - M) (1 - \gamma) > C(1 - \beta + \beta(1 - M) (1 - \gamma)) \quad (14.16)$$

$$\frac{U}{C} > \frac{1 - \beta + \beta(1 - M) (1 - \gamma)}{\beta(1 - M) (1 - \gamma)},$$

Therefore, a monetary equilibrium with $\Pi = 1$ exists if and only if

$$\frac{U}{C} > 1 + \frac{1 - \beta}{\beta(1 - M) (1 - \gamma)}.$$

1. From the expression $\frac{U}{C} > 1 + \frac{1 - \beta}{\beta(1 - M) (1 - \gamma)}$, notice that the monetary equilibrium with $\Pi = 1$ is possible when the ratio of utility gain to production cost is sufficiently large.

2. As $\beta$ decreases (households become less patient), the term $\frac{1 - \beta}{\beta(1 - M)(1 - \gamma)}$ increases, making a monetary equilibrium with $\Pi = 1$ less likely.

3. As the number of buyers $M$ increases, the term $\frac{1 - \beta}{\beta(1 - M)(1 - \gamma)}$ increases. Each buyer has a smaller chance of finding a seller willing to accept the 1 unit of currency. This makes a monetary equilibrium with $\Pi = 1$ less likely.

4. As the probability of double coincidence of wants $\gamma$ increases, the term $\frac{1 - \beta}{\beta(1 - M)(1 - \gamma)}$ increases. Each seller has a higher probability of barter, which reduces the appeal of trading with currency. This makes a monetary equilibrium with $\Pi = 1$ less likely.

### 14.1.9 Existence of a nonmonetary equilibrium

When is a nonmonetary equilibrium with $\Pi = 0$ possible? A nonmonetary equilibrium with $\Pi = 0$ exists provided that $D(\Pi) < C$, where the difference equation $D(\Pi)$ is evaluated at the value $\Pi = 0$. The strict inequality $D(\Pi) < C$ with $\Pi = 0$ is given by:

$$\frac{-\beta(1 - M)\gamma (U - C)}{1 - \beta} < C. \quad (14.17)$$
Since $U - C > 0$ by assumption, then $\frac{-\beta(1-M)\gamma(U-C)}{1-\beta} < 0$ always holds. This implies that $\frac{-\beta(1-M)\gamma(U-C)}{1-\beta} < C$ is always satisfied. Thus, there will always exist a nonmonetary equilibrium (where $\Pi = 0$).

### 14.2 Bargaining model

#### 14.2.1 Sneak peek

**Summary**

We now consider an extension of the previous monetary search model by allowing agents to bargain over the amount of consumption that will be exchanged for one unit of currency. We previously only allowed an exchange of one unit of currency for one unit of the commodity. With this bargaining setup, there remains one unit of currency, but the agents that are matched together are able to bargain over the number of units of commodity. We typically refer to value in terms of prices, which is the number of units of currency for every one unit of commodity. Here, the equilibrium variable is the inverse price, which measures the number of units of commodity for every one unit of currency.

With this setup, not only is it possible to find an equilibrium in which money has value, but there are actually two equilibria in which money has value: a high-price equilibrium and a low-price equilibrium.

**Notation**

The variables to be introduced in this section are given in the following table:

- $q$ units of the consumption good for each 1 unit of currency
- $Q$ average terms of trade
- $P$ nominal price of the consumption good, defined as $P = \frac{1}{Q}$
- $q^*$ units of the good traded under double coincidence of wants

**Main takeaways**

After completing this section, you will be able to answer the following questions:

- How many monetary equilibria exist and what are their properties?
14.2. BARGAINING MODEL

• Under what conditions does a stationary monetary equilibrium exist?

14.2.2 Model basics

In any buyer-seller match, the buyer makes a take-it-or-leave-it offer to the seller. This is the way in which the model handles bargaining; all bargaining power is granted to the buyer. The offer is that the buyer receives $q$ units of the consumption good and the seller receives one unit of currency.

The variable $Q$ is the average terms of trade in all buyer-seller transactions. This implies that the price of the consumption good is $P = \frac{1}{Q}$. Since each agent is a price taker (there are an infinite number of other agents, after all), they take the average terms of trade $Q$ as given.

Agents have utility over the consumption good given by the utility function $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that $u(0) = 0$ and $u$ is strictly increasing and strictly concave. Each agent incurs a linear production cost so that producing $q$ units results in a utility cost of $q$.

In any seller-seller transaction, define

$$q^* = \arg \max_{q \geq 0} u(q) - q.$$ (14.18)

The first order conditions are given by:

$$Du(q^*) - 1 = 0 \text{ if } q^* > 0.$$ (14.19)

From the first order conditions, $q^*$ solves $Du(q^*) = 1$ if $q^* > 0$. Otherwise, $q^* = 0$ is the optimal choice. If $q^* = 0$ is the optimal choice, then $u(q^*) - q^* = 0$. If $q^* > 0$, then the payout must be higher than with $q = 0$, meaning that $u(q^*) - q^* > 0$.

Under what conditions is $q^* > 0$? If $Du(0) > 1$, then there exists $q^* > 0$ such that $Du(q^*) = 1$ (since $u$ is strictly concave).

Define the following value functions for the buyer and the seller:

• $V_b$—value function for an unmatched buyer

• $V_s$—value function for an unmatched seller
14. MONETARY SEARCH THEORY

### 14.2.3 Buyer’s value function

The value function for the buyer is given by:

\[ V_b = \beta(1 - M) [u(q) + V_s] + \beta MV_b. \] (14.20)

With probability $1 - M$, the buyer meets a seller and the transaction occurs at the buyer’s chosen terms of trade $q$, meaning that the buyer has utility $u(q)$ and exits as a seller (no currency). With probability $M$, the buyer meets another buyer and no transaction is possible (each has 1 unit of currency), meaning that the buyer remains a buyer.

Adding and subtracting $V_b$ to the first term on the right-hand side of (14.20) and solving for $V_b$ yields:

\[ V_b = \frac{\beta(1 - M) [u(q) - (V_b - V_s)]}{1 - \beta}. \] (14.21)

### 14.2.4 Seller’s value function

The value function for the seller is given by:

\[ V_s = \beta(1 - M) \gamma [u(q^*) - q^* + V_s] + \beta(1 - M) (1 - \gamma) V_s + \beta M [V_b - Q]. \] (14.22)

With probability $(1 - M) \gamma$, the sellers meets another seller and they are able to each produce what the other consumes. This production and consumption results in utility $u(q^*) - q^*$ and the seller remains a seller (still does not have any units of currency). With probability $(1 - M) (1 - \gamma)$, the sellers meets another seller and they are unable to produce what the other consumes. Thus, no transaction occurs and the seller remains a seller. With probability $M$, the seller meets a buyer, pays the production cost $Q$ and exits the period as a buyer (with 1 unit of currency). The seller faces the average terms of trade $Q$ as the seller has no bargaining power.

