The Effects of Dependent Beliefs on Endogenous Leverage*

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Abstract

The number of financial markets and the beliefs about the relation between markets can have large effects on the access to credit in a model with collateralized borrowing. In the model, investors have beliefs about the payout likelihoods for assets. I vary the degree of dependence between the likelihoods for the asset payouts and solve for the endogenous leverage ratios. When investors believe that the payouts of the assets are more dependent, the model predicts higher leverage ratios for all assets. When the number of financial markets available to investors increases, a condition in terms of the belief elasticity characterizes whether or not the leverage ratios increase.

Keywords leverage – portfolio effects – copulas – collateral – heterogeneous beliefs

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1 Introduction

Leverage ratios have become important macroprudential variables following the recent 2007-2008 financial crisis. This paper analyzes the effects that the number of financial markets and

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the beliefs about the relation between markets can have on leverage. In analyzing leverage, it is important to consider models in which leverage is endogenously determined.[1] Investors in my model have multiple asset markets available to them and heterogeneous beliefs about asset payouts, where the beliefs are multi-dimensional and allow for dependence across assets. When investors believe that the payouts of the assets are more dependent, the model predicts higher leverage ratios for all assets. The effect of the number of financial markets on leverage depends crucially on the degree of belief dependence.

The term "financial fragility" is typically defined in terms of leverage ratios. In financial models under rational expectations, high leverage is not necessarily an undesirable outcome as it is associated with high levels of risk-sharing among investors. It is important, in these models, to understand the market conditions leading to high leverage, particularly if we take the remarks of policy makers following the 2007-2008 financial crisis at face value:

> "How could macroprudential policies be better integrated into the regulatory and supervisory system? One way would be for the Congress to direct and empower a governmental authority to monitor, assess, and, if necessary, address potential systemic risks within the financial system. The elements of such an authority's mission could include, for example, ... (2) assessing the potential for deficiencies in evolving risk-management practices, broad-based increases in financial leverage, or changes in financial markets or products to increase systemic risks..." Ben Bernanke, Council on Foreign Relations, March 10, 2009

The two effects analyzed in this paper are the effects of belief dependence on leverage and the effects of financial expansion on leverage. Financial expansion refers to an increase in the number of financial markets available to investors. Policy analysis must account for the indirect relation between policy and leverage via the belief dependence and the financial expansion channels. To see why it is essential to analyze these effects together, it is instructive to consider the findings from a canonical endogenous leverage model with a single asset.

Within the class of 2-period binomial economies, the canonical Geanakoplos (1997, 2003, 2010) model contains a unit mass of risk-neutral investors, each endowed with an equal amount of endowment in the initial period and one unit of the real asset. In the binomial economy, investors agree that the single asset has two possible dividend values in the final period, but they have heterogeneous beliefs about the likelihood of these dividends. Belief heterogeneity is characterized by a cumulative distribution function (cdf) that specifies the probability each investor assigns to the high dividend realization. In the simplest setting, investor beliefs are uniformly distributed.

[1] See Geanakoplos (1997, 2003, 2010), Geanakoplos and Fostel (2008, 2012, 2015), Simsek (2013), Geerolf (2015), and Phelan (2015).

The key mechanism in the canonical Geanakoplos (1997, 2003, 2010) model is driven by wealth effects, specifically the amount of wealth available for buyers to invest in the asset market. If investor endowment increases, the wealth in the asset market increases, meaning that the asset price will increase and the leverage (by definition) will decrease. If the low dividend value increases, wealth in the asset market increases as investors can borrow more, and this causes the asset price to increase.

My model considers a 2-period binomial economy with N assets. The economy is binomial as each asset has one of two possible dividend values in the final period. Risk-neutral investors have heterogeneous beliefs over all possible states of realizations in the final period, where a state of realization is a distinct portfolio of dividends. The model makes two simplifying assumptions, while maintaining enough structure to analyze the effects of belief dependence. First, investors agree that the dividend realizations are random variables drawn from an independent distribution, but they do not agree on that distribution. With an independent distribution, the heterogeneous beliefs are characterized by a cumulative distribution with dimension equal to the number of assets N. The second simplifying assumption is to consider a special class of continuous cdfs called copulas. Copulas have the property that the marginal distributions are uniformly distributed, meaning that the marginal beliefs for any asset are uniformly distributed. These two simplifying assumptions, the latter to maintain consistency with the canonical Geanakoplos (1997, 2003, 2010) model, allow for a tractable analysis of the effects of belief dependence and financial expansion.

