Term Structure Targeting, Price Level Indeterminacy, and Taylor Rules*

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Abstract

This paper analyzes the conditions under which term structure targeting is able to uniquely determine the equilibrium path of prices in a stochastic setting. A set of sufficient conditions includes complete markets and monetary policy that adopts stationary targets for the entire term structure. When restricted to Taylor rules, in which the short-term interest rates are targeted as functions of the inflation rates, monetary policy is only able to implement price level determinacy for an extreme set of interest rate targets.

Keywords velocity – fiscal theory of the price level – determinacy – term structure

JEL Classification **E41, E43, E52, E63**

1 Introduction

The ability of monetary policy to uniquely pin down the equilibrium path of prices in a stochastic setting is an important property. Conventional monetary policy may be able to uniquely determine the expected price level, but not the entire vector of stochastic realizations. The resulting price level indeterminacy makes it impossible to analyze the financial

*The authors wish to thank Zhigang Feng, Udara Peiris, and anonymous referees for helpful feedback. R. Paez Villate wishes to thank the Krannert School of Management (Purdue University) for hospitality and financial support from June 2015 - December 2015. Corresponding author: M. Hoelle, address: Purdue University, Department of Economics, 403 W. State Street, West Lafayette, IN 47907 (USA), e-mail address: mhoelle@purdue.edu, telephone: (+1) 765-496-2737. The most recent version of this paper can be found at http://www.matthew-hoelle.com/1/75/resources/document_735_1.pdf effects of monetary policy. Attempting to fill the void, central banks in our setting have access to unconventional monetary policy in the form of term structure targeting. We characterize a strong set of sufficient condition for price level determinacy. When restricted to forms of monetary policy currently utilized by central banks, specifically a nonlinear form of Taylor rules, price level determinacy requires a large spread between the state-contingent bond prices in the future period.

If the price levels are not uniquely determined, fiscal policy has a direct effect on inflation rates. The Fiscal Theory of the Price Level asserts that fiscal policy can be chosen in such a way so as to uniquely determine the path of prices (Leeper 1991; Sims 1994; Woodford 1994; Woodford 1995). Both proponents and critics of the Fiscal Theory agree that monetary policy alone may be insufficient to uniquely pin down the path of prices in a stochastic setting, an outcome referred to as price level indeterminacy. The disagreement between the two sides centers on the role of fiscal policy in restoring price level determinacy: proponents argue that fiscal policy uniquely pins down the path of prices, while critics question the applicability of so-called non-Ricardian fiscal policy (Kocherlakota and Phelan 1999; Buiter 2002; Niepelt 2004; McCallum and Nelson 2006). The theory of non-Ricardian fiscal policy views the government budget constraints as equilibrium conditions that only need to be satisfied along the equilibrium path, rather than economic identities that must be satisfied along all possible paths of prices. The central debate concerns whether or not policies leading to explosive inflation may be supported as equilibria. Fiscal Theory proponents argue that such policies can be ruled out by government commitments off the equilibrium path. Critics respond that these commitments are not credible as they violate the government budget constraint in some period off-the-equilibrium path, and thus that equilibria with explosive inflation are viable predictions of the model.

We, like both sides of this debate before us, recognize that monetary policy may be unable to uniquely determine the path of prices in a stochastic setting. What can be done about this problem? Without a strict adherence to the Fiscal Theory, we acknowledge that price level determinacy may remain present even after all policy has been implemented. This has financial effects in a representative household setting such as ours, and these financial effects translate into real effects when the basic framework of our model is extended to include heterogeneous households. Our paper focuses on the role of term structure targeting for the determinacy of the price level.

We consider a pure-exchange economy with a representative household and stochastic endowment risk. Money is valued via a generalization of the classical cash-in-advance constraints (Clower 1967; Lucas and Stokey 1983). The asset structure consists of bonds of varying maturities. Our first result provides three sufficient conditions under which policies of term structure targeting are able to uniquely determine the path of prices. First, the central bank must adopt stationary targets for the interest rates of the entire term structure. Second, the term structure must have as many different maturities as the number of states of uncertainty. Third, the stationary interest rate targets must be linearly independent across all maturities. Under such conditions, fiscal policy satisfies a strong form of Ricardian equivalence as it has no effect on the price levels. With these three conditions, the price level determinacy results are not surprising as we have explicitly assumed a complete set of financial markets (Adão et al. 2014; Magill and Quinzii 2014). Adding heterogeneous households to the economy would not change the results as the complete asset structure would allow all households to perfectly insure against all states of uncertainty.

Out of the three sufficient conditions, we relax the first and allow central banks to formulate monetary policy in a more realistic fashion using nonlinear versions of otherwise standard Taylor rules. Taylor rules specify a relation between the short-term nominal interest rate and the inflation rate. In the New Keynesian literature (Galí 1992; Sims 1992; Bernanke and Mihov 1998; Christiano et al. 1999; Taylor 1999; Clarida et al. 2000; Woodford 2003; Schmitt-Grohé and Uribe 2004a; Uhlig 2005; Galí 2011; Galí 2015), equilibrium conditions are formulated as log-linear deviations from the zero inflation steady state. Included in the log-linearized system of equilibrium equations is a linear Taylor rule relation between the short-term nominal interest rate and the inflation rate. To capture all effects of indeterminacy, we use the original system of nonlinear equations together with a nonlinear Taylor rule.

In a setting of rational expectations in which households correctly form expectations about future inflation rates, Taylor rules imply a nonlinear relation between current and future bond prices. The mapping from current period bond prices to the vector of next period bond prices is the focus of our determinacy analysis. If the mapping yields a unique vector of bond prices, then the vector of monetary growth rates and inflation rates are uniquely determined. If the mapping yields multiple vectors of bond prices, then these are all consistent with equilibrium as the transversality conditions are trivially satisfied. Ricardian fiscal policy is permitted as the presence of a unique path of prices is unrelated to policy commitments that violate government budget constraints off the equilibrium path.

Focusing on a nonlinear mapping from current period bond prices to future bond prices requires us to incorporate a robust formulation for the demand for money. We adopt a framework in the spirit of Baumol (1952) and Tobin (1956) in which households endogenously choose the velocity of money. Consider velocity as the number of times that households exchange financial assets or market activity for money. In Baumol (1952) and Tobin (1956), each exchange incurs a transactions cost and thereby induces a trade-off between cash balances and the number of exchanges. Karni (1973) extends the analysis by modeling the transactions cost as units of lost time (and lost productivity). Recent literature utilizing this framework (Barnett 1980; Freemand and Kydland 2000; Begonia and Ireland 2014; Lucas and Nicolini 2015) generalizes the definition of money by incorporating multiple forms of money in order to better match empirical movements in monetary aggregates and predict the effects of policy on money demand. In all papers utilizing this framework, the trade-off between the size of the money holdings and the transactions cost from obtaining money implies that the endogenous velocity is an increasing function of the short-term nominal interest rate. This is intuitive since a substitute for money holdings is interest-bearing short-term nominal bonds.[1]

Our model of endogenous velocity is inspired by Sims (1994) and adopts the timing of a classical cash-in-advance model (Clower 1967; Lucas and Stokey 1983). Households in our model are divided into two parts: a storekeeper that remains behind to sell the endowment and a shopper that enters the market to purchase consumption. The household decides how many trips the shopper will make into the market, and incurs a transportation cost for each trip made. The velocity is the number of trips made into the market and is an endogenous choice of the household. Due to the symmetry of the model, the amount of money spent on each shopping trip by the shopper is equal to the amount collected by the storekeeper from selling endowment. In our setting, households are exchanging endowment for money, with a wedge introduced by the timing assumptions. Importantly, and different from the classical cash-in-advance theory, the size of the wedge is influenced by a 2-dimensional household choice: money holdings and shopping trips.

Consistent with models in the framework of Baumol (1952) and Tobin (1956), velocity in our model is proportional to the short-term nominal interest rate. Instead of a production economy as in the framework of Baumol (1952) and Tobin (1956), in which a transactions cost resulting in a loss of time leads to a loss of output, our pure-exchange economy requires that the transactions cost leads to a loss of the physical commodity. We view our transactions cost as a transportation cost, consistent with the notion of an iceberg transport cost from the trade literature. Not pressed with the need to match monetary aggregates in the data, our model remains as simple as possible by only having one physical commodity and one form of money. With this simple framework, we can analyze the entire equilibrium set and not only the steady states as in the recent literature on money demand (Lucas and Nicolini 2015).

[1] Palivos et al. (1993), Jha et al. (2002), Chen and Guo (2008), and Chang et al. (2011) each assume a reduced form relation between money demand and interest rate, without deriving it from first principles.

For our main results, we consider a binomial economy[2] in which two conditions are satisfied: (i) the term structure has as many different maturities as the number of states of uncertainty and (ii) the interest rates are linearly independent across all maturities. Monetary policy is implemented in the form of a Taylor rule, which is a nonlinear relation between the short-term nominal interest rate and the inflation rate. Log-linearizing this nonlinear Taylor rule results in the same Taylor rule employed in the New Keynesian system of log-linearized equilibrium equations. The Taylor rule imposes a recursive and nonlinear relation on the bond prices. If there is a unique mapping from current period bond prices to the vector of next period bond prices, then the path of prices is uniquely determined. For nearly all policy parameters in the Taylor rule, there either exists zero or multiple distinct solutions to the nonlinear system of equations. Within a narrow range of the intercept parameter in the Taylor rule, there exists a unique solution. The unique solution requires a large spread between the state-contingent bond prices in the future period. This means that the realization of household income (the aggregate shock in the model) must have a large effect, through policy interaction, on the resulting bond prices.

The nature of Taylor rules makes them problematic for price level determinacy as they include a feedback mechanism. The Taylor rule provides a relation between current period bond prices and inflation rates. Euler equations provide a relation between current period bond prices and the vector of next period bond prices and inflation rates. Both relations must be jointly satisfied. Even if Taylor rules were to rely on lagged data, implying a relation between the current period bond prices and previous period inflation rates, rational expectations requires that the simultaneous satisfaction of both Taylor rules and Euler equations would require a system of recursive equations involving more than two time periods, as compared to the current model in which the recursive equations only involve successive time periods.

Our analysis, albeit limited to binomial economies, shows that price level determinacy under Taylor rule policy is the exception and not the rule. The conditions required for price level determinacy, namely a large spread between the state-contingent bond prices in the future period, require a volatile time path for bond prices and hyper-responsive monetary policy.

By extending the price level determinacy debate in two directions, first with an endogenous money demand as in in the framework of Baumol (1952) and Tobin (1956) and second with monetary policy chosen in the style of nonlinear Taylor rules, we hope to shed light on the potential role for various forms of unconventional monetary policy, including the policies of term structure targeting modeled in this paper.

[2] In a binomial economy, there are two possible realizations of uncertainty in each period.

This paper is organized as follows. Section 2 introduces the model. Section 3 characterizes equilibrium. Section 4 proves that the price level is determinate under stationary interest rate targets and complete markets. Section 5 considers the special case of a binomial economy and analyzes price level determinacy under Taylor rules. Section 6 considers extensions, Section 7 concludes, and Appendix A contains detailed calculations.

2 The model

The model is inspired by the endogenous velocity model of Sims (1994).

Time is discrete and infinite with time periods $t \in \{0, 1, ...\}$. The filtration of uncertainty follows a one-period Markov process with finite state space $\mathbf{S} = \{1, ..., S\}$. The key assumptions in the model are that the set of all states is finite and that all agents in the model have perfect information about the history of state realizations.

The realized states of uncertainty follow a Markov process characterized by a transition matrix $\Gamma \in \mathbb{R}^{S,S}$ whose elements are $\Gamma(s, s')$ for row s and column s'.

The history of all realizations up to and including the current realization completely characterizes the date-event and is required to uniquely identify the markets, household decisions, and policy choices. Define the history of realizations up to and including the realization s_t in period t as $s^t = (s_0, s_1, ..., s_t)$. Additionally, let $s^{t+j} \succ s^t$ refer to the S^j histories $(s^t, \sigma_1, ..., \sigma_j)_{(\sigma_1,...,\sigma_j) \in \mathbf{S}^j}$ that are realized j periods from the date-event s^t.

2.1 Households

The model considers a unit mass of homogeneous household. In each time period, a single physical commodity is traded and consumed. Denote $c(s^t)$ as the consumption in the date-event s^t. Household preferences are given by:

$$U(c) = E_0 \sum_{t=0}^{\infty} \beta^t \ln \left(c\left(s^t\right)\right). \tag{1}$$

The parameter $\beta \in (0, 1)$ is the discount factor.

The household receives endowments $y(s^t) > 0$ in date-event s^t. We assume that the endowments are stationary. Define the stationary endowment mapping as $\mathbf{y} : \mathbf{S} \to \mathbb{R}_{++}$ such that $y(s^t) = \mathbf{y}(s_t)$ for all date-events.

The model does not permit households to consume their own endowments; it is taboo.

The nominal price level of the commodity is $p(s^t)$ in date-event s^t.

Households can also hold bonds. The available bonds are indexed $j \in \mathbf{J} = \{1, ..., J\}$. The bond holdings chosen in date-event s^t are denoted $b_j(s^t) \in \mathbb{R}$. Bond holdings for households can be either positive (saving) or negative (borrowing). For simplicity, define the portfolio as $b(s^t) = (b_j(s^t))_{j \in \mathbf{J}}$. The nominal payout of the $j-$period bond is 1 in all date-events $s^{t+j} \succ s^t$, and 0 otherwise. The price of the $j-$period bond is $q_j(s^t)$. For simplicity, define the price vector $q(s^t) = (q_j(s^t))_{j \in \mathbf{J}}$.