Adding and subtracting $V_s$ to the third term on the right-hand side of (14.22) and solving for $V_s$ yields:

\[ V_s = \frac{\beta(1 - M) \gamma [u(q^*) - q^*] + \beta M [V_b - V_s - Q]}{1 - \beta}. \] (14.23)
14.2. BARGAINING MODEL

14.2.5 Incentive to hold money

Define $D(Q) = V_b - V_s$. Using (14.21) and (14.23) and replacing $V_b - V_s$ in those equations with $D(Q)$:

$$D(Q) = \frac{\beta(1 - M) [u(q) - D(Q)] - \beta(1 - M)\gamma [u(q^*) - q^*] - \beta M [D(Q) - Q]}{1 - \beta}.$$  \hfill (14.24)

Let's collect the terms involving $D(Q)$ on the left-hand side:

$$D(Q) \left(1 + \frac{\beta(1 - M) + \beta M}{1 - \beta}\right) = \frac{\beta(1 - M)u(q) - \beta(1 - M)\gamma [u(q^*) - q^*] + \beta MQ}{1 - \beta}.$$  \hfill (14.25)

The factor multiplying $D(Q)$ on the left-hand side is

$$\left(1 + \frac{\beta(1 - M) + \beta M}{1 - \beta}\right) = \left(\frac{1}{1 - \beta}\right).$$  \hfill (14.26)

This implies that the difference equation $D(Q)$ is given by:

$$D(Q) = -\beta(1 - M)\gamma [u(q^*) - q^*] + \beta(1 - M)u(q) + \beta MQ.$$  \hfill (14.27)

14.2.6 Optimal bargaining offer

What is the optimal offer from a buyer in a buyer-seller match? Since the buyer is able to extract all of the surplus (it has the bargaining power), then the offer must be such that the seller is just willing to accept. If the seller rejects the offer, then it exits the period with value function $V_s$. If the seller accepts the offer, then it must pay the production cost of $-q$ to produce $q$ units and then exits the period with value function $V_b$. Thus, the seller is just willing to accept provided that $V_s = -q + V_b$ or:

$$q = \max \{V_b - V_s, 0\}.$$  \hfill (14.28)

Recalling that by definition, $D(Q) = V_b - V_s$, the optimal bargaining offer satisfies $q = \max \{D(Q), 0\}$. An equilibrium is such that $q$, the terms of trade in any one buyer-seller transaction, must be equal to $Q$, the average terms of trade in all buyer-seller transactions. This is because all buyers are identical and all sellers are identical. So in equilibrium it must be that $q = \max \{D(Q), 0\} = Q$. 


This means that the difference equation $D(Q)$ from (14.27) is given by:

$$D(Q) = -\beta(1-M)\gamma [u(q^*) - q^*] + \beta(1-M)u(Q) + \beta MQ.$$  

(14.29)

The y-intercept of the function is $-\beta(1-M)\gamma [u(q^*) - q^*]$.

- The y-intercept $-\beta(1-M)\gamma [u(q^*) - q^*] = 0$ if $q^* = 0$ (as this implies $[u(q^*) - q^*] = 0$).
- The y-intercept $-\beta(1-M)\gamma [u(q^*) - q^*] < 0$ if $q^* > 0$. We know that $q^* > 0$ if $Du(0) > 1$.

The function depends upon $Q$ according to $\beta(1-M)u(Q) + \beta MQ$. This means that $D(Q)$ is a strictly increasing and strictly concave function (exactly the same properties as the utility function $u$).

### 14.2.7 Equilibrium properties

Figure 14.2.1 contains $Q$ on the x-axis and plots $\max\{D(Q), 0\}$ on the y-axis. An equilibrium is any intersection of $\max\{D(Q), 0\}$ with the 45-degree line.

For the particular economy shown in Figure 14.2.1, there exist 3 equilibria: 1 nonmonetary equilibrium with $Q = 0$ and 2 monetary equilibria with $Q > 0$. When $Q = 0$, the price $P = \frac{1}{Q}$ is infinite, meaning that money has no value. This equilibrium is a nonmonetary equilibrium. The other two equilibria are monetary equilibria since the fiat currency is valued. Out of the 2 monetary equilibria, the one with the lower $Q$ corresponds to a higher nominal price. This monetary equilibrium is unstable. Small deviations will cause the equilibrium terms $Q$ to converge either to 0 or to the monetary equilibrium with the higher $Q$. The monetary equilibrium with the higher $Q$ corresponds to a lower nominal price. This equilibrium is stable.

The conclusion is that out of the 2 monetary equilibria, the one with the lower nominal price is the stable one. The one with the higher nominal price can be thought of as a "bubble" and is unstable.

Additionally, the multiplicity of monetary equilibria occurs because the value of currency is based upon the endogenous expectations of agents for future bargaining opportunities. There exist 2 possible expectations that are self-fulfilling. Of course, only 1 is stable, but they are both equilibria.
14.2.8 Existence of a monetary equilibrium guaranteed?

A monetary equilibrium fails to exist if the function $D(Q)$ never intersects the 45-degree line, specifically the function $D(Q)$ lies strictly below the 45-degree line. If the function $D(Q)$ does cross the 45-degree line, it will first cross the 45-degree line from below (since the intercept is strictly negative). Since $D(Q)$ is strictly concave, after the curve intersects the 45-degree line from below it is guaranteed to intersect it again from above. Thus, we are looking at either zero or two monetary equilibria. The only exception is if the function $D(Q)$ is tangent to the 45-degree line, but this case is atypical.
14. MONETARY SEARCH THEORY
Bibliography


15

Labor Market Search Theory

15.1 Random labor search

15.1.1 Sneak peek

Summary

The search and matching framework can also be used to study labor market displacement by incorporating the search and matching friction into labor market transactions. There are two margins for labor market transactions: the extensive margin refers to whether a job is created by a firm and accepted by a worker, whereas the intensive margin refers to the equilibrium choices of hours worked and labor compensation.

The first model of labor search contains random labor search and no matching. The model only contains one side of the market, namely the workers’ labor supply side, and does not consider the firms’ labor demand decisions. In this simple framework, we can identify a reservation wage below which an unemployed worker is not willing to accept a job offer. Such a setting is important to begin to think about policies related to labor markets and serves as the foundation for the subsequent models with competitive labor search and matching.
Notation

The variables/parameters to be introduced in this section are given in the following table:

- $w$: wage of worker
- $c$: unemployment benefits
- $Y_t$: worker income
- $\alpha$: probability that an unemployed worker receives a wage offer
- $\delta$: probability that an employed worker loses its job
- $V(w)$: value function for a worker hired at wage $w$
- $U$: value function for an unemployed worker
- $R$: reservation wage
- $ED$: expected duration of unemployment

Main takeaways

After completing this section, you will be able to answer the following questions:

- What factors determine the value of the reservation wage at which a worker will accept a job?
- What factors determine the expected duration of unemployment?