The copulas considered in the paper are general enough to include comonotonic and independent beliefs as special cases. In terms of correlation, which is a narrow definition of the relation between two variables, comonotonic variables have correlation equal to 1 and independent variables have correlation equal to 0. The analysis focuses on belief dependence in the range between comonotonic beliefs and independent beliefs. The first main result shows that as the belief dependence increases (moving closer to comonotonicity), the leverage ratios increase for all assets.

Financial expansion is defined as an increase in the number of financial markets. As each market corresponds to a real asset in positive net supply, financial expansion also expands the total initial asset endowments of households. The commodity endowments remain fixed under financial expansion. On a per-market basis, financial expansion is a negative wealth effect as fewer funds are available for each market.

Two effects arise under financial expansion: the wealth effect and the belief dependence effect. With less wealth in each asset market, the asset price decreases and leverage increases. The belief dependence effect arises when the belief dependence between the new financial markets and the old financial markets is not perfectly dependent. With less-than-full dependence, investors are not able to predict the payout likelihoods of new financial markets using payout likelihoods from existing financial markets. The belief dependence effect states that an increase in the number of assets decreases the belief dependence over all markets, which in turn causes leverage to decrease.

The two effects work in opposite directions, where the latter belief dependence effect is only present in the multi-asset setting. Define the belief elasticity as the percent change in the cumulative distribution of household beliefs relative to the percent change in the number of financial markets. The belief effect is governed by this belief elasticity variable. The second main result of the paper derives a condition involving the belief elasticity to characterize the conditions under which each effect will dominate. With a high belief dependence, the wealth effect dominates and financial expansion leads to increased leverage. For low belief dependence, the belief dependence effect dominates and financial expansion leads to decreased leverage.

The effects of belief dependence on the financial expansion and leverage relation is strictly monotonic. This means that there exists a cutoff value for belief dependence above which financial expansion leads to increased leverage ratios and below which financial expansion leads to decreased leverage ratios. An example illustrates this mechanism.

1.1 Literature review

My paper belongs to the class of general equilibrium models with endogenous collateral, which have been developed using the general equilibrium framework in the Geanakoplos and Zame (2014) model with exogenous collateral constraints.

There are two strands of literature with endogenous collateral. In the first strand of literature, to which the model developed in the present paper belongs, all investors agree that there are a finite number of possible states of uncertainty that can be realized. Geanakoplos (1997, 2003, 2010) considers binomial economies with a continuum of different types of risk-neutral investors. The collateralized borrowing contracts that are traded in equilibrium in this setting are default-free, meaning that borrowers are able to borrow the maximum amount such that they will always repay their loans in all future states.

Within the class of binomial economies, Geanakoplos and Fostel (2015) extend the model to allow for risk-averse investors and show that there are no real effects from restricting the equilibrium set of borrowing contracts to the set of default-free contracts. Extending the present model to risk-averse investors is a natural extension and Appendix A.2 addresses the issues involved in extending the present model to the setting of risk aversion.

Geanakoplos and Fostel (2008, 2012) both consider multi-asset economies. Geanakoplos and Fostel (2008) consider an economy with three assets.[2] In that model, there are three time periods and information about only one asset is gathered from the initial period to the intermediate period. The random variables for dividend realizations are independently drawn, as in the current paper. Geanakoplos and Fostel (2008) is interested in generating leverage cycles (hence the 3-period model) and assumes a specific form of belief heterogeneity, namely two types of investors. Geanakoplos and Fostel (2012) consider an economy with two assets and a continuum of types of investors, but only one of the assets is risky, meaning that the only beliefs that matter are the beliefs for the single risky asset. A general cdf for investor beliefs is considered (corresponding to beliefs from a non-uniform distribution). The current paper considers a continuum of investors and any finite number of assets in a 2-period model in which beliefs are characterized by copulas.[3]

In the second strand of literature on endogenous collateral, 2-period models with a continuum of possible states of uncertainty are studied. In Simsek (2013), two types of risk-neutral investors have heterogeneous beliefs about the likelihood functions over states in the final period. In Geerolf (2015), a continuum of different types of risk-neutral investors have heterogeneous beliefs about the asset dividends in the final period. Both Simsek (2013) and Geerolf (2015) use belief disagreements among risk-neutral households to study leverage, an exercise I extend to the multi-asset setting. Phelan (2017) takes a different approach when studying leverage by employing a model with two types of risk-averse investors with common beliefs but heterogeneous risk aversion and endowments. Phelan (2017) shows that the effects of heterogeneity in either risk aversion or endowments can have ambiguous effects on leverage. By restricting my analysis to one form of heterogeneity, in terms of beliefs, I derive unambiguous effects on leverage.