The short-term nominal interest rate is denoted $i(s^t)$ and satisfies $1 + i(s^t) = \frac{1}{q_1(s^t)}$.

2.2 Monetary structure

In the morning of period t, the household receives the income consisting of real transfers $\tau(s^t)$ from the government, money carried over from the previous period, and bond payouts. Define $m(s^t)$ as the money held by the household at the end of date-event s^t. Define $\omega(s^t)$ as the household income at the beginning of date-event s^t, where the initial period income $\omega(s_0)$ is a parameter of the model. By definition:

$$\omega\left(s^t\right) = p\left(s^t\right)\tau\left(s^t\right) + m(s^{t-1}) + b_1\left(s^{t-1}\right) + \sum_{j>1} q_{j-1}\left(s^t\right) b_j\left(s^{t-1}\right).$$

Notice that the payout for a $j-$period bond is equal to the value (price) for a $(j-1)-$period bond, as the bond is 1 period closer to maturity. For simplicity, define $q_0(s^t) = 1$ for all date-events, meaning that the income can be written as:

$$\omega\left(s^t\right) = p\left(s^t\right)\tau\left(s^t\right) + m(s^{t-1}) + \sum_{j \in \mathbf{J}} q_{j-1}\left(s^t\right) b_j\left(s^{t-1}\right). \tag{2}$$

The household consists of two halves: a "shopper" half that travels to the market to purchase consumption and a "storekeeper" half that stays back at the store in order to sell the endowment.[3] Given the available income, the household decides on the amount of current-period bond holdings $(b_j(s^t))_{j \in \mathbf{J}}$ and the amount of money $\hat{m}(s^t)$ to give to the shopper for the first shopping trip. These choices must satisfy the budget constraint:

$$\sum_{j \in \mathbf{J}} q_j\left(s^t\right) b_j\left(s^t\right) + \hat{m}(s^t) \leq \omega\left(s^t\right). \tag{3}$$

The shopper leaves with $\hat{m}(s^t)$ and the storekeeper stays in the store to sell the endowment.

When the shopper returns to the store with purchased commodity and depleted supplies

[3]The market consists of all other stores in the economy. The key friction in a cash-in-advance model is that there does not exist a central market in which a household can simultaneously trade endowment for consumption (either with or without the use of currency).

of money, one shopping trip has been completed. During this period, the storekeeper has sold endowment with value identical to the amount that the shopper just spent (due to the symmetry of the model). The storekeeper replenishes the shopper's supply of money such that the shopper leaves with $\hat{m}(s^t)$ units of money for the second shopping trip.

The shopper incurs a real transportation cost for each shopping trip, which can be viewed as a cost of transporting purchased goods back to the store. Velocity $v(s^t)$ is the number of shopping trips and is an endogenous choice of the household. Given the parameter $A > 0$, the real transportation cost is equal to the fraction $\frac{v}{A+v}$ of the amount purchased.[4] This means that if $c(s^t)$ units are to be consumed, $c(s^t)\left(\frac{A+v}{A}\right)$ must be purchased as only the fraction $\left(1 - \frac{v}{A+v}\right) = \frac{A}{A+v}$ of the purchased amount remains after the transportation cost is extracted.

Since there are $v(s^t)$ shopping trips, the cash-in-advance constraint for the shopper is:

$$p\left(s^t\right) c\left(s^t\right) \left(\frac{A + v\left(s^t\right)}{A}\right) \leq v\left(s^t\right) \hat{m}\left(s^t\right). \tag{4}$$

This means that the shopper half of the household must take at least $\frac{p\left(s^t\right)c\left(s^t\right)}{v\left(s^t\right)}\left(\frac{A+v\left(s^t\right)}{A}\right)$ units of currency every time it leaves to go shopping in order to spend a total of $p\left(s^t\right)c\left(s^t\right)\left(\frac{A+v\left(s^t\right)}{A}\right)$ units of currency over $v(s^t)$ shopping trips. Every time the shopper goes off and spends $p\left(s^t\right)c\left(s^t\right)\left(\frac{A+v\left(s^t\right)}{A}\right)$ units of currency, the storekeeper is able to earn that exact amount by selling endowment.

After the last shopping trip, the shopper returns to the store.

Define $m(s^t)$ as the amount of money that the entire household has available at the end of the period to carry forward into the next period. By definition,

$$m\left(s^t\right) = \hat{m}\left(s^t\right) - \frac{p\left(s^t\right) c\left(s^t\right)}{v\left(s^t\right)} \left(\frac{A + v\left(s^t\right)}{A}\right) + \frac{p\left(s^t\right) \mathbf{y}\left(s_t\right)}{v\left(s^t\right)}. \tag{5}$$

On the final shopping trip, the shopper left with $\hat{m}(s^t)$ units of currency and spent $\frac{p\left(s^t\right)c\left(s^t\right)}{v\left(s^t\right)}\left(\frac{A+v\left(s^t\right)}{A}\right)$ units of currency. During that final shopping trip, the storekeeper sold the remaining endowment, which earned $\frac{p\left(s^t\right)\mathbf{y}(s_t)}{v\left(s^t\right)}$ units of currency. By symmetry, each trip by the shopper is the exact length of time needed for the storekeeper to sell $\frac{\mathbf{y}(s_t)}{v\left(s^t\right)}$ units of endowment. The

[4] This is the bounded functional form introduced by Sims (1994).

cash-in-advance constraint (4) can then be updated, using the definition (5):

$$m\left(s^{t}\right) \geq \frac{p\left(s^{t}\right) \mathbf{y}\left(s_{t}\right)}{v\left(s^{t}\right)}. \tag{6}$$

Inserting the definitions for $\hat{m}\left(s^{t}\right)$ from (5) and $\omega\left(s^{t}\right)$ from (2) into the original budget constraint (3) yields the updated household budget constraint:

$$\begin{aligned} & \frac{p\left(s^{t}\right) c\left(s^{t}\right)}{v\left(s^{t}\right)}\left(\frac{A+v\left(s^{t}\right)}{A}\right)+m\left(s^{t}\right)+\sum_{j \in \mathbf{J}} q_{j}\left(s^{t}\right) b_{j}\left(s^{t}\right) \\ \leq\ & \frac{p\left(s^{t}\right) \mathbf{y}\left(s_{t}\right)}{v\left(s^{t}\right)}+p\left(s^{t}\right) \tau\left(s^{t}\right)+m(s^{t-1})+\sum_{j \in \mathbf{J}} q_{j-1}\left(s^{t}\right) b_{j}\left(s^{t-1}\right). \end{aligned} \tag{7}$$

Households are permitted to short-sell the nominal bonds, so we require the following implicit debt constraint:

$$\inf_{t, s^{t}}\left(\sum_{j \in \mathbf{J}} q_{j}\left(s^{t}\right) b_{j}\left(s^{t}\right)\right)>-\infty. \tag{8}$$

The household optimization problem is given by:

$$\begin{array}{ll} \underset{\{c(s^t), m(s^t), b(s^t), v(s^t)\}}{max} & \sum_{t=0}^{\infty} \beta^{t} \Gamma\left(s^{t} | s_{0}\right) \ln \left(c\left(s^{t}\right)\right) \\ subj.\ to & \text{budget constraint (7) } \forall t, s^{t} \\ & \text{cash-in-advance constraint (6) } \forall t, s^{t} \\ & \text{implicit debt constraint (8)} \\ & c\left(s^{t}\right) \geq 0,\ m\left(s^{t}\right) \geq 0,\ v\left(s^{t}\right) \geq 0 \end{array}. \tag{9}$$

2.3 Monetary-fiscal authority constraints

The government will be modeled as one combined entity referred to as the monetary-fiscal authority. The monetary and fiscal authority can be combined into one entity when their objectives are perfectly aligned. The monetary-fiscal authority issues the money supply $M\left(s^{t}\right)$ in date-event s^{t}. Using the same bond markets that are available to households, the monetary-fiscal authority issues debt in bonds of all maturities. Denote $B_{j}\left(s^{t}\right) \geq 0$ as the debt issued in terms of the $j-$period bond in date-event s^{t}.

The initial financial obligations of the monetary-fiscal authority are $W\left(s_{0}\right)$, a parameter of the model. The monetary-fiscal authority constraints are given by:

$$\frac{M\left(s_{0}\right)}{p\left(s_{0}\right)}+\sum_{j \in \mathbf{J}} q_{j}\left(s_{0}\right) \frac{B_{j}\left(s_{0}\right)}{p\left(s_{0}\right)}=\tau\left(s_{0}\right)+\frac{W\left(s_{0}\right)}{p\left(s_{0}\right)}. \tag{10}$$

$$\frac{M\left(s^{t}\right)}{p\left(s^{t}\right)}+\sum_{j\in\mathbf{J}}q_{j}\left(s^{t}\right)\frac{B_{j}\left(s^{t}\right)}{p\left(s^{t}\right)}=\tau\left(s^{t}\right)+\frac{M\left(s^{t-1}\right)}{p\left(s^{t}\right)}+\sum_{j\in\mathbf{J}}q_{j-1}\left(s^{t}\right)\frac{B_{j}\left(s^{t-1}\right)}{p\left(s^{t}\right)}. \qquad (11)$$

For simplicity, define the portfolio as $B\left(s^{t}\right)=\left(B_{j}\left(s^{t}\right)\right)_{j\in\mathbf{J}}\in\mathbb{R}_{+}^{J}$.

2.4 Sequential competitive equilibrium

A sequential competitive equilibrium (SCE) is the household variables $\left\{c\left(s^{t}\right),m\left(s^{t}\right),b\left(s^{t}\right),v\left(s^{t}\right)\right\}$, the monetary-fiscal authority variables $\left\{B\left(s^{t}\right),M\left(s^{t}\right),\tau\left(s^{t}\right)\right\}$, and the price variables $\left\{p\left(s^{t}\right),q\left(s^{t}\right)\right\}$ such that:

1. Given $\left\{p\left(s^{t}\right),q\left(s^{t}\right),\tau\left(s^{t}\right)\right\}$, the household chooses $\left\{c\left(s^{t}\right),m\left(s^{t}\right),b\left(s^{t}\right),v\left(s^{t}\right)\right\}$ to solve the household problem (9).
2. The monetary-fiscal authority variables $\left\{B\left(s^{t}\right),M\left(s^{t}\right),\tau\left(s^{t}\right)\right\}$ satisfy the monetary-fiscal authority constraints (10) and (11).
3. Markets clear:
   (a) $c\left(s^{t}\right)=\mathbf{y}\left(s_{t}\right)\left(\frac{A}{A+v\left(s^{t}\right)}\right)$ for every t,s^{t}.
   (b) $m\left(s^{t}\right)=M\left(s^{t}\right)$ for every t,s^{t}.
   (c) $b_{j}\left(s^{t}\right)=B_{j}\left(s^{t}\right)$ for every j and for every t,s^{t}.

Using standard arguments, a SCE always exists. Velocity $v\left(s^{t}\right)$ is bounded above since consumption $c\left(s^{t}\right)\rightarrow 0$ as $v\left(s^{t}\right)\rightarrow\infty$ (from market clearing), which is not optimal for the household given the utility function. Since $p\left(s^{t}\right)>0$ and $m\left(s^{t}\right)$ is bounded above (from the market clearing condition and the monetary-fiscal authority constraint under the requirement that the debt positions $B\left(s^{t}\right)$ are nonnegative), then $v\left(s^{t}\right)>0$ in equilibrium.

In equilibrium, the cash-in-advance constraint binds:

$$m\left(s^{t}\right)=\frac{p\left(s^{t}\right)\mathbf{y}\left(s_{t}\right)}{v\left(s^{t}\right)}. \qquad (12)$$

This is the Quantity Theory of Money, which can be equivalently written as (using the commodity market clearing conditions):

$$m\left(s^{t}\right)=\frac{p\left(s^{t}\right)c\left(s^{t}\right)}{v\left(s^{t}\right)}\left(\frac{A+v\left(s^{t}\right)}{A}\right). \qquad (13)$$

3 Equilibrium characterization

3.1 First order conditions: Consumption and velocity

The first order conditions are evaluated for $v(s^t) > 0$, a necessary condition for equilibrium. The first order condition with respect to consumption $c(s^t)$ is given by:

$$\frac{\beta^t \Gamma(s^t|s_0)}{c(s^t)} = \frac{\lambda(s^t) p(s^t)}{v(s^t)} \left(\frac{A + v(s^t)}{A}\right), \tag{14}$$

where $\lambda(s^t)$ is the Lagrange multiplier associated with the household budget constraint (7). In equilibrium, $\lambda(s^t) > 0$. Using the fact that the cash-in-advance constraint is binding, (14) is equivalently expressed as:

$$\beta^t \Gamma\left(s^t|s_0\right) = \lambda\left(s^t\right) m\left(s^t\right). \tag{15}$$

The first order condition with respect to velocity $v(s^t)$ is given by:

$$\lambda\left(s^t\right) \left\{ \frac{p(s^t) \mathbf{y}(s_t)}{(v(s^t))^2} - \frac{p(s^t) c(s^t)}{(v(s^t))^2} \right\} = \mu\left(s^t\right) m\left(s^t\right), \tag{16}$$

where $\mu(s^t)$ is the Lagrange multiplier associated with the cash-in-advance constraint (6).