15.1.2 Setup of the model

Time is discrete and infinite. A unit mass of homogeneous workers live forever and receive income $Y_t$ in every period. If a worker is employed and hired with wage $w$, then $Y_t = w$. If the worker is unemployed, then $Y_t = c$.

Define $\alpha$ as the probability that an unemployed worker receives a wage offer. The wage offers that an unemployed worker receives are drawn from the cumulative distribution $F$. Given a job offer, an unemployed worker can either accept or reject the offer. Notice that the unemployed worker cannot take action in order to (i) get offers at a faster rate or (ii) get a higher offer.

For workers currently employed, define $\delta$ as the probability of job loss. It is not possible for a worker to quit the job. This means that a worker hired at wage $w$ will continue to receive the same wage until the match is split, which occurs with probability $\delta$. If a worker is fired in the current period, it does not have to wait before receiving a new offer. The
fired worker immediately joins the mass of unemployed workers and receives an offer with probability $\alpha$.

The discount factor for the workers is $\beta \in (0, 1)$. Workers are risk-neutral, meaning that their utility is equal to the expected discounted future income, which is the sum of all discounted incomes that they will earn in future periods, accounting for the possibilities of unemployment and re-employment.

### 15.1.3 Value functions

We need to specify two value functions to analyze this problem: a value function when employed and a value function when unemployed. When employed, the value function must depend upon the wage $w$, so we denote the employed value function as $V (w)$. When unemployed, all workers receive the same income, so we can denote the unemployed value simply as $U$. These two value functions are recursively defined in terms of each other.

For the employed value function, the current income is $w$. In the subsequent period, the employed worker experiences one of two outcomes: unemployment with probability $\delta$ or employment at the wage $w$ with the remaining probability $1 - \delta$. The employed value function $V (w)$ is recursively defined by:

$$ V (w) = w + \beta [\delta U + (1 - \delta) V (w)]. $$

This expression can be solved for $V (w)$:

$$ V (w) = \frac{w + \beta \delta U}{1 - \beta (1 - \delta)}. $$

Notice that the derivative $V' (w) = \frac{1}{1 - \beta (1 - \delta)} > 0$ meaning that a higher wage leads to a higher value of being employed.

For the unemployed value function, the current income is $c$. In the subsequent period, the unemployed worker experiences one of two outcomes: job offer with probability $\alpha$ or no job offer with probability $1 - \alpha$. Without a job offer, the worker remains unemployed. A job offer has value drawn from the distribution $F$. Given a job offer of value $w$, the unemployed worker can either choose to be employed at wage $w$ or can choose to remain unemployed. The unemployed value function $U$ is recursively defined by:

$$ U = c + \beta (1 - \alpha) U + \beta \alpha E \{ \max \{ V (w), U \} \}. $$
In this expression, the expectation is over the distribution $F$.

### 15.1.4 Reservation wage

Since $V(w)$ is strictly increasing in $w$ and $U$ is independent of $w$, then there exists a unique reservation wage $R$ defined such that with wage offer $w = R$, the unemployed worker is indifferent between accepting and rejecting the offer. In other words, the value of being employed at wage $w = R$ is equal to the value of unemployed $U$:

$$V(R) = U. \quad (15.4)$$

For any wage offer $w \geq R$, the unemployed worker will accept. For any wage offer $w < R$, the unemployed worker will reject. With the reservation wage, we can rewrite the expression for the unemployed value function:

$$U = c + \beta \alpha \int_0^R UdF(w) + \beta \alpha \int_R^\infty V(w)dF(w). \quad (15.5)$$

Notice that $\beta U$ appears in two of the terms on the right-hand side of the previous expression. Subtract $\beta U$ from both sides of the previous equation in order to simplify:

$$U (1 - \beta) = c + \beta \alpha \int_R^\infty (V(w) - U) dF(w). \quad (15.6)$$

The definition of the reservation wage, $V(R) = U$, together with the expression for the employed value function allows us to solve for $R$ as a function of $U$:

$$V(R) = \frac{R + \beta \delta U}{1 - \beta (1 - \delta)} = U. \quad (15.7)$$

The expression simplifies to:

$$R = U (1 - \beta). \quad (15.8)$$
15.1. RANDOM LABOR SEARCH

15.1.5 Fixed point equation

Using the expression for $U(1-\beta)$ from above, we obtain a fixed point equation in terms of the reservation wage variable $R$:

$$R = c + \beta \alpha \int_{R}^{\infty} (V(w) - U) dF(w).$$

(15.9)

The difference $V(w) - U$ is simplified to:

$$V(w) - U = \frac{w + \beta \delta \left( \frac{R}{1-\beta} \right)}{1-\beta (1-\delta)} - \left( \frac{R}{1-\beta} \right)$$

$$= \frac{w - (1-\beta) \left( \frac{R}{1-\beta} \right)}{1-\beta (1-\delta)}$$

$$= \frac{w - R}{1-\beta (1-\delta)}.$$  

(15.10)

The fixed point equation is then given by:

$$R = c + \beta \alpha \int_{R}^{\infty} (w - R) dF(w).$$

(15.11)

Define the excess wage function $\gamma(R) = \int_{R}^{\infty} (w - R) dF(w)$. To evaluate the solution, we need to understand the key properties of the excess wage function:

1. If $R = 0$, then all wages are accepted, so $\gamma(0) = \int_{0}^{\infty} wdF(w) = E[w]$, the average wage.

2. If $R = \infty$, then no wages are accepted, so $\gamma(\infty) = 0$.

3. The derivative $\gamma'(R) = [-(F(w) - R)]_{R}^{\infty} = -(F(\infty) - R) + (F(R) - R) = F(R) - 1 < 0$. The negative sign in the derivative (in front of the term $(F(w) - R)$) arises since the derivative of $w - R$ with respect to $R$ is equal to $-1$. The derivative tells us that an increase in the reservation wage leads to an decrease in the excess wage function.

4. The derivative $\gamma''(R) = F'(R) > 0$, since all cumulative distributions are strictly increasing. Since $\gamma''(R) > 0$, then the function is strictly convex.