The remainder of the paper is organized as follows. Section 2 introduces the model. Section 3 introduces the notion of a copula, which is how the model caputures belief dependence. Section 4, working with a general class of copulas, provides the main results relating belief dependence to leverage. Section 5 considers the special cases of perfectly independent and perfectly dependent beliefs. Section 6 provides concluding remarks, Appendix A contains supporting technical material, and Appendix B contains the proofs of the main results.

[2] The beliefs between two of the assets are assumed to be comonotonic, which reduces the model to one with two independent assets.

[3] The tractable model introduced in the present paper can be generalized to a 3-period model to analyze leverage cycles. However, such analysis is incomplete in a pure-exchange economy. With the properties of belief dependence established in the present paper, the more appropriate next step is to embed the model into a production economy with at least 3 periods that will allow for the possibility of leverage cycles (at least 3 periods).

2 The Model

The leverage cycle model was introduced by Geanakoplos (1997, 2003, 2010). This paper extends that model by considering an asset structure with multiple assets. To accomodate readers familiar with the Geanakoplos (1997, 2003, 2010) model, I adhere to his notation as much as possible.

The model contains two time periods. The economy contains $N \geq 2$ real assets. Denote the set of assets as $\mathbf{I} = \{1, ..., N\}$. These assets pay out dividends in the final period. The dividends for each asset can either be high (H) or low (L). With N assets, there are 2^N possible realizations of uncertainty in the final period. Label the date-events in the final period as $s \in \mathbf{S} = \{1, ..., 2^N\}$.

For each asset i, the dividends are normalized such that the high dividend equals 1 and the low dividend equals d_i, where $0 < d_i < 1$.

In each period, a single physical commodity is traded and consumed.

2.1 Households

The model contains a continuum of households with unit mass. The set of households is $\mathbf{H} = [0, 1]$ with typical element $h \in \mathbf{H}$.

In the initial period, each household receives an endowment of 1 unit of the commodity.

Denote household consumption the initial period as $c^h(0)$ and household consumption in date-event s in the final period as $c^h(s)$. For simplicity, denote $c^h = \left(c^h(0), \left(c^h(s)\right)_{s \in \mathbf{S}}\right)$.

2.1.1 Heterogeneous beliefs

Each household believes that the dividend realizations for all assets are random variables drawn from an independent distribution, but the households do not agree on that distribution.

The beliefs of a household are represented by the vector $(h_1, ..., h_N) \in [0, 1]^N$, where h_i is the belief associated with the random variable for asset i. A household with belief h_i assigns probability h_i to the event that asset i has high dividend and probability $1 - h_i$ to the event that asset i has low dividend.

For any household h, define $\pi^h(s)$ as the probability that date-event s occurs in the final period according to household h with beliefs $(h_1, ..., h_N)$. For instance, if $N = 2$ and $\mathbf{S} = \{1, 2, 3, 4\}$ corresponds to the dividend realizations $\{HH, HL, LH, LL\}$ for assets 1 and

2, respectively, then

$$\pi^h(s) = \begin{Bmatrix} h_1 h_2 & \text{for } s = 1 \\ h_1(1-h_2) & \text{for } s = 2 \\ (1-h_1)h_2 & \text{for } s = 3 \\ (1-h_1)(1-h_2) & \text{for } s = 4 \end{Bmatrix}.$$

The probabilities $(h_1, ..., h_N)$ are drawn from the joint distribution $F_{H_1,...,H_N}(h_1, ..., h_N)$. The distribution is symmetric. Moreover, it is assumed that the marginal distributions $F_{H_i}(h_i)$ are uniformly distributed for all i.

Households in this model are risk-neutral. Household h with beliefs $(h_1, ..., h_N)$ maximizes the following expected utility function:

$$U^h\left(c^h\right) = c^h(0) + \sum_{s\in\mathbf{S}} \pi^h(s)\, c^h(s).$$

2.1.2 Real assets

The model contains N real assets in unit net supply that are traded in the initial period.

Each household is initially endowed with 1 unit of each asset. Denote a_i^h as the asset i holdings chosen in the initial period. Short-selling the asset is not permitted, so the variables must satisfy $a_i^h \geq 0$ for all i and all h. The price of asset i is denoted p_i. For simplicity, denote $a^h = \left(a_i^h\right)_{i\in\mathbf{I}}$ and $p = (p_i)_{i\in\mathbf{I}}$.

2.1.3 Storage

Households have access to a perfect storage technology. Denote $s^h \in \mathbb{R}_+$ as the amount of the physical commodity stored by household h.