Combining (15) and (16), together with the fact that the cash-in-advance constraint is binding, yields:

$$\frac{\beta^t \Gamma(s^t|s_0)}{m(s^t)} \frac{1}{A + v(s^t)} = \mu\left(s^t\right). \tag{17}$$

3.2 Euler equations: Money

The first order condition with respect to money $m(s^t)$ is given by:

$$\lambda\left(s^t\right) = \mu_m\left(s^t\right) + \mu\left(s^t\right) v\left(s^t\right) + \sum_{\sigma \in \mathbf{S}} \lambda\left(s^t, \sigma\right), \tag{18}$$

where $\mu_m(s^t)$ is the Lagrange multiplier associated with the nonnegativity constraint $m(s^t) \geq 0$.

Using (15) and (17):

$$\frac{\beta^t \Gamma(s^t|s_0)}{m(s^t)} \frac{A}{A + v(s^t)} = \mu_m\left(s^t\right) + \sum_{\sigma \in \mathbf{S}} \frac{\beta^{t+1} \Gamma(s^t, \sigma|s_0)}{m(s^t, \sigma)}.$$

From the Quantity Theory of Money (12), since $p(s^t) > 0$ and endowments are strictly positive, a boundary solution $m(s^t) = 0$ corresponds to the case of $v(s^t) = \infty$. This is not possible as the velocity $v(s^t)$ is bounded above in equilibrium. This implies that $m(s^t) > 0$ and $\mu_m(s^t) = 0$.

Therefore, the Euler equation for money (18) is given by:

$$\frac{A}{A + v(s^t)} = \beta \sum_{\sigma \in \mathbf{S}} \Gamma(s_t, \sigma) \frac{m(s^t)}{m(s^t, \sigma)}. \tag{19}$$

3.3 Euler equations: Bonds

The first order condition with respect to the 1-period bond $b_1(s^t)$ is given by:

$$q_1\left(s^t\right) \lambda\left(s^t\right) = \sum_{\sigma \in \mathbf{S}} \lambda\left(s^t, \sigma\right). \tag{20}$$

Using (15) and (17), we derive the Euler equation for the short-term bond:

$$q_1\left(s^t\right) = \beta \sum_{\sigma \in \mathbf{S}} \Gamma(s_t, \sigma) \frac{m(s^t)}{m(s^t, \sigma)}. \tag{21}$$

From the Euler equation for money (19), this implies:

$$q_1\left(s^t\right) = \frac{A}{A + v(s^t)}.$$

By the definition of the short-term nominal interest rate $i(s^t)$, it must be that

$$v\left(s^t\right) = Ai\left(s^t\right).$$

The equilibrium velocity is a strictly increasing (and linear) function of the short-term nominal interest rate.

The first order condition with respect to the $j-$period bond $b_j(s^t)$ is given by:

$$q_j\left(s^t\right) \lambda\left(s^t\right) = \sum_{\sigma \in \mathbf{S}} q_{j-1}\left(s^t, \sigma\right) \lambda\left(s^t, \sigma\right). \tag{22}$$

Using (15) and (17), we derive the Euler equation for the long-term bonds:

$$q_j\left(s^t\right) = \beta \sum_{\sigma \in \mathbf{S}} \Gamma(s_t, \sigma) \, q_{j-1}\left(s^t, \sigma\right) \frac{m(s^t)}{m(s^t, \sigma)}. \tag{23}$$

4 Stationary interest rate targets

Under a policy of stationary interest rate targeting, the monetary-fiscal authority will target stationary interest rates for bonds of all maturities. Define the stationary asset price mappings $\mathbf{q}_j : \mathbf{S} \rightarrow \mathbb{R}_+$ such that $q_j(s^t) = \mathbf{q}_j(s_t)$ for all assets and all date-events.

Since $q_1(s^t) = \frac{A}{A+v(s^t)}$ is strictly decreasing in $v(s^t)$, there exists a stationary mapping $\mathbf{v} : \mathbf{S} \rightarrow \mathbb{R}_+$ such that $v(s^t) = \mathbf{v}(s_t)$ for all date-events, where $\mathbf{v}(s_t)$ solves the following equation:

$$\mathbf{q}_1(s_t) = \frac{A}{A + \mathbf{v}(s_t)} \quad \forall s_t \in \mathbf{S}. \tag{24}$$

From the commodity market clearing conditions, there exists a stationary mapping $\mathbf{c} : \mathbf{S} \rightarrow \mathbb{R}_+$ such that $c(s^t) = \mathbf{c}(s_t)$ for all date-events, where

$$\mathbf{c}(s_t) = \mathbf{y}(s_t) \frac{A}{A + \mathbf{v}(s_t)} \; \forall s_t \in \mathbf{S}. \tag{25}$$

Recall the Euler equation for the 1-period bond (20):

$$\mathbf{q}_1(s_t) = \beta \sum_{\sigma \in \mathbf{S}} \Gamma(s_t, \sigma) \frac{m(s^t)}{m(s^t, \sigma)}.$$

Defining $\alpha(s^t, \sigma) = \frac{m(s^t,\sigma)}{m(s^t)}$, it must be the case that there exists a mapping $\boldsymbol{\alpha} : \mathbf{S} \times \mathbf{S} \rightarrow \mathbb{R}_+$ such that $\boldsymbol{\alpha}(s^t) = \boldsymbol{\alpha}(s_{t-1}, s_t)$ for all date-events. With this stationary mapping, the Euler equation for the 1-period bond (20) is given by:

$$\mathbf{q}_1(s_t) = \beta \sum_{\sigma \in \mathbf{S}} \Gamma(s_t, \sigma) \frac{1}{\boldsymbol{\alpha}(s_t, \sigma)}.$$

The Euler equation for bond $j > 1$ is given by:

$$\mathbf{q}_j(s_t) = \beta \sum_{\sigma \in \mathbf{S}} \Gamma(s_t, \sigma) \frac{\mathbf{q}_{j-1}(\sigma)}{\boldsymbol{\alpha}(s_t, \sigma)}.$$

4.1 Matrix representation

Define the $S \times S$ matrix $\hat{\Gamma}$ as containing the elements $\hat{\Gamma}(s_t, \sigma) = \frac{\Gamma(s_t,\sigma)}{\boldsymbol{\alpha}(s_t,\sigma)}$. The matrix contains the probability transition parameters and the S^2 unknowns $(\boldsymbol{\alpha}(s, \sigma))_{(s,\sigma) \in \mathbf{S}^2}$. Define the bond price matrices:

$$Q_1^J = \begin{bmatrix} \mathbf{q}_1(1) & \mathbf{q}_2(1) & ... & \mathbf{q}_J(1) \\ : & : & & : \\ \mathbf{q}_1(S) & \mathbf{q}_2(S) & ... & \mathbf{q}_J(S) \end{bmatrix}.$$

$$Q_0^{J-1} = \begin{bmatrix} 1 & \mathbf{q}_1(1) & ... & \mathbf{q}_{J-1}(1) \\ : & : & & : \\ 1 & \mathbf{q}_1(S) & ... & \mathbf{q}_{J-1}(S) \end{bmatrix}.$$

If the short-term interest rates were constant, i.e., $\mathbf{q}_1(s) = \mathbf{q}_1(1)$ for all states $s \in \mathbf{S}$, then $(\mathbf{q}_1(s))_{s\in\mathbf{S}} \propto \overrightarrow{1}$ and from the Euler equations $(\mathbf{q}_j(s))_{s\in\mathbf{S}} \propto \overrightarrow{1}$ for any $j > 1$. The bond payouts would be linearly dependent.

Linear independence requires, at a minimum, that $\mathbf{q}_1(s) \neq \mathbf{q}_1(\sigma)$ for some states $s, \sigma \in \mathbf{S}$. The general assumption, though closely related to the weaker assumption that $\mathbf{q}_1(s) \neq \mathbf{q}_1(\sigma)$ for some states $s, \sigma \in \mathbf{S}$, is stated as Assumption 1.

Assumption 1 The matrix Q_0^{J-1} has full column rank, where

$$Q_0^{J-1} = \begin{bmatrix} 1 & \mathbf{q}_1(1) & ... & \mathbf{q}_{J-1}(1) \\ : & : & & : \\ 1 & \mathbf{q}_1(S) & ... & \mathbf{q}_{J-1}(S) \end{bmatrix}.$$

The second assumption requires that the number of assets must be large enough to span all states of uncertainty.

Assumption 2 $J = S$.

Given the matrix definitions, the Euler equations for all $(s,\sigma)_{(s,\sigma)\in\mathbf{S}^2}$ are given by:

$$Q_1^J = \beta\hat{\Gamma}Q_0^{J-1}.$$

Under Assumptions 1 and 2, a unique vector $(\boldsymbol{\alpha}(s,\sigma))_{(s,\sigma)\in\mathbf{S}^2}$ is defined according to:

$$\hat{\Gamma} = \beta^{-1}Q_1^J\left[Q_0^{J-1}\right]^{-1}.$$

Note that the money supply process is determined for any stationary interest rate targets satisfying Assumptions 1 and 2.

4.2 Price level determination

With the monetary growth rates uniquely determined, the price level is uniquely determined from our version of the Quantity Theory of Money (13):

$$p\left(s^t\right) = \frac{m\left(s^t\right)\mathbf{v}\left(s_t\right)}{\mathbf{c}\left(s_t\right)}\frac{A}{A+\mathbf{v}\left(s_t\right)}. \tag{26}$$

5 Binomial economies and Taylor rules

5.1 Binomial economies

This section considers binomial economies, which are economies with only $S = 2$ states of uncertainty. To satisfy Assumption 2 (complete markets), we assume $J = 2$ assets.

An economy is characterized by the following parameters: Markov transition matrix Γ, endowment vector $(\mathbf{y}(1), \mathbf{y}(2))$, discount factor β, and transportation cost parameter A. Denote $\Gamma = \begin{bmatrix} \Gamma_{11} & 1-\Gamma_{11} \\ 1-\Gamma_{22} & \Gamma_{22} \end{bmatrix}$. Suppose, without loss of generality, that $\mathbf{y}(1) < \mathbf{y}(2)$.

The example economy in this section is such that $\Gamma = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}$, $\frac{\mathbf{y}(2)}{\mathbf{y}(1)} = 2$, $\beta = 0.9$, and $A = 100$.[5]

5.2 Taylor rules

Define $\Pi\left(s^t\right) = \frac{p\left(s^t\right)}{p\left(s^{t-1}\right)}$ as the relative price level and $X\left(s^t\right) = \left(q_1\left(s^t\right), q_2\left(s^t\right), \alpha\left(s^t\right), \Pi\left(s^t\right)\right)$ as the vector of contemporaneous variables.

The monetary policy rules considered in this section are as follows:

$$\begin{aligned} q_1\left(s^t\right)\left(\Pi\left(s^t\right)\right)^{\theta_1} &= K_1. \\ q_2\left(s^t\right) &= F_2\left(q_1\left(s^t\right), \Pi\left(s^t\right), X\left(s^{t-1}\right), ..., X\left(s_0\right)\right). \end{aligned} \tag{27}$$

The coefficients (θ_1, K_1) are the Taylor rule policy parameters. The function F_2 is a general monetary policy rule such that $q_2\left(s^t\right)$ is uniquely determined as a function of $\left(q_1\left(s^t\right), \Pi\left(s^t\right)\right)$ and the history of variables.

Since $\Pi\left(s^t\right)$ as the relative price level, we can define $\pi\left(s^t\right)$ as the inflation rate such that

$$\Pi\left(s^t\right) = 1 + \pi\left(s^t\right).$$

[5]For the analysis, only the endowment ratio matters. The levels have no effect on prices.

New Keynesian models take log-linear approximations around the zero inflation steady state in order to derive their equilibrium equations. We adhere to this methodology and adopt the approximations $i(s^t) = \ln(1 + i(s^t))$ and $\pi(s^t) = \ln(\Pi(s^t)) = \ln(1 + \pi(s^t))$. Using the relation $\kappa_1 = -\ln(K_1)$, the log-linearization of the Taylor rule in (27) is consistent with the Taylor rules adopted in the New Keynesian literature:

$$i\left(s^t\right) = \kappa_1 + \theta_1 \pi\left(s^t\right). \tag{28}$$

The following assumption is consistent with the Taylor principle required for determinacy in the New Keynesian framework.

Assumption 3 $\quad \theta_1 > 1.$

While the Taylor principle is not a theoretical requirement in our model, we do acknowledge that it is an empirical regularity and thereby utilize Assumption 3. The example in this section considers $\theta_1 = 1.4$.

5.3 Equilibrium system of equations

5.3.1 Endogenous velocity case

Consider the price level determination in period t. The Taylor rule equations for the 1-period bond are given by:

$$\begin{aligned} q_1\left(s^{t-1},1\right)\left(\Pi\left(s^{t-1},1\right)\right)^{\theta_1} &= K_1. \\ q_1\left(s^{t-1},2\right)\left(\Pi\left(s^{t-1},2\right)\right)^{\theta_1} &= K_1. \end{aligned} \tag{29}$$

If the 1-period bond prices $(q_1(s^{t-1},1), q_1(s^{t-1},2))$ are uniquely determined, then the Taylor rule equations for the 2-period bond uniquely determine the 2-period bond prices.