Using the 4 properties above, Figure 15.1.1 shows how a fixed point to the equation $R = c + \frac{\beta \alpha}{1-\beta(1-\delta)} \gamma(R)$ is determined.
15.1.6 Expected duration of unemployment

We seek to compute the expected duration of unemployment. To do that, we have to compute the probability of being unemployed for \( d \) periods. A \( d \)-period unemployment requires \( d - 1 \) periods with either (a) no offer or (b) an offer \( w < R \) and then 1 period with both (a) an offer and (b) an offer \( w \geq R \). The probability of an offer such that \( w \geq R \) is \( \alpha (1 - F(R)) \). This means that the probability of the complement (either no offer or an offer \( w < R \)) must be:

\[
1 - \alpha (1 - F(R)).
\]  
(15.12)

The probability of a \( d \)-period unemployment is:

\[
Pr(d) = \alpha (1 - F(R)) (1 - \alpha (1 - F(R)))^{d-1}.
\]  
(15.13)

The expected duration of unemployment \( ED \) is then equal to the sum of the probability of being unemployed for \( d \) periods multiplied by \( d \):

\[
ED = \sum_{d=1}^{\infty} Pr(d) \cdot d
\]  
(15.14)

\[
= \sum_{d=1}^{\infty} \alpha d (1 - F(R)) (1 - \alpha (1 - F(R)))^{d-1}.
\]

By definition,

\[
\sum_{d=1}^{\infty} \alpha (1 - F(R)) (1 - \alpha (1 - F(R)))^{d-1} = 1,
\]  
(15.15)

since the probability of being unemployed for any length of time must be equal to 1. Differentiating the above expression with respect to the variable \( F(R) \) yields:

\[
\sum_{d=1}^{\infty} \left\{ \alpha^2 (d - 1) (1 - \alpha (1 - F(R)))^{d-2} (1 - F(R)) - \alpha (1 - \alpha (1 - F(R)))^{d-1} \right\} = 0.
\]  
(15.16)
15.1. RANDOM LABOR SEARCH

Cancel $\alpha$ from both terms and expand this equation:

$$
\sum_{d=1}^{\infty} \alpha d (1 - F(R)) (1 - \alpha (1 - F(R)))^{d-2} = \sum_{d=1}^{\infty} \alpha (1 - \alpha (1 - F(R)))^{d-2} (1 - F(R)) \quad (15.17)
+ \sum_{d=1}^{\infty} (1 - \alpha (1 - F(R)))^{d-1}.
$$

Recall the expected duration equation, where we now have an expression for the term

$$
\sum_{d=1}^{\infty} \alpha d (1 - F(R)) (1 - \alpha (1 - F(R)))^{d-2}.
$$

The expected duration expression is simply this expression multiplied by $(1 - \alpha (1 - F(R)))$, meaning that

$$
ED = \sum_{d=1}^{\infty} \alpha (1 - \alpha (1 - F(R)))^{d-1} (1 - F(R)) \quad (15.19)
+ \sum_{d=1}^{\infty} (1 - \alpha (1 - F(R)))^{d}.
$$

Factoring out the common term $(1 - \alpha (1 - F(R)))^{d-1}$ leads to:

$$
ED = \sum_{d=1}^{\infty} (1 - \alpha (1 - F(R)))^{d-1} \{\alpha (1 - F(R)) + 1 - \alpha (1 - F(R))\}. \quad (15.20)
$$

Since the term $\{\alpha (1 - F(R)) + 1 - \alpha (1 - F(R))\} = 1$ and the definition

$$
\sum_{d=1}^{\infty} \alpha (1 - F(R)) (1 - \alpha (1 - F(R)))^{d-1} = 1 \quad (15.21)
$$

implies

$$
\sum_{d=1}^{\infty} (1 - \alpha (1 - F(R)))^{d-1} = \frac{1}{\alpha (1 - F(R))}, \quad (15.22)
$$

then expected duration is given by:

$$
ED = \frac{1}{\alpha (1 - F(R))}. \quad (15.23)
$$
The expected duration of unemployment is equal to the inverse of the probability of an offer \( w \geq R \). This is a common property with hazard rates. As you might then expect, the expected duration of unemployment depends in a natural way upon: (i) the probability of receiving an offer and (ii) the reservation wage below which workers are not willing to accept an offer:

1. Expected duration of unemployment is decreasing in \( \alpha \), which as you recall is the probability of receiving an offer when unemployed.

2. Expected duration of unemployment is increasing in \( R \), where the reservation wage \( R \) is endogenously determined in the model as a function of the parameters \( (\alpha, \beta, \delta, c, F) \).

### 15.2 Search with matching

#### 15.2.1 Sneak peek

**Summary**

The previous model only addressed the search problem of the worker, a so-called 1-sided search problem. In reality, labor markets are comprised of both workers and firms, where both sides are tasked with searching. The process by which workers and firms search and are ultimately matched together in employment is the search and matching model introduced in this section.

There are a few important assumptions of the model that allow it to be solved. First, firms exist in a setting of free entry. This means that the expected profit for any firm entering the market and searching for a worker to hire is equal to 0. The free entry condition, namely the existence of a bunch of potential firms not in the market that have the possibility to enter, provides an extra equation (the zero profit equation) that allows the model to be solved. The second key assumption is that if a worker and firm have the opportunity to negotiate a labor contract, the wage is determined according to Nash bargaining. Nash bargaining is a simple mechanism in which the surplus of a match is split between the two parties according to an exogenous Nash bargaining weight.

This model is commonly referred to as the Mortensen-Pissarides model. In recognition of this model, Dale Mortensen and Christopher Pissarides, along with Peter Diamond, received the 2010 Nobel Prize in Economics.
Notation

The variables/parameters to be introduced in this section are given in the following table:

- $j$: index for either $j = w$ (worker) or $j = f$ (firm)
- $e$: index for either $e = 0$ (no match) or $e = 1$ (match)
- $y$: output of a match
- $\delta$: exogenous probability of match destruction
- $\lambda^j$: arrival rate of a match for agent $j$
- $F$: ratio of number of firms to number of workers
- $k$: cost to post a vacancy
- $u$: number of workers searching for a match
- $v$: number of firms searching for a match
- $\tau$: market tightness
- $N(u, v)$: matching technology
- $\eta$: Nash bargaining weight

Main takeaways

After completing this section, you will be able to answer the following questions:

- What factors determine when firms and workers search for a match?
- How is the equilibrium wage determined?
- Under what condition is the equilibrium outcome Pareto efficient?

15.2.2 Model setup

The model contains two types of agents: workers and firms. The workers are homogeneous with unit mass. The potential firms are homogeneous with mass equal to $F > 1$. It is required that more potential firms exist than workers in order to maintain the free entry condition. The free entry condition states that there are always potential firms that are inactive in the market that may decide to enter if the expected profit is strictly positive. The free entry condition therefore implies that the expected profit of all unmatched firms in the market equals 0.
Both workers and firms have the same discount factor $\beta \in (0, 1)$. Both workers and firms are risk-neutral, meaning that the lifetime utility function is equal to the expected discounted income (worker) or expected discounted profit (firm).