2.1.4 Collateralized borrowing

In addition to the assets, markets exist for the households to trade noncontingent borrowing contracts. The borrowing contracts are non-recourse loans. In order to protect the interests of the lenders, the loans must be secured with collateral. Otherwise, lenders would not be willing to lend as the borrowers would renege on their obligations ex-post.

Though intuitive to normalize the promised repayment to 1 and consider borrowing contracts as (interest rate, collateral) pairs, it is actually more convenient (mathematically) to adopt an equivalent representation of borrowing contracts as (promised repayment, interest rate) pairs with normalized collateral. The collateral requirement is 1 unit of cumulative asset holdings, meaning that any combination of the real assets is possible, provided that the holdings sum to 1. This accounts for collateral requirements in terms of mixtures across assets. The markets will ultimately determine which collateral mixtures support nonzero borrowing in equilibrium, together with the promised repayment and the interest rate for such borrowing.

In the unit simplex Δ^{N-1}, the vector $\theta \in \Delta^{N-1}$ characterizes one possible collateral mixtrue with θ_1 units of asset 1, θ_2 units of asset 2, and so forth. For each collateral mixture $\theta \in \Delta^{N-1}$, consider all possible promised repayments $j \geq 0$. These two features characterize a borrowing contract (the interest rate r_j is inversely related to the borrowing price q_j and is determined from market clearing (4)).

When $\theta = (1, 0, ..., 0)$, I denote $(\theta, j) = (1, j)$, and similarly for other assets.

I allow the possibility for all borrowing contracts, using the notation $\theta \in \Delta^{N-1}$ and $j \in \mathbb{R}_+$. Define $\mathcal{B}$ as the Borel algebra over $\Delta^{N-1} \times \mathbb{R}_+$. I denote the contract prices $q : \Delta^{N-1} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and require that q is $\mathcal{B}-$measurable. For each household h, the net borrowing contracts are represented by the measure $\mu^h : \mathcal{B} \rightarrow \mathbb{R}$. Denote μ^h_+ as the measure consisting only of the nonnegative positions and μ^h_- as the measure consisting only of the nonpositive positions.

2.2 Equilibrium

2.2.1 Initial period constraints

The households face both a budget constraint and collateral constraints in the initial period. The budget constraint is given by:

$$c^h(0) + s^h + \sum_{i \in \mathbf{I}} p_i a_i^h \leq 1 + \sum_{i \in \mathbf{I}} p_i + \int_{(\theta,j) \in \Delta^{N-1} \times \mathbb{R}_+} q(\theta, j) \mu^h(d(\theta, j)). \tag{1}$$

The collateral constraints (one for each asset) are given by:

$$\int_{(\theta,j) \in \Delta^{N-1} \times \mathbb{R}_+} \theta_i \cdot \mu^h_+(d(\theta, j)) \leq a_i^h \quad \forall i \in \mathbf{I}. \tag{2}$$

The constraints specify that lending on one contract cannot serve as collateral for borrowing on another contract. Collateral must be chosen from the set of real assets.

2.2.2 Final period constraints

For simplicity, denote $d_i(s)$ as the dividend for asset i in date-event s. A contract (θ, j) borrower has the option in the final period to either repay the loan (pay the promise j) or default and forfeit the value of collateral, where the value of collateral varies across states $s \in \mathbf{S}$. The default decision by borrowers determines the repayment received by lenders.

The budget constraint in states $s \in \mathbf{S}$ captures these default decisions:

$$c^h(s) \leq s^h + \sum_{i\in\mathbf{I}} d_i(s)a_i^h - \int_{(\theta,j)\in\Delta^{N-1}\times\mathbb{R}_+} \min\left\{j, \sum_{i\in\mathbf{I}}\theta_i d_i(s)\right\}\mu^h\left(d\left(\theta,j\right)\right). \tag{3}$$

2.2.3 Equilibrium definition

An equilibrium consists of the household consumption choices $\left(c^h\right)_{h\in\mathbf{H}}$, the household storage choices $\left(s^h\right)_{h\in\mathbf{H}}$, the household asset choices $\left(a^h\right)_{h\in\mathbf{H}}$, the net borrowing contract measures $\left(\mu^h\right)_{h\in\mathbf{H}}$, the asset prices p, and the borrowing contract prices q such that:

1. Given the prices p and q, each household $h \in \mathbf{H}$ solves the problem:

$$\begin{array}{cc} \underset{c^h,s^h,a^h,\mu^h}{max} & U^h\left(c^h\right) \\ \text{subject to} & (1), (2), \text{and } (3) \end{array}.$$

2. The commodity markets clear:

$$\int_{h\in\mathbf{H}}\left(c^h(0)+s^h\right)dh = 1.$$
$$\int_{h\in\mathbf{H}} c^h(s)\,dh = \sum_{i\in\mathbf{I}} d_i(s) + \int_{h\in\mathbf{H}} s^h dh \text{ for all } \in \mathbf{S}.$$

3. The asset markets clear:

$$\int_{h\in\mathbf{H}} a_i^h dh = 1 \text{ for all } i \in \mathbf{I}. \tag{4}$$

4. The markets for the borrowing contracts clear:

$$\int_{h\in\mathbf{H}} \mu_+^h dh = \int_{h\in\mathbf{H}} \mu_-^h dh.$$

3 Copulas

To introduce the concept of a copula, consider the set of economies with $N = 2$ assets.