The relative price levels are found from the Quantity Theory of Money:

$$\begin{aligned} \Pi\left(s^{t-1},1\right) &= \alpha\left(s^{t-1},1\right)\frac{\nu\left(s^{t-1},1\right)}{\nu\left(s^{t-1}\right)}\frac{\mathbf{y}\left(s_{t-1}\right)}{\mathbf{y}\left(1\right)}. \\ \Pi\left(s^{t-1},2\right) &= \alpha\left(s^{t-1},2\right)\frac{\nu\left(s^{t-1},2\right)}{\nu\left(s^{t-1}\right)}\frac{\mathbf{y}\left(s_{t-1}\right)}{\mathbf{y}\left(2\right)}. \end{aligned} \tag{30}$$

The monetary growth rates are found from inverting the Euler equations:

$$\left(q_1\left(s^{t-1}\right), q_2\left(s^{t-1}\right)\right) = \left(\frac{\beta\Gamma\left(s_{t-1},1\right)}{\alpha\left(s^{t-1},1\right)}, \frac{\beta\Gamma\left(s_{t-1},2\right)}{\alpha\left(s^{t-1},2\right)}\right)\begin{bmatrix}1 & q_1\left(s^{t-1},1\right) \\ 1 & q_1\left(s^{t-1},2\right)\end{bmatrix}. \tag{31}$$

From the optimal condition for velocity:

$$\begin{aligned}\nu\left(s^{t-1},1\right) &= Ai\left(s^{t-1},1\right)=\frac{A\left(1-q_1\left(s^{t-1},1\right)\right)}{q_1\left(s^{t-1},1\right)}. \\ \nu\left(s^{t-1},2\right) &= Ai\left(s^{t-1},2\right)=\frac{A\left(1-q_1\left(s^{t-1},2\right)\right)}{q_1\left(s^{t-1},2\right)}.\end{aligned} \tag{32}$$

The system of equations (29) determines the 2 unknown variables $(q_1(s^{t-1},1), q_1(s^{t-1},2))$ as a mapping from the previous period variables $(s_{t-1}, q_1(s^{t-1}), q_2(s^{t-1}))$ and all parameters $(\Gamma, \mathbf{y}(1), \mathbf{y}(2), \beta, A)$.

We seek to determine if there exists policy parameters (θ_1, K_1) such that the price variables $(q_1(s^{t-1},1), q_1(s^{t-1},2))$ are uniquely determined as a function of $(s_{t-1}, q_1(s^{t-1}), q_2(s^{t-1}))$. If the 1-period bond prices are uniquely determined, then the Taylor rule for the 2-period bonds uniquely determines the 2-period bond prices. If all bond prices are uniquely determined, then the monetary growth rates and inflation rates are uniquely determined. In such scenarios, the economy exhibits price level determinacy.

5.3.2 Constant velocity case

It is important in this model to determine velocity endogenously. This will be evident by comparing the results in our model of endogenous velocity to an otherwise identical model with constant velocity. If the model is such that the velocity is constant in all time periods, the Taylor rule equations (29) and the Euler equations (31) remain unchanged, while the Quantity Theory of Money equations (30) are updated as:

$$\begin{aligned}\Pi\left(s^{t-1},1\right) &= \alpha\left(s^{t-1},1\right)\frac{\mathbf{y}\left(s_{t-1}\right)}{\mathbf{y}\left(1\right)}. \\ \Pi\left(s^{t-1},2\right) &= \alpha\left(s^{t-1},2\right)\frac{\mathbf{y}\left(s_{t-1}\right)}{\mathbf{y}\left(2\right)}.\end{aligned} \tag{33}$$

5.4 Numerical example

The parameters are given by $\Gamma = \begin{bmatrix}0.9 & 0.1\\ 0.1 & 0.9\end{bmatrix}$, $\frac{\mathbf{y}(2)}{\mathbf{y}(1)} = 2$, $\beta = 0.9$, and $A = 100$. We fix the policy parameter $\theta_1 = 1.4$. We fix the state $s_{t-1} = 1$ and the previous period bond prices $(q_1(s^{t-1}), q_2(s^{t-1})) = (0.9, 0.8)$.

5.4.1 Constant velocity case

We first analyze the baseline case of constant velocity. Define the ratio $z = \frac{q_1\left(s^{t-1},1\right)}{q_1\left(s^{t-1},2\right)}$. The monetary authority can find a value for the policy parameter K_1 that guarantees a unique solution for $\left(q_1\left(s^{t-1},1\right), q_1\left(s^{t-1},2\right)\right)$ only when $z > 1.146$. Define

$$\begin{aligned}
\xi'(1) &= \beta\Gamma\left(s_{t-1},1\right)\frac{\mathbf{y}\left(s_{t-1}\right)}{\mathbf{y}(1)}. \\
\xi'(2) &= \beta\Gamma\left(s_{t-1},2\right)\frac{\mathbf{y}\left(s_{t-1}\right)}{\mathbf{y}(2)}.
\end{aligned}$$

As a function of z, the bond prices are such that:

$$\begin{aligned}
q_1\left(s^{t-1},2\right) &= \frac{q_2\left(s^{t-1}\right)}{q_1\left(s^{t-1}\right)}\frac{\left(\xi'(1)+\xi'(2)\,z^{-\frac{1}{\theta}}\right)}{\left(\xi'(1)\,z+\xi'(2)\,z^{-\frac{1}{\theta}}\right)}. \\
q_1\left(s^{t-1},1\right) &= zq_1\left(s^{t-1},2\right).
\end{aligned}$$

When the spread is the smallest, i.e., $z = 1.146$, then $\left(q_1\left(s^{t-1},1\right), q_1\left(s^{t-1},2\right)\right) = (0.894, 0.780)$. For values $z > 1.146$, the bond prices $q_1\left(s^{t-1},1\right) > 0.894$ and $q_1\left(s^{t-1},2\right) < 0.780$. For this reason, 0.894 is a lower bound for $q_1\left(s^{t-1},1\right)$ and 0.780 is an upper bound for $q_1\left(s^{t-1},2\right)$.

The details surrounding this calculation are contained in Appendix A.1.

5.4.2 Endogenous velocity case

The same definition for z is used for this analysis. The monetary authority can find a value for the policy parameter K_1 that guarantees a unique solution for $\left(q_1\left(s^{t-1},1\right), q_1\left(s^{t-1},2\right)\right)$ only when $z \in \left[\frac{q_2\left(s^{t-1}\right)}{q_1\left(s^{t-1}\right)}, 0.895\right)$, where $\frac{q_2\left(s^{t-1}\right)}{q_1\left(s^{t-1}\right)} = \frac{8}{9}$. Define

$$\begin{aligned}
\xi(1) &= \frac{\beta\Gamma\left(s_{t-1},1\right)}{i\left(s^{t-1}\right)}\frac{\mathbf{y}\left(s_{t-1}\right)}{\mathbf{y}(1)}. \\
\xi(2) &= \frac{\beta\Gamma\left(s_{t-1},2\right)}{i\left(s^{t-1}\right)}\frac{\mathbf{y}\left(s_{t-1}\right)}{\mathbf{y}(2)}.
\end{aligned} \tag{34}$$

As a function of z, the bond price $q_1\left(s^{t-1},2\right)$ is found as a solution to the following quadratic equation:

$$a\left(q_1\left(s^{t-1},2\right)\right)^2 + bq_1\left(s^{t-1},2\right) + c = 0,$$

where the coefficients are given by:

$$
\begin{aligned}
a &= q_1\left(s^{t-1}\right)\left(\xi(1)\,z^2+\xi(2)\,z^{\frac{\theta-1}{\theta}}\right). \\
b &= -\left(q_1\left(s^{t-1}\right)+q_2\left(s^{t-1}\right)\right)\left(\xi(1)\,z+\xi(2)\,z^{\frac{\theta-1}{\theta}}\right). \\
c &= q_2\left(s^{t-1}\right)\left(\xi(1)+\xi(2)\,z^{\frac{\theta-1}{\theta}}\right).
\end{aligned}
$$

By definition, the bond price $q_1\left(s^{t-1},1\right)=zq_1\left(s^{t-1},2\right)$.

At the upper bound $z=0.895$, the bond prices are $\left(q_1\left(s^{t-1},1\right),q_1\left(s^{t-1},2\right)\right)=(0.888,0.991)$. For values $z\in\left[\frac{q_2\left(s^{t-1}\right)}{q_1\left(s^{t-1}\right)},0.895\right)$, the bond prices $q_1\left(s^{t-1},1\right)<0.888$ and $q_1\left(s^{t-1},2\right)>0.991$. For this reason, 0.888 is an upper bound for $q_1\left(s^{t-1},1\right)$ and 0.991 is a lower bound for $q_1\left(s^{t-1},2\right)$. The details surrounding this calculation are contained in Appendix A.2.

Comparison to the constant velocity case It is important to note two importance differences (and these are robust for changes in the parameter values):

1. The range of values of z for which a unique solution exists is much larger under constant velocity compared to endogenous velocity.

2. Under constant velocity, a unique solution exists only when $q_1\left(s^{t-1},1\right)>q_1\left(s^{t-1},2\right)$. Under endogenous velocity, a unique solution exists only when $q_1\left(s^{t-1},1\right)<q_1\left(s^{t-1},2\right)$.

Robustness check The economy considers values $\left(s_{t-1},q_1\left(s^{t-1}\right),q_2\left(s^{t-1}\right)\right)=(1,0.9,0.8)$ and $\left(\Gamma_{11},\frac{\mathbf{y}(2)}{\mathbf{y}(1)},\theta_1\right)=(0.9,2,1.4)$. When $s_{t-1}=1$, a unique solution occurs only if $z\in\left[\frac{q_2\left(s^{t-1}\right)}{q_1\left(s^{t-1}\right)},z^*\right)$, where the values for z^* depend only on $\left(q_1\left(s^{t-1}\right),q_2\left(s^{t-1}\right)\right)$ and $\left(\Gamma_{11},\frac{\mathbf{y}(2)}{\mathbf{y}(1)},\theta_1\right)$. The findings, i.e., the range in which a unique solution exists, is independent of (β,A).

If $s_{t-1}=2$, a unique solution exists only if $z\in\left(z^{**},\frac{q_1\left(s^{t-1}\right)}{q_2\left(s^{t-1}\right)}\right]$, where the values for z^{**} depend only on $\left(q_1\left(s^{t-1}\right),q_2\left(s^{t-1}\right)\right)$ and $\left(\Gamma_{22},\frac{\mathbf{y}(2)}{\mathbf{y}(1)},\theta_1\right)$. There is a lower bound for $q_1\left(s^{t-1},1\right)$ and an upper bound for $q_1\left(s^{t-1},2\right)$. The size of the spreads is quantitatively similar to the case with $s_{t-1}=1$.

Tables I-V tabulate the range of values for z under which a unique solution exists. The range is such that $z\in\left[\frac{q_2\left(s^{t-1}\right)}{q_1\left(s^{t-1}\right)},z^*\right)$ for varying values of z^*. The tables also compute the lower bound for $q_1\left(s^{t-1},2\right)$ and the upper bound for $q_1\left(s^{t-1},1\right)$. Both of these values are computed as the solutions when $z=z^*$. In the tables below, the baseline economy values are **bold**.

Holding fixed $\left(\frac{\mathbf{y}(2)}{\mathbf{y}(1)}, \theta_1, q_1\left(s^{t-1}\right), q_2\left(s^{t-1}\right)\right) = (2, 1.4, 0.9, 0.8)$, the effect of a change in Γ_{11} is minimal (see Table I).

Γ_{11}	$z \in \left[\frac{q_2\left(s^{t-1}\right)}{q_1\left(s^{t-1}\right)}, z^*\right)$	Lower bound for $q_1\left(s^{t-1}, 2\right)$	Upper bound for $q_1\left(s^{t-1}, 1\right)$
0.9	**0.896**	**0.991**	**0.888**
0.7	0.894	0.992	0.887
0.5	0.892	0.994	0.886

Table I

Holding fixed $\left(\Gamma_{11}, \theta_1, q_1\left(s^{t-1}\right), q_2\left(s^{t-1}\right)\right) = (0.9, 1.4, 0.9, 0.8)$, the effect of a change in $\frac{\mathbf{y}(2)}{\mathbf{y}(1)}$ is minimal (see Table II).

$\frac{\mathbf{y}(2)}{\mathbf{y}(1)}$	$z \in \left[\frac{q_2\left(s^{t-1}\right)}{q_1\left(s^{t-1}\right)}, z^*\right)$	Lower bound for $q_1\left(s^{t-1}, 2\right)$	Upper bound for $q_1\left(s^{t-1}, 1\right)$
1	0.896	0.991	0.888
2	**0.896**	**0.991**	**0.888**
4	0.896	0.992	0.889

Table II

Holding fixed $\left(\Gamma_{11}, \frac{\mathbf{y}(2)}{\mathbf{y}(1)}, q_1\left(s^{t-1}\right), q_2\left(s^{t-1}\right)\right) = (0.9, 2, 0.9, 0.8)$, the effect of a change in θ_1 is minimal (see Table III).