### 15.2.3 Worker value functions

For workers, there are two possibilities $e = 0$ (no match) or $e = 1$ (match). Without a match, worker income equals $c$. With a match, workers receive the wage rate $w$, which we later show is the outcome of a bargaining process with the firm that employs them. Denote $\delta$ as the exogenous probability of match destruction, which can either be worker quitting or a firm firing a worker. The model does not permit a worker to search for other jobs while employed. The match destruction occurs randomly. Denote $\lambda^w$ as the rate at which a new match arrives for the worker. When a new match arrives, we show in equilibrium that the outcome will always be employment at the wage rate $w$. The value of the wage rate $w$ will be determined in equilibrium, but given that both agents discount the future at rate $\beta$, it is in the best interest of both the worker and the firm to reach an agreement on the wage rate.

There are two value functions for the worker: one corresponding to unemployment $e = 0$ (no match) and one corresponding to employment $e = 1$ (match). Denote $V^w_0$ as the unemployment value function and $V^w_1$ as the employment value function. For the employed value function, the current income is $w$. In the subsequent period, the employed worker experiences one of two outcomes: unemployment with probability $\delta$ or employment with the remaining probability $1 - \delta$. The employed value function $V^w_1$ is recursively defined by:

$$V^w_1 = w + \beta [\delta V^w_0 + (1 - \delta) V^w_1].$$

Subtracting $\beta V^w_1$ from both sides, the expression can be simplified:

$$V^w_1 (1 - \beta) = w - \beta \delta (V^w_1 - V^w_0).$$

(15.24)

For the unemployed value function, the current income is $c$. In the subsequent period, the unemployed worker experiences one of two outcomes: a job offer and employment with probability $\lambda^w$ or unemployment with the remaining probability $1 - \lambda^w$. The unemployed value function $V^w_0$ is recursively defined by:

$$V^w_0 = c + \beta [\lambda^w V^w_1 + (1 - \lambda^w) V^w_0].$$

(15.25)
Subtracting \( \beta V_1^w \) from both sides, the expression can be simplified:

\[
V_0^w (1 - \beta) = c + \beta \lambda^w (V_1^w - V_0^w).
\]  

15.2.4 Firm value functions

The output of any match between a firm and a worker equals \( y \). The firm can either be matched (active) or unmatched (inactive). The profit of a matched firm is equal to \( y - w \), which is equal to output \( y \) minus the labor cost \( w \). The output \( y \) is a parameter of the model, but the wage rate \( w \) is the endogenous outcome of a bargaining process.

There are two value functions for the firm: one corresponding to \( e = 0 \) (no match) and one corresponding to \( e = 1 \) (match). Denote \( V_0^f \) as the unmatched value function and \( V_1^f \) as the matched value function. For the matched value function, the current profit is \( y - w \). In the subsequent period, the matched firm experiences one of two outcomes: unmatched with probability \( \delta \) or matched with the remaining probability \( 1 - \delta \). The matched value function \( V_1^f \) is recursively defined by:

\[
V_1^f = y - w + \beta \left[ \delta V_0^f + (1 - \delta) V_1^f \right].
\]  

This expression can be solved for \( V_1^f \):

\[
V_1^f = \frac{y - w + \beta \delta V_0^f}{1 - \beta (1 - \delta)}.
\]  

Any firm can enter the market by paying the vacancy cost \( k \). For the unmatched value function, the current profit is \(-k\) as the vacancy cost must be paid one period before a potential match can be realized. In the subsequent period, the unmatched firm experiences one of two outcomes: matched with probability \( \lambda^f \) or unmatched with the remaining probability \( 1 - \lambda^f \). The unmatched value function \( V_0^f \) is recursively defined by:

\[
V_0^f = -k + \beta \left[ \lambda^f V_1^f + (1 - \lambda^f) V_0^f \right].
\]  

The free entry condition requires that the expected value of entering the market equals 0:

\[
V_0^f = 0.
\]
If \( V_0^f < 0 \), then no firms would enter and the market tightness (to be introduced shortly) would adjust so that any firm that did enter would make a strictly positive profit. If \( V_0^f > 0 \), then all the potential firms (of mass greater than the mass of workers) would enter and the market tightness would adjust so that any firm in the market would make negative profit. Equilibrium therefore requires that the optimal number of firms enter the market such that \( V_0^f = 0 \).

From the definition of \( V_0^f \), the free entry condition implies:

\[
k = \beta \lambda V_1^f.
\]  

(15.32)

### 15.2.5 Matching

Denote \( u \) as the number of workers searching for a match and \( v \) as the number of firms searching for a match. The total number of matches that will occur is given by the matching technology function \( N(u, v) \). The matching technology function satisfies constant returns to scale, meaning that if the numbers of both workers and firms doubled, then twice as many matches would be formed.

Denote \( \tau = \frac{v}{u} \) as the market tightness. A high value of \( \tau \) is good for workers (the market is tight for firms as there are many firms searching) and a low value of \( \tau \) is good for firms (the market is tight for workers as there are many workers searching).

If the total number of matches equals \( N(u, v) \), then the probability that any given worker receives a match is equal to \( \lambda^w = \frac{N(u, v)}{u} \), which is simply the total number of matches divided by the total number of workers searching. Likewise, the probability that any firm receives a match is equal to \( \lambda^f = \frac{N(u, v)}{v} \), which is simply the total number of matches divided by the total number of firms searching. Since the matching technology function satisfies constant returns to scale, then \( \frac{N(u, v)}{u} = N(1, \frac{v}{u}) \) and \( \frac{N(u, v)}{v} = \frac{N(1, \frac{u}{v})}{\tau} \). Using the market tightness definition \( \tau = \frac{v}{u} \), then

\[
\lambda^w = N(1, \tau).
\]  

\[
\lambda^f = \frac{N(1, \tau)}{\tau}.
\]  

(15.33)

We use the market tightness variable as the raw numbers of workers and firms searching does not matter, only their ratio.
15.2.6 Bargaining

When a worker and a firm are matched, they bargain over the value of the wage rate \( w \). Workers want a higher wage rate and firms want a lower wage rate. The wage rate is determined as the outcome of Nash bargaining. Nash bargaining is a process by which the surplus between a worker and a firm is split. The surplus for a worker is equal to \( V^w - V'^w \), as this is the gain from reaching agreement. The surplus for the firm is equal to \( V^f - V'^f \), as this is the gain from reaching agreement. The Nash bargaining weight for the firm is equal to \( \frac{2}{1-\eta} \) and the Nash bargaining weight for the worker is equal to \( \frac{1}{1-\eta} \).

In the perfectly equitable situation in which \( \eta = 0.5 \), the surplus is evenly shared between the worker and the firm, meaning that \( V^w - V'^w = V^f - V'^f \).