The belief structures in this paper are represented by a cumulative distribution function $F_{H_1,H_2}(h_1,h_2)$ with the following properties: (i) $(h_1,h_2) \in [0,1]^2$ and (ii) the marginal distributions are uniformly distributed:

$$F_{H_1,H_2}(h_1,1) = h_1.$$
$$F_{H_1,H_2}(1,h_2) = h_2.$$

Any cumulative distribution with this property is called a copula (Sklar 1959; Nelsen 2006; Embrechts et al. 1999; Embrechts et al. 2005; Eling and Toplek 2009).

Consider any cdf $G(x_1, x_2)$ with marginal distribution functions $G_1(x_1)$ and $G_2(x_2)$. A copula is the cumulative distribution function $C : [0,1]^2 \to [0,1]$ such that:

$$G(x_1, x_2) = C(G_1(x_1), G_2(x_2)).$$

If the marginal distributions are continuous, then C is unique (Sklar 1959). For any $(h_1, h_2) \in [0,1]^2$, the copula associated with G can then be defined using the inverse distributions G_1^{-1} and G_2^{-1}:

$$C(h_1, h_2) = G\left(G_1^{-1}(h_1), G_2^{-1}(h_2)\right).$$

By construction, $C(h_1, h_2)$ is a cdf with uniform marginal distribution functions.

3.1 Archimedean copulas

This paper focuses on copulas in the class of Archimedian copulas, which can be written in the form

$$C^{Arch}(h_1, h_2) = \psi\left(\psi^{-1}(h_1) + \psi^{-1}(h_2)\right)$$

for a generator function ψ satisfying the following properties: (i) ψ is strictly decreasing, (ii) ψ is convex, and (iii) $\psi(0) = 1$. Leading examples in this class include (i) the Clayton copula, (ii) the Frank copula, and (iii) the Joe copula.[4]

For values of the parameter $\alpha > 0$, the Clayton copula is defined by:

$$C_\alpha^{Cl}(h_1, h_2) = \left(h_1^{-\alpha} + h_2^{-\alpha} - 1\right)^{-1/\alpha}.$$

The generator function for the Clayton copula is $\psi(x) = (1 + \alpha x)^{-1/\alpha}$. As $\alpha \to 0$, the Clayton copula approaches the special case with independent beliefs. As $\alpha \to \infty$, the Clayton copula approaches the special case of perfectly dependent beliefs (also called comonotonic beliefs).

Kendall's tau (τ) is a measure of nonparametric rank correlation between two variables. A Kendall's tau value of $\tau = 0$ indicates zero rank correlation and a value of $\tau = 1$ indicates perfect positive rank correlation. The Clayton parameter α is positively related to Kendall's

[4] A well-known copula in this class is the Gumbel copula with cdf $C_\alpha^{Gu}(h_1, h_2) = \exp\left\{-\left((-\ln(h_1))^\alpha + (-\ln(h_2))^\alpha\right)^{1/\alpha}\right\}$ and generator function $\psi(x) = \exp\left(-(x)^{1/\alpha}\right)$. For the Gumbel copula, the generator function is not monotonic in the parameter α, a property required for Theorem 1.

tau:

$$\tau = \frac{\alpha}{2+\alpha}.^{5}$$

For values of the parameter $\alpha > 0$, the Frank copula is defined by:

$$C_{\alpha}^{Fr}(h_1, h_2) = -\frac{1}{\alpha} \ln \left(1 + \frac{(\exp(-\alpha h_1) - 1)(\exp(-\alpha h_2) - 1)}{(\exp(-\alpha) - 1)}\right).$$

The generator function for the Frank copula is $\psi(x) = -\frac{1}{\alpha}\ln(1 + \exp(-x)(\exp(-\alpha) - 1))$. When $\alpha \to 0$, the Frank copula approaches the special case with independent beliefs. As $\alpha \to \infty$, the Frank copula approaches the special case with comonotonic beliefs.