θ_1	$z \in \left[\frac{q_2\left(s^{t-1}\right)}{q_1\left(s^{t-1}\right)}, z^*\right)$	Lower bound for $q_1\left(s^{t-1}, 2\right)$	Upper bound for $q_1\left(s^{t-1}, 1\right)$
1.1	0.898	0.989	0.888
1.4	**0.896**	**0.991**	**0.888**
1.7	0.895	0.993	0.889

Table III

Holding fixed $\left(\Gamma_{11}, \frac{\mathbf{y}(2)}{\mathbf{y}(1)}, \theta_1, q_2\left(s^{t-1}\right)\right) = (0.9, 2, 1.4, 0.8)$, the effect of a change in $q_1\left(s^{t-1}\right)$ is minimal on the size of the range $\left[\frac{q_2(s^{t-1})}{q_1(s^{t-1})}, z^*\right)$, but scales the bond prices (see Table IV).

$q_1(s^{t-1})$	$z \in \left[\frac{q_2(s^{t-1})}{q_1(s^{t-1})}, z^*\right)$	Lower bound for $q_1(s^{t-1},2)$	Upper bound for $q_1(s^{t-1},1)$
0.86	0.933	0.997	0.930
0.9	**0.896**	**0.991**	**0.888**
0.94	0.864	0.984	0.850
0.98	0.836	0.976	0.815

Table IV

Holding fixed $\left(\Gamma_{11}, \frac{\mathbf{y}(2)}{\mathbf{y}(1)}, \theta_1, q_1(s^{t-1})\right) = (0.9, 2, 1.4, 0.9)$, the effect of a change in $q_2(s^{t-1})$ is minimal on the size of the range $\left[\frac{q_2(s^{t-1})}{q_1(s^{t-1})}, z^*\right)$, but scales the bond prices (see Table V).

$q_2(s^{t-1})$	$z \in \left[\frac{q_2(s^{t-1})}{q_1(s^{t-1})}, z^*\right)$	Lower bound for $q_1(s^{t-1},2)$	Upper bound for $q_1(s^{t-1},1)$
0.72	0.823	0.971	0.799
0.76	0.859	0.983	0.844
0.8	**0.896**	**0.991**	**0.888**
0.84	0.936	0.997	0.933

Table V

6 Extensions to general economies

With binomial economies and complete markets ($S = J = 2$), the form for the monetary policy rule determining the 2-period bond prices $(q_2(s^{t-1},1), q_2(s^{t-1},2))$ has no bearing on the presence of price level determinacy. All that is required is that the 2-period bond prices are uniquely determined as functions of the 1-period bond prices $(q_1(s^{t-1},1), q_1(s^{t-1},2))$, the relative price levels $(\Pi(s^{t-1},1), \Pi(s^{t-1},2))$, and any variables from previous periods. This is a minimal requirement for any form of monetary policy as it only requires that current policy targets are well-defined functions of the history of variables up until that point.

With binomial economies and complete markets ($S = J = 2$), the relative price levels $(\Pi(s^{t-1},1), \Pi(s^{t-1},2))$ are independent of the 2-period bond prices $(q_2(s^{t-1},1), q_2(s^{t-1},2))$. It is only the 1-period bond prices $(q_1(s^{t-1},1), q_1(s^{t-1},2))$, along with the relative price levels $(\Pi(s^{t-1},1), \Pi(s^{t-1},2))$, that are present in the recursive system of equilibrium equations.

With more general economies under complete markets ($S = J > 2$), the form of term structure targeting, namely how the monetary policy rule is specified for 2-period, 3-period, and all the way up to $(S-1)-$period bond prices, has consequences for price level determinacy.

The extension to a trinomial economy with complete markets ($S = J = 3$) contains all mechanisms present in more general extensions. The system of equations includes the Taylor rules together with the Quantity Theory of Money, the Euler equations, and the optimal conditions for velocity. The latter 3 types of equations are given in (30), (31), and (32), respectively. The Euler equations, in particular, provide a relation between the relative price levels $\left(\Pi\left(s^{t-1}, s_t\right)\right)_{s_t\in\{1,2,3\}}$ and both the current 1-period bond prices $\left(q_1\left(s^{t-1}, s_t\right)\right)_{s_t\in\{1,2,3\}}$ and the 2-period bond prices $\left(q_2\left(s^{t-1}, s_t\right)\right)_{s_t\in\{1,2,3\}}$:

$$\left(q_1\left(s^{t-1}\right), q_2\left(s^{t-1}\right), q_3\left(s^{t-1}\right)\right) = \left(\frac{\beta\Gamma\left(s_{t-1},1\right)}{\alpha\left(s^{t-1},1\right)}, \frac{\beta\Gamma\left(s_{t-1},2\right)}{\alpha\left(s^{t-1},2\right)}, \frac{\beta\Gamma\left(s_{t-1},3\right)}{\alpha\left(s^{t-1},3\right)}\right) \begin{bmatrix} 1 & q_1\left(s^{t-1},1\right) & q_2\left(s^{t-1},1\right) \\ 1 & q_1\left(s^{t-1},2\right) & q_2\left(s^{t-1},2\right) \\ 1 & q_1\left(s^{t-1},3\right) & q_2\left(s^{t-1},3\right) \end{bmatrix}.$$

Suppose that the Taylor rule relationship for the 1-period bond prices is maintained as in (29):

$$q_1\left(s^{t-1}, s_t\right)\left(\Pi\left(s^{t-1}, s_t\right)\right)^{\theta_1} = K_1 \;\; \forall s_t \in \{1,2,3\}.$$

Assume that the 3-period bond prices are uniquely determined as a function of the other contemporaneous bond prices, the relative price levels, and the history of variables:

$$q_3\left(s^t\right) = F_3\left(q_1\left(s^t\right), q_2\left(s^t\right), \Pi\left(s^t\right), X\left(s^{t-1}\right), ..., X\left(s_0\right)\right).$$

Under such conditions, the monetary policy rule for the 3-period bond prices does not have any effects on price level determinacy.

The key dimension for analysis is the monetary policy rules for the 2-period bond prices:

$$q_2\left(s^t\right) = F_2\left(q_1\left(s^t\right), \Pi\left(s^t\right), X\left(s^{t-1}\right), ..., X\left(s_0\right)\right).$$

In the simplest scenario, the monetary policy rule F_2 may be an extension of the Taylor rule used for the 1-period bond:

$$q_2\left(s^{t-1}, s_t\right)\left(\Pi\left(s^{t-1}, s_t\right)\right)^{\theta_2} = K_2 \;\; \forall s_t \in \{1,2,3\}.$$

Provided that $\theta_2 \neq \theta_1$, an equilibrium is a solution to a system of recursive equations in terms of 6 variables: $\left(q_1\left(s^{t-1}, s_t\right), q_2\left(s^{t-1}, s_t\right)\right)_{s_t\in\{1,2,3\}}$.[6] Provided $\theta_2 \neq \theta_1$, the role of term

[6] If $\theta_2 = \theta_1$, the system of recursive equations is only in terms of the 1-period bond prices $\left(q_1\left(s^{t-1}, s_t\right)\right)_{s_t\in\{1,2,3\}}$ as the 2-period bond prices are linear functions of the 1-period bond prices.

structure targeting has effects on price level determinacy above and beyond what is predicted for conventional monetary policies that only target the 1-period bond prices.

7 Concluding remarks

This paper has considered a simple extension of the Sims (1994) model with endogenous velocity to allow for monetary policy to target the entire term structure of interest rates. If monetary policy adopts stationary targets for the entire term structure and a complete set of financial instruments is available, the price level is determinate. Monetary policy, however, is typically implemented through Taylor rules. We adopt Taylor rules consistent with those implemented in the log-linearized solutions of New Keynesian models. Under such Taylor rules, price level determinacy only occurs for economies with large spreads in the state-contingent bond prices. Endogenous velocity plays an important role in exacerbating the price level indeterminacy problem.

Given the findings of this paper, the Fiscal Theory of the Price Level, namely the reliance on fiscal policy to determine the price level, appears to be a fundamental feature of monetary economies. Price level determinacy requires either a strong assumption on the ability of the central bank to implement monetary policy, namely through stationary targets for the entire term structure, or a large spread in the state-contingent short-term interest rates when Taylor rules are implemented. In an extension of the model with heterogeneous households and financial frictions, such as incomplete markets, we conjecture that the price level indeterminacy would manifest into real indeterminacy. For this reason, it is important to understand which features of the model and what forms of monetary policy are consistent with price level determinacy. Such knowledge is essential for an analysis of the effects, both real and financial, of unconventional monetary policy.

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A Appendix

A.1 Constant velocity determinacy analysis

For simplicity, denote $\theta = \theta_1$, $(q(1), q(2)) = (q_1(s^{t-1}, 1), q_1(s^{t-1}, 2))$, and

$$\begin{aligned} \xi'(1) &= \beta\Gamma(s_{t-1}, 1)\frac{\mathbf{y}(s_{t-1})}{\mathbf{y}(1)}. \\ \xi'(2) &= \beta\Gamma(s_{t-1}, 2)\frac{\mathbf{y}(s_{t-1})}{\mathbf{y}(2)}. \end{aligned} \tag{35}$$

The vector of stochastic monetary growth rates is uniquely determined only if Assumption 1 is satisfied. Under Assumption 1, $Q_0^{J-1}(s^{t-1}) = \begin{bmatrix} 1 & q_1(s^{t-1}, 1) \\ 1 & q_1(s^{t-1}, 2) \end{bmatrix}$ has full rank, implying that $q_1(s^{t-1}, 1) \neq q_1(s^{t-1}, 2)$. Inverting the Euler equations (31) and solving for $(\alpha(s^{t-1}, 1), \alpha(s^{t-1}, 2))$:

$$\begin{aligned} \alpha\left(s^{t-1}, 1\right) &= \beta\Gamma(s_{t-1}, 1)\left(\frac{q_1(s^{t-1}, 2) - q_1(s^{t-1}, 1)}{q_1(s^{t-1})\,q_1(s^{t-1}, 2) - q_2(s^{t-1})}\right). \\ \alpha\left(s^{t-1}, 2\right) &= \beta\Gamma(s_{t-1}, 2)\left(\frac{q_1(s^{t-1}, 2) - q_1(s^{t-1}, 1)}{-q_1(s^{t-1})\,q_1(s^{t-1}, 1) + q_2(s^{t-1})}\right). \end{aligned} \tag{36}$$

From (30) and (36), the system of equations (29) is given by:

$$\begin{aligned} q\left(1\right)\left(\xi'\left(1\right)\left(\frac{q(2)-q(1)}{q_1\left(s^{t-1}\right)q(2)-q_2\left(s^{t-1}\right)}\right)\right)^{\theta} &= K_1. \\ q\left(2\right)\left(\xi'\left(2\right)\left(\frac{q(2)-q(1)}{-q_1\left(s^{t-1}\right)q(1)+q_2\left(s^{t-1}\right)}\right)\right)^{\theta} &= K_1. \end{aligned} \tag{37}$$

Dividing the first equation by the second equation yields the following equality:

$$\left(\frac{q(1)}{q(2)}\right)^{-\frac{1}{\theta}} = \frac{\xi'\left(1\right)\left(-q_1\left(s^{t-1}\right)q(1)+q_2\left(s^{t-1}\right)\right)}{\xi'\left(2\right)\left(q_1\left(s^{t-1}\right)q(2)-q_2\left(s^{t-1}\right)\right)}. \tag{38}$$

Define $z=\frac{q(1)}{q(2)}$. By definition, $q(1)=zq(2)$, meaning (38) is updated as:

$$z^{-\frac{1}{\theta}} = \frac{\xi'\left(1\right)\left(-q_1\left(s^{t-1}\right)zq(2)+q_2\left(s^{t-1}\right)\right)}{\xi'\left(2\right)\left(q_1\left(s^{t-1}\right)q(2)-q_2\left(s^{t-1}\right)\right)}. \tag{39}$$

From equation (39), we can solve for $q(2)$ as a function of z :

$$q(2) = \frac{q_2\left(s^{t-1}\right)\left(\xi'\left(1\right)+\xi'\left(2\right)z^{-\frac{1}{\theta}}\right)}{q_1\left(s^{t-1}\right)\left(\xi'\left(1\right)z+\xi'\left(2\right)z^{-\frac{1}{\theta}}\right)}. \tag{40}$$

Since $q(2)\leq 1$, then

$$\frac{q_2\left(s^{t-1}\right)\left(\xi'\left(1\right)+\xi'\left(2\right)z^{-\frac{1}{\theta}}\right)}{q_1\left(s^{t-1}\right)\left(\xi'\left(1\right)z+\xi'\left(2\right)z^{-\frac{1}{\theta}}\right)} \leq 1. \tag{41}$$

By definition,

$$q(1) = zq(2) = \frac{q_2\left(s^{t-1}\right)\left(\xi'\left(1\right)z+\xi'\left(2\right)z^{\frac{\theta-1}{\theta}}\right)}{q_1\left(s^{t-1}\right)\left(\xi'\left(1\right)z+\xi'\left(2\right)z^{-\frac{1}{\theta}}\right)}.$$

Since $q(1)\leq 1$, then

$$\frac{q_2\left(s^{t-1}\right)\left(\xi'\left(1\right)z+\xi'\left(2\right)z^{\frac{\theta-1}{\theta}}\right)}{q_1\left(s^{t-1}\right)\left(\xi'\left(1\right)z+\xi'\left(2\right)z^{-\frac{1}{\theta}}\right)} \leq 1. \tag{42}$$

The inequalities (41) and (42) narrow the range of equilibrium values for z. For the economy parameters provided, the range of equilibrium values is $z\geq\tilde{z}$, where $\tilde{z}<1$. This finding is robust for all economies with $s_{t-1}=1$. The result is symmetric for economies with $s_{t-1}=2$, i.e., the range of equilibrium values is $z\leq\tilde{z}$, where $\tilde{z}>1$.