The optimization problem that determines the wage rate is given by:

\[
\max_{w} \left( V^f - V'^f \right) \left( V^w - V'^w \right)^{1-\eta}.
\] (15.34)

As with any unconstrained maximization problem, the solution is determined via the first order condition. The first order condition of the above problem is given by:

\[
-\eta \left( \frac{V^w - V'^w}{V^f - V'^f} \right)^{1-\eta} + (1 - \eta) \left( \frac{V^f - V'^f}{V^w - V'^w} \right)^{\eta} = 0,
\] (15.35)

which implies the following relation

\[
(1 - \eta) \left( V^f - V'^f \right) = \eta \left( V^w - V'^w \right).
\] (15.36)

15.2.7 Equilibrium

We now start to fill in the equations for the value functions that we found above. Recall \( V'^f = 0 \) and

\[
V^f = \frac{y - w + \beta \delta V^f}{1 - \beta (1 - \delta)} = \frac{y - w}{1 - \beta (1 - \delta)}.
\] (15.37)

Using the equations for \( V^w \) and \( V'^w \), the difference

\[
(V^w - V'^w) = \frac{w - c}{1 - \beta (1 - \delta) + \beta \lambda^w}.
\] (15.38)
15. LABOR MARKET SEARCH THEORY

The Nash bargaining equation is then given by:

\[
(1 - \eta) \left( \frac{y - w}{1 - \beta (1 - \delta)} \right) = \eta \left( \frac{w - c}{1 - \beta (1 - \delta) + \beta \lambda^w} \right). \tag{15.39}
\]

Cross-multiply:

\[
(1 - \eta) (1 - \beta (1 - \delta) + \beta \lambda^w) (y - w) = \eta (1 - \beta (1 - \delta)) (w - c). \tag{15.40}
\]

Bring the term \( \eta (1 - \beta (1 - \delta)) (y - w) \) to the right-hand side and simplify:

\[
(1 - \beta (1 - \delta) + (1 - \eta) \beta \lambda^w) (y - w) = \eta (1 - \beta (1 - \delta)) (y - c). \tag{15.41}
\]

Use the expression \( \lambda^w = N (1, \tau) \) to solve for \( y - w \) as a function of \( y - c \):

\[
y - w = \frac{\eta (1 - \beta (1 - \delta)) (y - c)}{1 - \beta (1 - \delta) + (1 - \eta) \beta N (1, \tau)}. \tag{15.42}
\]

From the free entry condition, \( k = \beta \lambda^f V_1^f \). Recall that \( \lambda^f = \frac{N(1, \tau)}{\tau} \) and

\[
V_1^f = \frac{y - w}{1 - \beta (1 - \delta)}. \tag{15.43}
\]

The updated free entry condition is a fixed point equation in terms of market tightness \( \tau \):

\[
k = \frac{\beta \eta (y - c) \frac{N(1, \tau)}{\tau}}{1 - \beta (1 - \delta) + (1 - \eta) \beta N (1, \tau)}. \tag{15.44}
\]

The fixed point equation is equivalently written as:

\[
k \tau = \frac{\beta (y - c)}{\eta N (1, \tau)} = \frac{1 - \beta (1 - \delta) + (1 - \eta) \beta N (1, \tau)}{\eta N (1, \tau)}. \tag{15.45}
\]

The matching function \( N (1, \tau) \) has the following properties:

1. \( N (1, 0) = 0 \).

2. \( \frac{d}{dt} N (1, \tau) > 0 \). Since \( \tau = \frac{u}{w} \), then an increase in \( \tau \) leads to an increase in the probability that a worker will be matched.

3. \( \frac{d}{ds} \left( \frac{N(1, \tau)}{\tau} \right) < 0 \). Since \( \tau = \frac{\tau}{u} \), then an increase in \( \tau \) leads to a decrease in the probability
that a firm will be matched.

With these three properties, the left-hand side of the fixed point equation above is strictly increasing in $\tau$ and the right-hand side is strictly decreasing in $\tau$. This implies that there exists a unique equilibrium value $\tau$.

In a stationary equilibrium, the number of workers leaving the marketplace must be equal to the number of workers entering the marketplace. Since $u$ is the number of workers searching for a match at any point in time, then $u$ is the unemployment rate. The number of unmatched workers transitioning to employment equals $uN(1, \tau)$ and the number of matched workers transitioning to unemployment equals $\delta(1 - u)$. In a stationary equilibrium, $uN(1, \tau) = \delta(1 - u)$, so the equation for the unemployment rate is equal to:

$$u = \frac{\delta}{N(1, \tau) + \delta}. \quad (15.46)$$

### 15.2.8 Pareto efficiency

There exists a Pareto efficient market tightness $\tau$ that would be chosen by a social planner in order to maximize the total surplus in the economy. The social planner would solve the following recursive problem where the state variable in the current period is the current unemployment rate (number of workers searching for a match):

$$J(u) = \max_{\tau} uc + (1 - u)y - k\tau u + \beta J(u') \quad (15.47)$$

s.t. $u' = u + \delta(1 - u) - uN(1, \tau)$

Notice that the planner’s problem considers an equilibrium in general and not a stationary equilibrium. The number of workers searching next period, $u'$, is equal to the number of workers searching this period, $u$, plus matched workers that lose a job, $\delta(1 - u)$, minus unmatched workers that find a job, $uN(1, \tau)$. The total surplus in the current period is equal to the total income of workers, $uc + (1 - u)w$, plus the total profit of firms, $(1 - u)(y - w)$, minus the entry costs paid by unmatched firms in the market, $kv = k\tau u$. Notice that the total income of workers includes both unmatched and matched workers. The total profit of firms only includes matched firms as unmatched firms earn 0 profit.

The Bellman equation is well-defined if the value function for the social planner is the same on the left-hand side and the right-hand side. Define the mapping $T : CBL[0, 1] \rightarrow$
The set $CBL[0,1]$ is the set of continuous, bounded, and linear functions on the compact domain $[0,1]$. To show that a fixed point exists, namely there exists a unique linear function $J$ such that $TJ = J$, we guess and check that $J$ is a linear function.