For values of the parameter $\alpha \geq 1$, the Joe copula is defined by:

$$C_{\alpha}^{Joe}(h_1, h_2) = 1 - \left((1-h_1)^{\alpha} + (1-h_2)^{\alpha} - (1-h_1)^{\alpha}(1-h_2)^{\alpha}\right)^{1/\alpha}.$$

The generator function for the Joe copula is $\psi(x) = 1 - (1 - \exp(-x))^{1/\alpha}$. When $\alpha = 1$, the Joe copula reduces to the special case with independent beliefs. As $\alpha \to \infty$, the Joe copula approaches the special case with comonotonic beliefs.

The parameter α will be labeled the 'belief dependence parameter', with higher values signifying greater belief dependence.

3.2 Copulas in higher dimensions

Consider economies with a general number of $N > 2$ assets. The convenient feature about copulas is that the two-asset marginal distributions, i.e., $\Pr\left(h_i \leq h_i^*, h_j \leq h_j^*\right)$, are identical to the cdfs from the 2-asset economies:

$$F_{H_1,\ldots,H_N}\left(1, \ldots, 1, h_i^*, 1, .., 1, h_j^*, 1, .., 1\right) = F_{H_i,H_j}\left(h_i^*, h_j^*\right).$$

For Archimedean copulas, the cdf is defined by:

$$C^{Arch}(h_1, \ldots., h_N) = \psi\left(\psi^{-1}(h_1) + \ldots + \psi^{-1}(h_N)\right).$$

The fact that the definition includes the summation $\psi^{-1}(h_1) + \ldots + \psi^{-1}(h_N)$ allows me to easily generalize results from the 2-asset case to the $N-$asset case. To illustrate how the definition extends, the Clayton copula in higher dimensions is given by:

$$C_{\alpha}^{Cl}(h_1, \ldots., h_N) = \left(h_1^{-\alpha} + \ldots. + h_N^{-\alpha} - 1\right)^{-1/\alpha}.$$

[5] See Clayton (1978).

Similar extensions hold for the Frank and Joe copulas.

4 The Effects of Beliefs on Leverage

Consider any of the three copulas from the Archimedean class. The key assumption for the results is that the cdf is differentiable and the generator function ψ is monotonic in the belief dependence parameter α.

4.1 Belief dependence and leverage

Define the cutoff household h_i^* such that the household with beliefs $h_i = h_i^*$ is indifferent between buying and selling asset i. From the first order conditions for this cutoff household:

$$p_i = h_i^* + (1 - h_i^*)\, d_i. \tag{5}$$

All borrowing contracts can be traded, but I assume at this time that the only ones actually traded in equilibrium are the following: contract $(1, d_1)$ with collateral of 1 unit of asset 1 and promise to repay d_1 units, contract $(2, d_2)$ with collateral of 1 unit of asset 2 and promise to repay d_2 units, and so forth for all assets $i \in \mathbf{I}$.

I impose this assumption. I later verify that the assumption holds in equilibrium.

> **Equilibrium Assumption (EA)** The only contracts traded in equilibrium are the N borrowing contracts $\{(1, d_1), ..., (N, d_N)\}$ and any linear combinations of these contracts.

For simplicity, denote the household contract choices for (i, d_i) as b_i^h and the contract price as q_i. The payout of contract (i, d_i) is equal to d_i in all date-events in the final period (a no-default loan). For all households, the first order conditions require that:

$$q_i = d_i.$$

For simplicity, define $P = \sum\limits_{i \in \mathbf{I}} p_i$ and $D = \sum\limits_{i \in \mathbf{I}} d_i$.

Define $Fr\left\{a_i^h > 0\right\}$ as the fraction of households that hold asset i. For households with beliefs $(h_1, ..., h_N) \le (h_1^*, ..., h_N^*)$, the asset positions $a_i^h = 0$ for all $i \in \mathbf{I}$. The fraction of households with beliefs $(h_1, ..., h_N) \le (h_1^*, ..., h_N^*)$ is equal to $F_{H1}, ..., H_N\,(h_1^*, ..., h_N^*)$. Therefore,

$$\sum_{i \in \mathbf{I}} Fr\left\{a_i^h > 0\right\} = 1 - F_{H_1,...,H_N}\,(h_1^*, ..., h_N^*). \tag{6}$$

To fix ideas, consider an example with $N=2$ assets. In the set of households with beliefs $(h_1,h_2)\geq(h_1^*,h_2^*)$, a full measure subset of households strictly prefer one asset over another. There is a measure zero subset of households that are indifferent between the two assets. Suppose, without loss of generality, that $h_1^*\geq h_2^*$. From Appendix A.1, the fraction $\frac{1}{2}\frac{h_2^*(1-h_1^*)}{h_1^*(1-h_2^*)}$ of the households with beliefs $(h_1,h_2)\geq(h_1^*,h_2^*)$ purchase only asset 1 and the remaining households in that set choose asset 2.[6]

This idea extends to a general economy with $N>2$ assets. A full measure subset of all asset purchasers will purchase at most one asset. Closed-form expressions can be derived for $Fr\left\{a_i^h>0\right\}$ as a function of $(h_1^*,....,h_N^*)$ (see Appendix A.1).