Using the equation (40) for $q(2)$, the second equation in the system of equations (37) simplifies to:

$$\frac{zq_2\left(s^{t-1}\right)}{q_1\left(s^{t-1}\right)}\frac{\left(\xi'\left(1\right)+\xi'\left(2\right)z^{-\frac{1}{\theta}}\right)^{1+\theta}}{\left(\xi'\left(1\right)z+\xi'\left(2\right)z^{-\frac{1}{\theta}}\right)}=K_1. \tag{43}$$

The task is to determine if the monetary authority can choose values for the policy parameter K_1 such that a unique solution for z to equation (43) exists.

Lemma 1 *The left-hand side of (43) is strictly increasing for $z<1$ and strictly decreasing for $z>1$.*

For the economy parameters considered, there exist two solutions for $z\in[\tilde{z},1)\cup(1,z^*]$ and a unique solution for $z=1$ and $z>z^*$. Assumption 1 requires $q_1\left(s^{t-1},1\right)\neq q_1\left(s^{t-1},2\right)$, meaning that $z\neq1$. Thus, a unique solution only occurs for $z>z^*$, where $z^*>1$.

The value z^* determines the lower bound for z under which a unique solution exists. For the bond prices $q_1\left(s^{t-1},1\right)$, the lower bound under which a unique solution exists occurs when $z=z^*$. For the bond prices $q_1\left(s^{t-1},2\right)$, the upper bound under which a unique solution exists occurs when $z=z^*$.

A.1.1 Proof of Lemma 1

Denote LHS as the left-hand side of (43), which is a function only of the variable z. Define

$$\begin{aligned}
T1 &= q_1\left(s^{t-1}\right)q_2\left(s^{t-1}\right)\left(\xi'\left(1\right)z+\xi'\left(2\right)z^{-\frac{1}{\theta}}\right)\left(\xi'\left(1\right)+\xi'\left(2\right)z^{-\frac{1}{\theta}}\right)^{1+\theta}.\\
T2 &= \frac{1+\theta}{\theta}q_1\left(s^{t-1}\right)q_2\left(s^{t-1}\right)\xi'\left(2\right)z^{-\frac{1}{\theta}}\left(\xi'\left(1\right)z+\xi'\left(2\right)z^{-\frac{1}{\theta}}\right)\left(\xi'\left(1\right)+\xi'\left(2\right)z^{-\frac{1}{\theta}}\right)^{\theta}.\\
T3 &= q_1\left(s^{t-1}\right)q_2\left(s^{t-1}\right)\left(\xi'\left(1\right)+\xi'\left(2\right)z^{-\frac{1}{\theta}}\right)^{1+\theta}\left(\xi'\left(1\right)z-\frac{1}{\theta}\xi'\left(2\right)z^{-\frac{1}{\theta}}\right).
\end{aligned}$$

Using the quotient rule:

$$\frac{dLHS(z)}{dz}=\frac{T1-T2-T3}{\left[q_1\left(s^{t-1}\right)\left(\xi'\left(1\right)z+\xi'\left(2\right)z^{-\frac{1}{\theta}}\right)\right]^2}.$$

The term

$$T1-T3=\frac{1+\theta}{\theta}q_1\left(s^{t-1}\right)q_2\left(s^{t-1}\right)\xi'\left(2\right)z^{-\frac{1}{\theta}}\left(\xi'\left(1\right)+\xi'\left(2\right)z^{-\frac{1}{\theta}}\right)^{1+\theta}.$$

This means that

$$T1-T3-T2=\left(\frac{1+\theta}{\theta}q_1\left(s^{t-1}\right)q_2\left(s^{t-1}\right)\xi'\left(2\right)z^{-\frac{1}{\theta}}\right)\left(\xi'\left(1\right)+\xi'\left(2\right)z^{-\frac{1}{\theta}}\right)^{\theta}\xi'\left(1\right)\left(1-z\right).$$

When $z < 1$, then $T1 - T3 - T2 > 0$ and $\frac{dLHS(z)}{dz} > 0$. When $z > 1$, then $T1 - T3 - T2 < 0$ and $\frac{dLHS(z)}{dz} < 0$.

A.2 Endogenous velocity determinacy analysis

For simplicity, denote $\theta = \theta_1$, $(q(1), q(2)) = (q_1(s^{t-1}, 1), q_1(s^{t-1}, 2))$, and

$$\begin{aligned} \xi(1) &= \frac{\beta\Gamma(s_{t-1}, 1)}{i(s^{t-1})} \frac{\mathbf{y}(s_{t-1})}{\mathbf{y}(1)}. \\ \xi(2) &= \frac{\beta\Gamma(s_{t-1}, 2)}{i(s^{t-1})} \frac{\mathbf{y}(s_{t-1})}{\mathbf{y}(2)}. \end{aligned} \tag{44}$$

From (30), (36), and (32), the Taylor rules (29) for the 1-period bond in each state $s_t \in \{1, 2\}$ are given by:

$$\begin{aligned} q(1)\left(\xi(1)\frac{1-q(1)}{q(1)}\left(\frac{q(2)-q(1)}{q_1(s^{t-1})q(2)-q_2(s^{t-1})}\right)\right)^{\theta} &= K_1. \\ q(2)\left(\xi(2)\frac{1-q(2)}{q(2)}\left(\frac{q(2)-q(1)}{-q_1(s^{t-1})q(1)+q_2(s^{t-1})}\right)\right)^{\theta} &= K_1. \end{aligned} \tag{45}$$

Dividing the first equation of (45) by the second yields the following equality:

$$\left(\frac{q(1)}{q(2)}\right)^{\frac{\theta-1}{\theta}} = \frac{\xi(1)(1-q(1))(-q_1(s^{t-1})q(1)+q_2(s^{t-1}))}{\xi(2)(1-q(2))(q_1(s^{t-1})q(2)-q_2(s^{t-1}))}. \tag{46}$$

Define $z = \frac{q(1)}{q(2)}$. By definition, $q(1) = zq(2)$, meaning (46) is updated as:

$$z^{\frac{\theta-1}{\theta}} = \frac{\xi(1)(1-zq(2))(-q_1(s^{t-1})zq(2)+q_2(s^{t-1}))}{\xi(2)(1-q(2))(q_1(s^{t-1})q(2)-q_2(s^{t-1}))}. \tag{47}$$

From equation (47), we can solve for $q(2)$ as a function of z :

$$z^{\frac{\theta-1}{\theta}}\xi(2)(1-q(2))\left(q_1\left(s^{t-1}\right)q(2)-q_2\left(s^{t-1}\right)\right) = \xi(1)(1-zq(2))\left(-q_1\left(s^{t-1}\right)zq(2)+q_2\left(s^{t-1}\right)\right).$$

The equation in terms of $q(2)$ is in the form of a quadratic equation

$$a(q(2))^2 + bq(2) + c = 0$$

with coefficients given by:

$$
\begin{aligned}
a &= q_1\left(s^{t-1}\right)\left(\xi(1)\, z^2 + \xi(2)\, z^{\frac{\theta-1}{\theta}}\right). \\
b &= -\left(q_1\left(s^{t-1}\right) + q_2\left(s^{t-1}\right)\right)\left(\xi(1)\, z + \xi(2)\, z^{\frac{\theta-1}{\theta}}\right). \\
c &= q_2\left(s^{t-1}\right)\left(\xi(1) + \xi(2)\, z^{\frac{\theta-1}{\theta}}\right).
\end{aligned}
$$

Notice that $a > 0$, $b < 0$, and $c > 0$. There exists a real solution to the quadratic equation provided that $b^2 \geq 4ac$, which is equivalent to:

$$
\left(q_1\left(s^{t-1}\right) - q_2\left(s^{t-1}\right)\right)^2\left(\xi(1)\, z + \xi(2)\, z^{\frac{\theta-1}{\theta}}\right)^2 \geq 4\xi(1)\,\xi(2)\, q_1\left(s^{t-1}\right) q_2\left(s^{t-1}\right) z^{\frac{\theta-1}{\theta}}\,(z-1)^2. \tag{48}
$$

Define

$$
q_-(2) = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad q_+(2) = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.
$$

Since $a > 0$, $b < 0$, and $c > 0$, then $0 < q_-(2) \leq \frac{-b}{2a} \leq q_+(2)$.

A.2.1 Range of equilibrium variables

To analyze price level determinacy, we focus on economies in which the following conditions are satisfied.

Condition 1 $\quad \frac{\Gamma_{11}}{1-\Gamma_{11}} > \frac{\mathbf{y}(1)}{\mathbf{y}(2)} \frac{q_1\left(s^{t-1}\right)}{q_2\left(s^{t-1}\right)}$.

Condition 2 $\quad \frac{\Gamma_{22}}{1-\Gamma_{22}} > \frac{\mathbf{y}(2)}{\mathbf{y}(1)} \frac{q_1\left(s^{t-1}\right)}{q_2\left(s^{t-1}\right)}$.

Condition 3 $\quad \sqrt{\frac{\theta_1-1}{\theta_1}} < \left(2 - \frac{q_2\left(s^{t-1}\right)}{q_1\left(s^{t-1}\right)}\right) \frac{q_2\left(s^{t-1}\right)}{q_1\left(s^{t-1}\right)}$.

We isolate the range of values of z under which there exists a unique solution $q(2)$ for the given value of z.

Case 1: $z < \frac{q_2\left(s^{t-1}\right)}{q_1\left(s^{t-1}\right)}$. The sum of the coefficients

$$
\begin{aligned}
a + b + c &= \xi(1)\left(q_1\left(s^{t-1}\right)\left(z^2 - z\right) - q_2\left(s^{t-1}\right)(z-1)\right) \\
&= \xi(1)\,(z-1)\left(q_1\left(s^{t-1}\right) z - q_2\left(s^{t-1}\right)\right).
\end{aligned}
$$

Since $z < \frac{q_2\left(s^{t-1}\right)}{q_1\left(s^{t-1}\right)}$, then $a + b + c > 0$. This implies

$$
\begin{aligned}
4a\,(a + b + c) &> 0. \\
4a^2 + 4ab + b^2 &> b^2 - 4ac.
\end{aligned}
$$

Notice that $4a^2 + 4ab + b^2 = (2a+b)^2$. This implies that

$$\sqrt{b^2 - 4ac} < +(2a+b) \quad \text{or} \quad \sqrt{b^2 - 4ac} < -(2a+b).$$

One of the inequalities must hold. If $2a + b \geq 0$, then $\sqrt{b^2 - 4ac} < (2a+b)$ implies $q_+(2) = \frac{-b+\sqrt{b^2-4ac}}{2a} < 1$ and $q_+(1) = zq_+(2) < 1$. There are then multiple solutions ($q_-(2)$ and $q_-(1) = zq_-(2)$ are also solutions). If $2a + b < 0$, then $\sqrt{b^2 - 4ac} < -(2a+b)$ implies $q_-(2) = \frac{-b-\sqrt{b^2-4ac}}{2a} > 1$. There do not exist any solutions in this scenario. In either scenario, there does not exist a unique solution for Case 1.

Case 2: $z \in \left[\frac{q_2(s^{t-1})}{q_1(s^{t-1})}, 1\right)$.

A real solution exists When $z \in \left[\frac{q_2(s^{t-1})}{q_1(s^{t-1})}, 1\right)$, there exists a real solution as $b^2 \geq 4ac$. First, we know that

$$\left(q_1\left(s^{t-1}\right) - q_2\left(s^{t-1}\right)\right)^2 \left(\xi(1) z + \xi(2) z^{\frac{\theta-1}{\theta}}\right)^2 \geq \left(q_1\left(s^{t-1}\right) - q_2\left(s^{t-1}\right)\right)^2 4\xi(1)\xi(2) z z^{\frac{\theta-1}{\theta}}$$

by the fact that

$$\left(q_1\left(s^{t-1}\right) - q_2\left(s^{t-1}\right)\right)^2 \left(\xi(1) z - \xi(2) z^{\frac{\theta-1}{\theta}}\right)^2 \geq 0.$$

Second, since $z \geq \frac{q_2(s^{t-1})}{q_1(s^{t-1})}$, then both of the inequalities below hold:

$$\begin{aligned} \left(q_1\left(s^{t-1}\right) - q_2\left(s^{t-1}\right)\right)^2 4\xi(1)\xi(2) z z^{\frac{\theta-1}{\theta}} &\geq \left(q_1\left(s^{t-1}\right) - q_2\left(s^{t-1}\right)\right)^2 4\xi(1)\xi(2) \frac{q_2\left(s^{t-1}\right)}{q_1\left(s^{t-1}\right)} z^{\frac{\theta-1}{\theta}} \\ &\geq 4\xi(1)\xi(2) q_1\left(s^{t-1}\right) q_2\left(s^{t-1}\right) z^{\frac{\theta-1}{\theta}} (z-1)^2. \end{aligned}$$

This verifies that inequality (48) is satisfied.