Guess that $J$ is a linear function of the form $J(u^0) = a_0 + a_1 u^0$. Plug this into the Bellman equation and take the first order condition:

$$-k u - a_1 u \beta N_2(1, \tau) = 0.$$  

Solving for $k$ yields:

$$k = -a_1 \beta N_2(1, \tau).$$  

The optimal choice $\tau$ is independent of $u$. This means that the Bellman equation can be written as:

$$TJ(u) = uc + (1 - u)y - k \tau u + \alpha_0 \beta + a_1 \beta (u + \delta(1 - u) - u N(1, \tau)).$$  

If we can show that $TJ(u) = a_0 + a_1 u$, then we have found the fixed point $J = TJ$. In the equation

$$a_0 + a_1 u = uc + (1 - u)y - k \tau u + \alpha_0 \beta + a_1 \beta (u + \delta(1 - u) - u N(1, \tau)), $$

there are constant terms and there are terms that multiply $u$. This leads to two equations that must be satisfied (first the constant terms and second the terms that multiply $u$):

$$a_0 = y + \alpha_0 \beta + a_1 \beta \delta.$$  

$$a_1 = c - y - k \tau + a_1 \beta ((1 - \delta) - N(1, \tau)).$$

The second equation can be solved directly for $a_1$:

$$a_1 = -\frac{y - c + k \tau}{1 - \beta(1 - \delta) + \beta N(1, \tau)}.$$
The first equation can then be solved for $a_0$, but we are really only interested in solving the first order condition from above:

$$k = -a_1 \beta N_2 (1, \tau). \quad (15.56)$$

Using the expression for $a_1$:

$$k = \frac{(y - c + k \tau) \beta N_2 (1, \tau)}{1 - \beta (1 - \delta) + \beta N (1, \tau)}. \quad (15.57)$$

This recursive equation in terms of $k$ can be simplified to:

$$\frac{k}{N_2 (1, \tau)} = \frac{\beta (y - c)}{1 - \beta (1 - \delta) + \beta N (1, \tau) - \beta \tau N_2 (1, \tau)}. \quad (15.58)$$

Compare the Pareto efficiency equation to the equilibrium equation under Nash bargaining:

$$\frac{k \tau}{\eta N (1, \tau)} = \frac{\beta (y - c)}{1 - \beta (1 - \delta) + (1 - \eta) \beta N (1, \tau)}. \quad (15.59)$$

The equilibrium under Nash bargaining is Pareto efficient when the so-called Hosios condition is satisfied:

$$\frac{N (1, \tau)}{\tau} = N_2 (1, \tau). \quad (15.60)$$

This condition is exactly the one such that the equilibrium equation under Nash bargaining reduces to the Pareto efficiency equation. Under the Hosios condition, the left-hand sides of both equations trivially coincide. Additionally, under the Hosios condition:

$$\beta N (1, \tau) - \beta \tau N_2 (1, \tau) = \beta N (1, \tau) - \beta \eta N (1, \tau)$$
$$= (1 - \eta) \beta N (1, \tau), \quad (15.61)$$

meaning that the right-hand sides of both equations coincide.

The Hosios condition sets the firms’ bargaining equal to the elasticity of the matching function for firms:

$$\eta = \frac{N_2 (1, \tau)}{N (1, \tau)}. \quad (15.62)$$

In the equilibrium with Nash bargaining, it is only this value of the bargaining weight that is able to support a Pareto efficient equilibrium allocation.
15.3 Competitive search with matching

15.3.1 Sneak peek

Summary

In the previous section, we introduced a search and matching model in which the terms of trade (the labor contract) is determined according to Nash bargaining. This section extends the Mortensen-Pissarides model to allow for competitive search. In competitive search, firms post a menu of different labor contracts that differ according to the wage rate. With different wage rates, the workers have to decide which jobs to apply to. The easiest way to view the setting is containing an infinite number of submarkets, each with a different wage. In each submarket, the submarket tightness is endogenously determined. The submarket tightness will adjust depending upon the wage rate. This happens in response to the following trade-off from the point of view of a worker: (i) a higher-wage job pays more, but (ii) a higher-wage job will also have more workers applying for the position, which reduces the probability of receiving an offer.

In competitive search, we have removed the bargaining friction as workers and firms are able to optimally choose all aspects of the terms of their match contract (rather than being bound to Nash bargaining). Consequently, we derive a very powerful result in which the outcome of competitive search is always consistent with Pareto efficiency. Left to their own devices, namely the ability of workers to search for the jobs that they want, the outcome is identical to what a social planner would achieve.

There are two key aspects of each side of the labor market that allow for such a powerful result to hold. First, the firms operate in a setting with free entry. This means that the expected profit for any firm that enters the market is zero, regardless of which labor contract is offered. Second, the unmatched workers search in a competitive setting in which the tightnesses for all submarkets immediately adjust based upon the aggregate decisions of unmatched workers. Each worker has no effect on the tightness in any submarket as it is only one worker in a continuum of workers searching in that particular submarket. The model contains a continuum of submarkets and a continuum of firms and workers searching in each submarket.

Main takeaways

After completing this section, you will be able to answer the following questions:
15.3. COMPETITIVE SEARCH WITH MATCHING

- When search is competitive, what is the relation between the wage rate and the market tightness?
- Is the outcome of competitive search Pareto efficient?

### 15.3.2 Extending the search and matching model

Wages are no longer determined by bargaining. In the current setting, firms post a continuum of different wage contracts, where each contract differs in terms of the wage being offered. At any wage $w$, the terms of the contract must be such that the free entry condition is satisfied for the firms:

$$k = \beta \lambda^f V^f_1 = \frac{N(1,\tau) \beta (y - w)}{1 - \beta (1 - \delta)}.$$  \hspace{1cm} (15.63)

The free entry condition is a function of $(\tau, w)$.

Define

$$F(\tau, w) = \frac{N(1,\tau)}{\tau} \beta (y - w) - k (1 - \beta (1 - \delta))$$ \hspace{1cm} (15.64)

such that $F(\tau, w) = 0$ when the free entry condition is satisfied. For any given $w$, the marginal effect of a change in $w$ on the market tightness $\tau$ can be evaluated using the Implicit Function Theorem:

$$\tau'(w) = -\frac{F_2(\tau, w)}{F_1(\tau, w)} = -\frac{-\frac{N(1,\tau)}{\tau} \beta}{\beta (y - w) \left[ \frac{N_2(1,\tau)}{\tau} - \frac{N(1,\tau)}{\tau^2} \right]}$$ \hspace{1cm} (15.65)

$$= \frac{N(1,\tau)}{(y - w) \left[ N_2(1,\tau) - \frac{N(1,\tau)}{\tau} \right]}.$$

### 15.3.3 Workers choose which jobs to apply for

Unmatched workers choose which jobs to apply for. Since all jobs are characterized by the posted wages, then workers choose the wages. For a wage $w$, the value function for an unmatched worker is:

$$V^w_0 = c + \beta \left[ N(1,\tau) V^w_1 + (1 - N(1,\tau)) V^w_0 \right].$$ \hspace{1cm} (15.66)

The value function $V^w_0$ is a function of both $(\tau, w)$, where we have already evaluated the derivative of the implicit function $\tau(w)$. Using the previous evaluation of the unmatched
value function $V^w_0 = \frac{c + \beta N(1, \tau)(V^w_1 - V^w_0)}{1 - \beta}$, then the right-hand side can be written only in terms of the difference $V^w_1 - V^w_0$:

$$V^w_0 = \frac{c}{1 - \beta} + \frac{\beta N(1, \tau)(V^w_1 - V^w_0)}{1 - \beta}. \quad (15.67)$$

From the previous section, the difference $V^w_1 - V^w_0$ was evaluated and can be inserted into the above equation:

$$V^w_0 = \frac{c}{1 - \beta} + \frac{\beta N(1, \tau)(w - c)}{(1 - \beta)(1 - \beta(1 - \delta) + \beta N(1, \tau))}. \quad (15.68)$$

All of the wage contracts must, in equilibrium, offer identical value to the workers. If not, supposing that one wage contract offered strictly higher value than the others, than all workers would flock toward that contract. The market tightness $\tau$ would decrease making it more difficult for a worker to receive an offer. This drives down the value until it is in line with the other wage contracts. The possible wages belong to a continuum. Notice that a worker would rather be the only applicant for a job with wage $29.99 than the 10th applicant for a job with wage $30. Since the wages belong to a continuum, then $\frac{\partial V^w_0}{\partial w} = 0$ as a marginal change in $w$ does not lead to a change in the value $V^w_0$.