Households such that $h_i\geq h_i^*$ for some $i\in\mathbf{I}$ set $c^h(0)=s^h=0$ in order to purchase as many units of their preferred asset as possible.

Households with beliefs $(h_1,....,h_N)\leq(h_1^*,....,h_N^*)$ sell all assets to the point where the short-sale constraints bind: $a_i^h=0$ for all $i\in\mathbf{I}$.

Risk-neutral households that choose to borrow will borrow up until the point that the collateral constraint binds, as shown in the following claim.

Claim 1 *If $a_i^h>0$, then $b_i^h=a_i^h$.*

Proof. See Section B.1. ■

Add up the initial period budget constraints (1) for all households with $a_i^h>0$:

$$(p_i-d_i)\int_{a_i^h>0}a_i^h dh=Fr\left\{a_i^h>0\right\}(1+P). \tag{7}$$

From the market clearing conditions (4), I obtain:

$$p_i-d_i=Fr\left\{a_i^h>0\right\}(1+P). \tag{8}$$

The following claim provides a necessary and sufficient condition for $h_1^*>...>h_N^*$.

Claim 2 *If $d_i>d_j$, then $h_i^*>h_j^*$.*

Proof. See Section B.2. ■

Using (5), the equilibrium equations (8) can be written only in terms of $(h_1^*,....,h_N^*)$:

$$h_i^*(1-d_i)=Fr\left\{a_i^h>0\right\}\left(1+D+\sum_{i\in\mathbf{I}}h_i^*(1-d_i)\right). \tag{9}$$

[6] The measure zero subset of households that are indifferent between two or more assets are irrelevant when we add the budget constraints over all households and apply the market clearing conditions.

For an $N-$asset economy with N distinct dividends $(d_1, ..., d_N)$, the system of equations (9) contains N equations in terms of the N unknowns $(h_1^*, ..., h_N^*)$.

I can now verify the Equilibrium Assumption (EA).

Claim 3 *Verification of EA: All contracts* $(\theta, j) \in \Delta^{N-1} \times \mathbb{R}_+ \backslash \{(1, d_1), ..., (N, d_N)\}$ *are either not traded in equilibrium or are redundant to one of the contracts in the set* $\{(1, d_1), ..., (N, d_N)\}$.

Proof. See Section B.3. ∎

Denote lev_i as the leverage ratio for asset i. The price of asset i equals p_i and the loan size is equal to the low dividend payout d_i. By definition,

$$lev_i = \frac{p_i}{p_i - d_i}. \tag{10}$$

Using the equilibrium price equations, the leverage ratios are equivalently expressed as:

$$lev_i = \frac{h_i^* (1 - d_i) + d_i}{h_i^* (1 - d_i)}.$$

Leverage lev_i is strictly decreasing in h_i^*. The first result analyzes how changes in the belief dependence α affect leverage for all assets.

Theorem 1 *When the cdf for household beliefs is either a Clayton, Frank, or Joe copula, the leverage ratios for all assets are strictly increasing functions of the belief dependence* α.

Proof. See Section B.4. ∎

4.2 Financial expansion

4.2.1 Wealth effect

To analyze how the number of financial markets affects the leverage ratios, I employ the concept of replica economies. The base economy will contain N assets with dividends $(d_1, ..., d_N)$. The financial side of the economy is scaled up by the factor $m \geq 1$, meaning that there are a total of mN financial markets, m with dividends equal to d_1, m with dividends equal to d_2, and so forth.

A replication of the financial side of the model leads to an equilibrium that must account for two competing effects: (i) the wealth effect and (ii) the belief dependence effect. The wealth effect occurs because more financial markets leads to a smaller investment of the fixed resource (endowment) in each asset market (wealth per asset market decreases).

The belief dependence effect refers to the cumulative distribution of household beliefs. I can isolate the wealth effect by shutting off the belief dependence effect. To do this, I assume that all assets with the same dividends (there are m assets with identical dividends) have perfect belief dependence (comonotonic beliefs within this set of assets). The belief dependence across assets with different dividends will continue to be characterized by a copula with the belief dependence α.