Unique solution for a given z The sum of the coefficients

$$\begin{aligned} a+b+c &= \xi(1)\left(q_1\left(s^{t-1}\right)\left(z^2 - z\right) - q_2\left(s^{t-1}\right)(z-1)\right) \\ &= \xi(1)(z-1)\left(q_1\left(s^{t-1}\right) z - q_2\left(s^{t-1}\right)\right). \end{aligned}$$

Since $z \in \left[\frac{q_2(s^{t-1})}{q_1(s^{t-1})}, 1\right)$, then $a + b + c \leq 0$. This implies

$$\sqrt{b^2 - 4ac} \geq +(2a+b) \text{ and } \sqrt{b^2 - 4ac} \geq -(2a+b).$$

If $2a+b \geq 0$, then the inequality of interest is $\sqrt{b^2-4ac} \geq 2a+b$, which implies $q_+(2) = \frac{-b+\sqrt{b^2-4ac}}{2a} \geq 1$. Additionally, $\frac{-b}{2a} < 1$ and so $q_-(2) < 1$. Since $z < 1$, then $q_-(1) = zq_-(2) < 1$.

If $2a+b < 0$, then $\frac{-b}{2a} > 1$ and so $q_+(2) > 1$. Additionally, $\sqrt{b^2-4ac} \geq -2a-b$ implies $q_-(2) = \frac{-b-\sqrt{b^2-4ac}}{2a} \leq 1$. Since $z < 1$, then $q_-(1) = zq_-(2) < 1$.

For any $z \in \left[\frac{q_2(s^{t-1})}{q_1(s^{t-1})}, 1\right)$, $q_-(2) \in (0,1)$ and $q_-(1) = zq_-(2) \in (0,1)$ are the only economic solutions of the quadratic equation.

Case 3: $z \in \left(1, \frac{q_1(s^{t-1})}{q_2(s^{t-1})}\right]$. Since $z > 1$, then $q_-(1) = zq_-(2) > q_-(2)$ and $q_+(1) = zq_+(2) > q_+(2)$. The task is to determine the magnitudes of $q_-(1)$ and $q_+(1)$. By definition:

$$q_-(1) = \frac{-bz - \sqrt{b^2z^2 - 4acz^2}}{2a} \quad \text{and } q_+(1) = \frac{-bz + \sqrt{b^2z^2 - 4acz^2}}{2a}.$$

Define the new coefficients such that

$$\begin{aligned} \tilde{a} &= a. \\ \tilde{b} &= bz. \\ \tilde{c} &= cz^2. \end{aligned}$$

With these coefficients, $q_-(1)$ and $q_+(1)$ are solutions to the quadratic equation

$$\tilde{a}\,(q(1))^2 + \tilde{b}q(1) + \tilde{c} = 0$$

and given by:

$$q_-(1) = \frac{-\tilde{b} - \sqrt{\tilde{b}^2 - 4\tilde{a}\tilde{c}}}{2\tilde{a}} \quad \text{and } q_+(1) = \frac{-\tilde{b} + \sqrt{\tilde{b}^2 - 4\tilde{a}\tilde{c}}}{2\tilde{a}}.$$

Since $\tilde{a} > 0$, $\tilde{b} < 0$, and $\tilde{c} > 0$, then $0 < q_-(1) \leq \frac{-\tilde{b}}{2\tilde{a}} \leq q_+(1)$.

A real solution exists There exists a real solution provided that $\tilde{b}^2 \geq 4\tilde{a}\tilde{c}$. By construction, $\tilde{b}^2 \geq 4\tilde{a}\tilde{c}$ whenever $b^2 \geq 4ac$.

When $z \in \left(1, \frac{q_1(s^{t-1})}{q_2(s^{t-1})}\right]$, there exists a real solution as $b^2 \geq 4ac$. When $z \in \left(1, 2 - \frac{q_2(s^{t-1})}{q_1(s^{t-1})}\right]$, a similar argument as in Case 2 is applied and the inequality (48) is satisfied as

$$\begin{aligned} 4\xi(1)\xi(2)q_1(s^{t-1})q_2(s^{t-1})z^{\frac{\theta-1}{\theta}}(z-1)^2 &\leq 4\xi(1)\xi(2)\frac{q_2(s^{t-1})}{q_1(s^{t-1})}z^{\frac{\theta-1}{\theta}}\left(q_1(s^{t-1}) - q_2(s^{t-1})\right)^2 \\ &\leq 4\xi(1)\xi(2)zz^{\frac{\theta-1}{\theta}}\left(q_1(s^{t-1}) - q_2(s^{t-1})\right)^2. \end{aligned}$$

When $z \in \left(2-\frac{q_2\left(s^{t-1}\right)}{q_1\left(s^{t-1}\right)}, \frac{q_1\left(s^{t-1}\right)}{q_2\left(s^{t-1}\right)}\right]$, inequality (48) is equivalent to:

$$\left(\xi(1) z-\xi(2) z^{\frac{\theta-1}{\theta}}\right)^2 \geq 4 \xi(1) \xi(2) z^{\frac{\theta-1}{\theta}}\left(\frac{q_1\left(s^{t-1}\right)}{q_2\left(s^{t-1}\right)}-z\right). \tag{49}$$

Define the function

$$h(z)=\left(\xi(1) z-\xi(2) z^{\frac{\theta-1}{\theta}}\right)^2-4 \xi(1) \xi(2) z^{\frac{\theta-1}{\theta}}\left(\frac{q_1\left(s^{t-1}\right)}{q_2\left(s^{t-1}\right)}-z\right).$$

At both $z=2-\frac{q_2\left(s^{t-1}\right)}{q_1\left(s^{t-1}\right)}$ and $z=\frac{q_1\left(s^{t-1}\right)}{q_2\left(s^{t-1}\right)}$, $h(z) \geq 0$ and there exists a real solution. If the function $h(z)$ is monotonic in the range $z \in \left(2-\frac{q_2\left(s^{t-1}\right)}{q_1\left(s^{t-1}\right)}, \frac{q_1\left(s^{t-1}\right)}{q_2\left(s^{t-1}\right)}\right]$, then there exists a real solution for all $z \in \left(2-\frac{q_2\left(s^{t-1}\right)}{q_1\left(s^{t-1}\right)}, \frac{q_1\left(s^{t-1}\right)}{q_2\left(s^{t-1}\right)}\right]$. The derivative

$$h'(z)=2\left(\xi(1) z-\xi(2) z^{\frac{\theta-1}{\theta}}\right)\left(\xi(1)-\frac{\theta-1}{\theta} \xi(2) z^{-\frac{1}{\theta}}\right)+4 \xi(1) \xi(2) z^{-\frac{1}{\theta}}\left(z-\frac{\theta-1}{\theta} \frac{q_1\left(s^{t-1}\right)}{q_2\left(s^{t-1}\right)}\right).$$

We claim that $h'(z)>0$, which holds iff $\frac{h'(z) z}{2}>0$, where

$$\frac{h'(z) z}{2}=(\xi(1) z)^2+\frac{\theta-1}{\theta}\left(\xi(2) z^{\frac{\theta-1}{\theta}}\right)^2+\xi(1) \xi(2) z z^{\frac{\theta-1}{\theta}}-\frac{\theta-1}{\theta} \xi(1) \xi(2) z^{\frac{\theta-1}{\theta}}\left(z+2 \frac{q_1\left(s^{t-1}\right)}{q_2\left(s^{t-1}\right)}\right).$$

The expression

$$\frac{h'(z) z}{2}>\left(\xi(1) z-\sqrt{\frac{\theta-1}{\theta}} \xi(2) z^{\frac{\theta-1}{\theta}}\right)^2 \geq 0$$

if

$$z-\frac{\theta-1}{\theta}\left(z+2 \frac{q_1\left(s^{t-1}\right)}{q_2\left(s^{t-1}\right)}\right)>-2 z \sqrt{\frac{\theta-1}{\theta}}. \tag{50}$$

Strict inequality (50) is equivalent to:

$$z+2 z \sqrt{\frac{\theta-1}{\theta}}>\frac{\theta-1}{\theta}\left(z+2 \frac{q_1\left(s^{t-1}\right)}{q_2\left(s^{t-1}\right)}\right).$$

Since $\theta>1$, then $z>\frac{\theta-1}{\theta} z$. Since $z>2-\frac{q_2\left(s^{t-1}\right)}{q_1\left(s^{t-1}\right)}$, then $2 z \sqrt{\frac{\theta-1}{\theta}}>\frac{2(\theta-1)}{\theta} \frac{q_1\left(s^{t-1}\right)}{q_2\left(s^{t-1}\right)}$ as

$$2-\frac{q_2\left(s^{t-1}\right)}{q_1\left(s^{t-1}\right)}>\sqrt{\frac{\theta-1}{\theta}} \frac{q_1\left(s^{t-1}\right)}{q_2\left(s^{t-1}\right)}. \tag{51}$$

Strict inequality (51) is satisfied by Condition 3. Under Condition 3, $b^2 \geq 4ac$ for all $z \in \left(1, \frac{q_1\left(s^{t-1}\right)}{q_2\left(s^{t-1}\right)}\right]$.

Unique solution for a given z The sum of the coefficients

$$\tilde{a} + \tilde{b} + \tilde{c} = \xi\left(2\right)\left(1 - z\right) z^{\frac{\theta - 1}{\theta}} \left(q_1\left(s^{t-1}\right) - q_2\left(s^{t-1}\right) z\right).$$

Since $z \in \left(1, \frac{q_1\left(s^{t-1}\right)}{q_2\left(s^{t-1}\right)}\right]$, then $\tilde{a} + \tilde{b} + \tilde{c} \leq 0$. This implies

$$\begin{aligned} 4\tilde{a}\left(\tilde{a} + \tilde{b} + \tilde{c}\right) &\leq 0. \\ 4\tilde{a}^2 + 4\tilde{a}\tilde{b} + \tilde{b}^2 &\leq \tilde{b}^2 - 4\tilde{a}\tilde{c}. \end{aligned}$$

Notice that $4\tilde{a}^2 + 4\tilde{a}\tilde{b} + \tilde{b}^2 = \left(2\tilde{a} + \tilde{b}\right)^2$. This implies that

$$\sqrt{\tilde{b}^2 - 4\tilde{a}\tilde{c}} \geq +\left(2\tilde{a} + \tilde{b}\right) \text{ and } \sqrt{\tilde{b}^2 - 4\tilde{a}\tilde{c}} \geq -\left(2\tilde{a} + \tilde{b}\right).$$

If $2\tilde{a} + \tilde{b} \geq 0$, then the inequality of interest is $\sqrt{\tilde{b}^2 - 4\tilde{a}\tilde{c}} \geq 2\tilde{a} + \tilde{b}$, which implies $q_+(1) = \frac{-\tilde{b} + \sqrt{\tilde{b}^2 - 4\tilde{a}\tilde{c}}}{2\tilde{a}} \geq 1$. Additionally, $\frac{-\tilde{b}}{2\tilde{a}} < 1$ and so $q_-(1) < 1$. Since $z > 1$, then $q_-(2) = \frac{q_-(1)}{z} < 1$.

If $2\tilde{a} + \tilde{b} < 0$, then $\frac{-\tilde{b}}{2\tilde{a}} > 1$ and so $q_+(1) > 1$. Additionally, $\sqrt{\tilde{b}^2 - 4\tilde{a}\tilde{c}} \geq -2\tilde{a} - \tilde{b}$ implies $q_-(1) = \frac{-\tilde{b} - \sqrt{\tilde{b}^2 - 4\tilde{a}\tilde{c}}}{2\tilde{a}} \leq 1$. Since $z > 1$, then $q_-(2) = \frac{q_-(1)}{z} < 1$.

For any $z \in \left(1, \frac{q_1\left(s^{t-1}\right)}{q_2\left(s^{t-1}\right)}\right]$, $q_-(1) \in (0,1)$ and $q_-(2) = \frac{q_-(1)}{z} \in (0,1)$ are the only economic solutions of the quadratic equation.

Case 4: $z > \frac{q_1\left(s^{t-1}\right)}{q_2\left(s^{t-1}\right)}$. The sum of the coefficients is $\tilde{a} + \tilde{b} + \tilde{c} > 0$. This implies

$$\sqrt{\tilde{b}^2 - 4\tilde{a}\tilde{c}} < +\left(2\tilde{a} + \tilde{b}\right) \text{ or } \sqrt{\tilde{b}^2 - 4\tilde{a}\tilde{c}} < -\left(2\tilde{a} + \tilde{b}\right).$$

One of the inequalities must hold. If $2\tilde{a} + \tilde{b} \geq 0$, then $\sqrt{\tilde{b}^2 - 4\tilde{a}\tilde{c}} < 2\tilde{a} + \tilde{b}$ implies $q_+(1) = \frac{-\tilde{b} + \sqrt{\tilde{b}^2 - 4\tilde{a}\tilde{c}}}{2\tilde{a}} < 1$ and $q_+(2) = \frac{q_+(1)}{z} < 1$. There are then multiple solutions ($q_-(1)$ and $q_-(2) = \frac{q_-(1)}{z}$ are also solutions). If $2\tilde{a} + \tilde{b} < 0$, then $\sqrt{\tilde{b}^2 - 4\tilde{a}\tilde{c}} < -\left(2\tilde{a} + \tilde{b}\right)$ implies $q_-(1) = \frac{-\tilde{b} - \sqrt{\tilde{b}^2 - 4\tilde{a}\tilde{c}}}{2\tilde{a}} > 1$. There do not exist any solutions in this scenario. In either scenario, there does not exist a unique solution for Case 4.