### 15.3.4 First order condition

When evaluating the first order condition, we only focus on the terms in $V^w_0$ that include either $\tau$ or $w$, so we take the derivative of $\frac{N(1, \tau)(w - c)}{1 - \beta(1 - \delta) + \beta N(1, \tau)}$. The first order condition of this expression is given by:

$$N_2(1, \tau) \tau'(w) \frac{(w - c)}{1 - \beta(1 - \delta) + \beta N(1, \tau)} + \frac{N(1, \tau)}{1 - \beta(1 - \delta) + \beta N(1, \tau)} - \frac{N(1, \tau)(w - c) \beta N_2(1, \tau) \tau'(w)}{(1 - \beta(1 - \delta) + \beta N(1, \tau))^2} = 0. \quad (15.69)$$

This is the equilibrium equation under competitive search. The term $(1 - \beta(1 - \delta) + \beta N(1, \tau))$ cancels out from the denominator and we combine the terms involving $N_2(1, \tau) \tau'(w)$:

$$N_2(1, \tau) \tau'(w)(w - c) \left(1 - \frac{\beta N(1, \tau)}{1 - \beta(1 - \delta) + \beta N(1, \tau)}\right) + N(1, \tau) = 0. \quad (15.70)$$
Simplifying the competitive search equilibrium equation leads to:

\[ N_2 (1, \tau_0) (w - c) \left( \frac{1 - \beta (1 - \delta)}{1 - \beta (1 - \delta) + \beta N (1, \tau)} \right) + N (1, \tau) = 0. \]  

This equation is equivalent to:

\[ N_2 (1, \tau) (\tau (w - c) (1 - \beta (1 - \delta)) + \frac{N (1, \tau)}{\tau' (w)} (1 - \beta (1 - \delta) + \beta N (1, \tau)) = 0. \]  

This equation is equivalent to:

\[
(1 - \beta (1 - \delta)) \left( N_2 (1, \tau) (y - c) - N_2 (1, \tau) (y - w) + \frac{N (1, \tau)}{\tau' (w)} \right) \\
+ \beta N (1, \tau) \frac{N (1, \tau)}{\tau' (w)} = 0.
\]

### 15.3.5 Inserting the Implicit Function Theorem

Recall from the Implicit Function Theorem

\[
\tau' (w) = \frac{N (1, \tau)}{(y - w) \left[ N_2 (1, \tau) - \frac{N (1, \tau)}{\tau} \right]}.
\]

This leads to several expressions involving the derivative \( \tau' (w) \):

1. \( N_2 (1, \tau) (y - w) - \frac{N (1, \tau)}{\tau' (w)} = \frac{N (1, \tau)}{\tau} (y - w). \)

2. \( \frac{N (1, \tau)}{\tau' (w)} = (y - w) \left[ N_2 (1, \tau) - \frac{N (1, \tau)}{\tau} \right]. \)

Using these two facts, the competitive search equilibrium equation is given by:

\[
(1 - \beta (1 - \delta)) \left( N_2 (1, \tau) (y - c) - \frac{N (1, \tau)}{\tau} (y - w) \right) \]

\[
+ \beta N (1, \tau) (y - w) \left[ N_2 (1, \tau) - \frac{N (1, \tau)}{\tau} \right] = 0.
\]
Gathering all the terms involving $\frac{N(1, \tau)}{\tau} (y - w)$ leads to:

$$N_2 (1, \tau) (y - c) (1 - \beta (1 - \delta)) = \frac{N (1, \tau)}{\tau} (y - w) (1 - \beta (1 - \delta) + \beta N (1, \tau) - \beta \tau N_2 (1, \tau)).$$  \quad (15.76)

The expression is better expressed as:

$$\frac{N(1, \tau)}{\tau} (y - w) \frac{N_2 (1, \tau) (1 - \beta (1 - \delta))}{N_2 (1, \tau) (1 - \beta (1 - \delta))} = \frac{y - c}{1 - \beta (1 - \delta) + \beta N (1, \tau) - \beta \tau N_2 (1, \tau)}. \quad (15.77)$$

### 15.3.6 Free entry condition

From the free entry condition for the particular submarket under consideration:

$$k = \frac{N(1, \tau)}{\tau} \beta (y - w).$$ \quad (15.78)

Inserting the free entry parameter leads to the competitive search equilibrium equation:

$$\frac{k}{N_2 (1, \tau)} = \frac{\beta (y - c)}{1 - \beta (1 - \delta) + \beta N (1, \tau) - \beta \tau N_2 (1, \tau)}. \quad (15.79)$$

### 15.3.7 Pareto efficiency

Recall the Pareto efficiency condition:

$$\frac{k}{N_2 (1, \tau)} = \frac{\beta (y - c)}{1 - \beta (1 - \delta) + \beta N (1, \tau) - \beta \tau N_2 (1, \tau)}. \quad (15.80)$$

This proves that the outcome under competitive search is Pareto efficient.

In the social planner problem, there is only one market and the number of workers searching $u$ is taken as given. The solution to the social planner dictates the level of market tightness, namely the number of firms that should be actively searching in the market. The social planner is only concerned with allocation and is not concerned with the wage rate $w$.

In the competitive search setting, there are a continuum of submarkets, each with a different wage rate $w$. As a function of the wage rate, the competitive search outcome specifies a submarket tightness $\tau$. In equilibrium, we now know the entire distribution of wages and submarket tightnesses.
The workers are indifferent between all possible submarkets. It is equivalent to have the workers divide up evenly among all submarkets. This means that the number of workers searching in any particular submarket is known. In equilibrium, the submarket tightnesses are determined, which determines the number of firms that are posting wages in all submarkets. The competitive markets perform the same function as a social planner in each submarket, namely optimally choosing the number of firms that enter the submarket, for a given number of workers searching. The competitive markets use the wage rate $w$ to sort firms in order to support the Pareto efficient allocation.
Bibliography