A.2.2 Necessary condition for a unique solution for z

The range of equilibrium values for z is $z \in \left[\frac{q_2(s^{t-1})}{q_1(s^{t-1})}, 1\right) \cup \left(1, \frac{q_1(s^{t-1})}{q_2(s^{t-1})}\right]$. For each $z \in \left[\frac{q_2(s^{t-1})}{q_1(s^{t-1})}, 1\right) \cup \left(1, \frac{q_1(s^{t-1})}{q_2(s^{t-1})}\right]$, there exists a unique solution $(q(1), q(2))$. We know have to determine how many values of z in that range are consistent with the remaining system in our system of equations. The remaining equation is either of the equations in (45).

Lower endpoint for z When $z = \frac{q_2(s^{t-1})}{q_1(s^{t-1})}$, then $b^2 - 4ac = (2a+b)^2$.

Case A1 Suppose that $\xi(1) z \leq \xi(2) z^{\frac{\theta-1}{\theta}}$. By definition, when $z = \frac{q_2(s^{t-1})}{q_1(s^{t-1})}$,

$$2a + b = \left(q_1\left(s^{t-1}\right) - q_2\left(s^{t-1}\right)\right)\left(-\xi(1) z + \xi(2) z^{\frac{\theta-1}{\theta}}\right).$$

Since $\xi(1) z \leq \xi(2) z^{\frac{\theta-1}{\theta}}$, then $2a + b \geq 0$, and $\sqrt{b^2 - 4ac} = 2a + b$. This implies that

$$q_-(2) = \frac{z\left(\xi(1) + \xi(2) z^{\frac{\theta-1}{\theta}}\right)}{\xi(1) z^2 + \xi(2) z^{\frac{\theta-1}{\theta}}}.$$

We have previously shown that $q_-(2) \leq 1$, which implies

$$z\left(\xi(1) + \xi(2) z^{\frac{\theta-1}{\theta}}\right) \leq \xi(1) z^2 + \xi(2) z^{\frac{\theta-1}{\theta}}$$

at $z = \frac{q_2(s^{t-1})}{q_1(s^{t-1})}$. This inequality reduces to

$$(z-1)\left(\xi(1) z - \xi(2) z^{\frac{\theta-1}{\theta}}\right) \geq 0,$$

which holds at $z = \frac{q_2(s^{t-1})}{q_1(s^{t-1})}$ given that $\xi(1) z \leq \xi(2) z^{\frac{\theta-1}{\theta}}$.

The requirement $\xi(1) z \leq \xi(2) z^{\frac{\theta-1}{\theta}}$ implies:

$$\frac{q_2\left(s^{t-1}\right)}{q_1\left(s^{t-1}\right)} \leq \left(\frac{\xi(2)}{\xi(1)}\right)^{\theta}.$$

Case B1 Suppose that $\xi(1) z > \xi(2) z^{\frac{\theta-1}{\theta}}$. Then $2a + b < 0$, $\sqrt{b^2 - 4ac} = -2a - b$, and $q_-(2) = 1$. The requirement $\xi(1) z > \xi(2) z^{\frac{\theta-1}{\theta}}$ implies:

$$\frac{q_2\left(s^{t-1}\right)}{q_1\left(s^{t-1}\right)} > \left(\frac{\xi(2)}{\xi(1)}\right)^{\theta}.$$

Upper endpoint for z When $z = \frac{q_1(s^{t-1})}{q_2(s^{t-1})}$, then $\tilde{b}^2 - 4\tilde{a}\tilde{c} = \left(2\tilde{a} + \tilde{b}\right)^2$.

Case A2 Suppose that $\xi(1) z \geq \xi(2) z^{\frac{\theta-1}{\theta}}$. By definition, when $z = \frac{q_1(s^{t-1})}{q_2(s^{t-1})}$,

$$2\tilde{a} + \tilde{b} = \left(q_1\left(s^{t-1}\right) - q_2\left(s^{t-1}\right)\right) z \left(\xi(1) z - \xi(2) z^{\frac{\theta-1}{\theta}}\right).$$

Since $\xi(1) z \geq \xi(2) z^{\frac{\theta-1}{\theta}}$, then $2\tilde{a} + \tilde{b} \geq 0$, and $\sqrt{\tilde{b}^2 - 4\tilde{a}\tilde{c}} = 2\tilde{a} + \tilde{b}$. This implies that

$$q_-(1) = \frac{z\left(\xi(1) + \xi(2) z^{\frac{\theta-1}{\theta}}\right)}{\xi(1) z^2 + \xi(2) z^{\frac{\theta-1}{\theta}}}.$$

We have previously shown that $q_-(1) \leq 1$, which implies

$$z\left(\xi(1) + \xi(2) z^{\frac{\theta-1}{\theta}}\right) \leq \xi(1) z^2 + \xi(2) z^{\frac{\theta-1}{\theta}}$$

at $z = \frac{q_1(s^{t-1})}{q_2(s^{t-1})}$. This inequality reduces to

$$(z-1)\left(\xi(1) z - \xi(2) z^{\frac{\theta-1}{\theta}}\right) \geq 0,$$

which holds at $z = \frac{q_1(s^{t-1})}{q_2(s^{t-1})}$ given that $\xi(1) z \geq \xi(2) z^{\frac{\theta-1}{\theta}}$.

The requirement $\xi(1) z \geq \xi(2) z^{\frac{\theta-1}{\theta}}$ implies:

$$\frac{q_1\left(s^{t-1}\right)}{q_2\left(s^{t-1}\right)} \geq \left(\frac{\xi(2)}{\xi(1)}\right)^{\theta}.$$

Case B2 Suppose that $\xi(1) z < \xi(2) z^{\frac{\theta-1}{\theta}}$. Then $2\tilde{a} + \tilde{b} < 0$, $\sqrt{\tilde{b}^2 - 4\tilde{a}\tilde{c}} = -2\tilde{a} - \tilde{b}$, and $q_-(1) = 1$. The requirement $\xi(1) z < \xi(2) z^{\frac{\theta-1}{\theta}}$ implies:

$$\frac{q_1\left(s^{t-1}\right)}{q_2\left(s^{t-1}\right)} < \left(\frac{\xi(2)}{\xi(1)}\right)^{\theta}.$$

Possible regimes Combining the results from Cases A1, B1, A2, and B2, then one of the three following regimes must occur in each period:

1. Regime 1: $\left(\frac{\xi(2)}{\xi(1)}\right)^{\theta} < \frac{q_2(s^{t-1})}{q_1(s^{t-1})}$ (Cases B1 and A2 hold).

2. Regime 2: $\frac{q_2(s^{t-1})}{q_1(s^{t-1})} \leq \left(\frac{\xi(2)}{\xi(1)}\right)^{\theta} \leq \frac{q_1(s^{t-1})}{q_2(s^{t-1})}$ (Cases A1 and A2 hold).

3. Regime 3: $\frac{q_1\left(s^{t-1}\right)}{q_2\left(s^{t-1}\right)} < \left(\frac{\xi(2)}{\xi(1)}\right)^{\theta}$ (Cases A1 and B2 hold).

By definition, $\frac{\xi(2)}{\xi(1)} = \frac{\Gamma(s_{t-1},2)}{\Gamma(s_{t-1},1)}\frac{\mathbf{y}(1)}{\mathbf{y}(2)}$. If $s_{t-1}=1$, Condition 1 implies that only Regime 1 is possible. If $s_{t-1}=2$, Condition 2 implies that only Regime 3 is possible.

Necessary condition for $s_{t-1}=1$ Regime 1 occurs. When $z=\frac{q_1\left(s^{t-1}\right)}{q_2\left(s^{t-1}\right)}$ and Case A2 holds, then $\sqrt{\tilde{b}^2-4\tilde{a}\tilde{c}}=\tilde{b}+2\tilde{a}$ implying that

$$q_-(1)=\frac{-\tilde{b}-\sqrt{\tilde{b}^2-4\tilde{a}\tilde{c}}}{2\tilde{a}}=\frac{-\tilde{a}-\tilde{b}}{\tilde{a}}.$$

Using the definitions of $\tilde{a}$ and $\tilde{b}$:

$$q_-(1)=\frac{z\left(\xi\left(1\right)+\xi\left(2\right)z^{\frac{\theta-1}{\theta}}\right)}{\xi\left(1\right)z^2+\xi\left(2\right)z^{\frac{\theta-1}{\theta}}}.$$

The 1st equation from (45) will be the remaining equation in the system of equations:

$$q_-(1)\left(\xi\left(1\right)\frac{\left(1-q_-(1)\right)\left(1-z\right)}{q_1\left(s^{t-1}\right)q_-(1)-zq_2\left(s^{t-1}\right)}\right)^{\theta}=K_1. \tag{52}$$

The function $g_2(z)=q_-(1)\left(\xi\left(1\right)\frac{\left(1-q_-(1)\right)\left(1-z\right)}{q_1\left(s^{t-1}\right)q_-(1)-zq_2\left(s^{t-1}\right)}\right)^{\theta}$ is a continuous function of z and equilibrium equation (52) is satisfied when $g_2(z)=K_1$.When $z=\frac{q_1\left(s^{t-1}\right)}{q_2\left(s^{t-1}\right)}$,

$$g_2\left(\frac{q_1\left(s^{t-1}\right)}{q_2\left(s^{t-1}\right)}\right)=q_-(1)\left(\xi\left(1\right)\frac{q_1\left(s^{t-1}\right)-q_2\left(s^{t-1}\right)}{q_1\left(s^{t-1}\right)q_2\left(s^{t-1}\right)}\right)^{\theta}.$$

Near the other endpoint $z=\frac{q_2\left(s^{t-1}\right)}{q_1\left(s^{t-1}\right)}$, the 2nd equation from (45) will be the remaining equation in the system of equations:

$$q_-(2)\left(\xi\left(2\right)\frac{\left(1-q_-(2)\right)\left(1-z\right)}{-q_1\left(s^{t-1}\right)zq_-(2)+q_2\left(s^{t-1}\right)}\right)^{\theta}=K_1. \tag{53}$$

As a function of z,

$$q_-(2)=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

The function $g_1(z)=q_-(2)\left(\xi\left(2\right)\frac{\left(1-q_-(2)\right)\left(1-z\right)}{-q_1\left(s^{t-1}\right)zq_-(2)+q_2\left(s^{t-1}\right)}\right)^{\theta}$ is a continuous function of z and equilibrium equation (53) is satisfied when $g_1(z)=K_1$.

Under Regime 1, there exists $z_1 > \frac{q_2(s^{t-1})}{q_1(s^{t-1})}$ such that $g_1(z_1) = g_2\left(\frac{q_1(s^{t-1})}{q_2(s^{t-1})}\right)$. For all values $z \geq z_1$, there either exists zero or two or more solutions for z (over the range of all possible policy parameter values for K_1). There exists a unique solution only if $z < z_1$ (and for a well-chosen value for the policy parameter K_1). The lower bound $\underline{q}(2)$ is determined such that $q_-(2) = \underline{q}(2)$ at $z = z_1$. The upper bound $\overline{q}(1)$ is given by $\overline{q}(1) = z_1 \underline{q}(2)$.

Necessary condition for $s_{t-1} = 2$ Regime 3 occurs. When $z = \frac{q_2(s^{t-1})}{q_1(s^{t-1})}$ and Case A1 holds, $\sqrt{b^2 - 4ac} = b + 2a$ implying that

$$q_-(2) = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-a - b}{a}.$$

Using the definitions of a and b :

$$q_-(2) = \frac{z\left(\xi(1) + \xi(2)\, z^{\frac{\theta-1}{\theta}}\right)}{\xi(1)\, z^2 + \xi(2)\, z^{\frac{\theta-1}{\theta}}}.$$

When $z = \frac{q_2(s^{t-1})}{q_1(s^{t-1})}$,

$$g_1\left(\frac{q_2(s^{t-1})}{q_1(s^{t-1})}\right) = q_-(2)\left(\xi(2)\, \frac{q_1(s^{t-1}) - q_2(s^{t-1})}{q_1(s^{t-1})\, q_2(s^{t-1})}\right)^{\theta}.$$

Near $z = \frac{q_1(s^{t-1})}{q_2(s^{t-1})}$, $q_-(1)$ is determined as a function of z :

$$q_-(1) = \frac{-\tilde{b} - \sqrt{\tilde{b}^2 - 4\tilde{a}\tilde{c}}}{2\tilde{a}}.$$

Under Regime 3, there exists $z_2 < \frac{q_1(s^{t-1})}{q_2(s^{t-1})}$ such that $g_1\left(\frac{q_2(s^{t-1})}{q_1(s^{t-1})}\right) = g_2(z_2)$. For all values $z \leq z_2$, there either exists zero or two or more solutions for z. There exists a unique solution only if $z > z_2$. The upper bound $\overline{q}(2)$ is determined such that $q_-(2) = \overline{q}(2)$ at $z = z_2$. The lower bound $\underline{q}(1)$ is then given by $\underline{q}(1) = z_2 \overline{q}(2)$.