Claim 2 states that all financial markets with identical dividends d_i have identical cutoffs h_i^*. The assumption that household beliefs are comonotonic between financial markets with identical dividends is mathematically expressed as:

$$F_{H_1,\ldots,H_{mN}}\left(\overrightarrow{h_1},\ldots.,\overrightarrow{h_N}\right) = F_{H_1,\ldots,H_N}\left(h_1,\ldots.,h_N\right),$$

where $\overrightarrow{h_i} = (h_i,\ldots,h_i)$ is the $m-$dimensional vector representing the beliefs for all m replications of the financial market with dividend d_i.

The wealth effect implies a positive relation between financial expansion and leverage, as verified by the following result.

Theorem 2 *Consider a base economy with N assets and dividends $(d_1,\ldots,d_N)$. Consider $m \geq 1$ replications of the base economy such that $F_{H_1,\ldots,H_{mN}}\left(\overrightarrow{h_1},\ldots.,\overrightarrow{h_N}\right) = F_{H_1,\ldots,H_N}\left(h_1,\ldots.,h_N\right)$. The leverage ratios for all assets are a strictly increasing function of m.*

Proof. See Section B.5. ■

4.2.2 Financial expansion and leverage

The isolated comparative statics for the two effects of financial expansion are very clear:

1. Holding fixed the number of assets, if the belief dependence α decreases, the leverage ratios decrease for all assets.

2. Increasing the number of replications of the financial side of the economy, while isolating the wealth effect (shutting off the belief dependence effect), leads to higher leverage ratios for all assets.

Following financial expansion, I maintain the belief dependence of α between all assets, both those with the same dividends and those with different dividends.[7]

[7] The qualitative findings remain unchanged if the belief dependence between asset markets with different dividends is smaller than the belief dependence between asset markets with identical dividends, provided that the latter is not comonotonic. The assumption of symmetry across markets continues to be used for simplicity.

Total dividends after expansion equal mD. Claim 2 states that all financial markets with identical dividends d_i have identical cutoffs h_i^*. Define $\overrightarrow{h_i^*}$ as the $m-$dimensional vector $(h_i^*, ..., h_i^*)$. For brevity, omit the subscripts on the cumulative distribution function. The fraction $1 - F\left(\overrightarrow{h_1^*}, , ..., \overrightarrow{h_N^*}\right)$ is the fraction of households that purchase an asset.

The total amount of equilibrium borrowing equals mD (recall Claim 1). The total commodity endowment in the economy equals 1. Therefore, the potential new investment equals $1 + mD$. Each asset purchaser has at most $\frac{1+mD}{1-F\left(\overrightarrow{h_1^*},...,\overrightarrow{h_N^*}\right)}$ for investment, which can be viewed as a measure of individual liquidity.

For all Archimedean copulas, the effects of beliefs are governed by the generator function ψ, where $F\left(\overrightarrow{h_1^*}, ..., \overrightarrow{h_N^*}\right) = \psi\left(m\psi^{-1}\left(h_1^*\right) + ... + m\psi^{-1}\left(h_N^*\right)\right)$. Define $x(m, \alpha) = m\psi^{-1}\left(h_1^*\right) + ... + m\psi^{-1}\left(h_N^*\right)$ and the elasticity of the generator function $\psi(x(m, \alpha))$ as

$$\xi_\psi(x(m, \alpha)) = \frac{\psi'(x(m, \alpha))\, x(m, \alpha)}{\psi(x(m, \alpha))}.$$

In words, the elasticity is defined as the percentage change in $\psi(x(m, \alpha))$ relative to the percentage change in $x(m, \alpha)$. The properties characterizing Archimedean copulas guarantee that $\xi_\psi(x(m, \alpha)) < 0$ and $|\xi_\psi(x(m, \alpha))|$ is strictly increasing in $x(m, \alpha)$.

To evaluate the effects of financial expansion, I evaluate the marginal effects of m when $m = 1$.

Theorem 3 *If* $\frac{1+D}{1-F\left(h_1^*,...,h_N^*\right)} < \frac{1}{\left|\xi_\psi(x(1,\alpha))\right|}$, *then the belief dependence effect is small and financial expansion increases the leverage ratios. If* $\frac{1+D}{1-F\left(h_1^*,...,h_N^*\right)} > \frac{1}{\left|\xi_\psi(x(1,\alpha))\right|}$, *then the belief dependence effect is large and financial expansion decreases the leverage ratios.*

Proof. See Section B.6. ■

As $\alpha \to \infty$ (comonotonic beliefs), then $\xi_\psi(x(m, \alpha)) \to 0$ for all m. Such beliefs are labeled 'perfectly inelastic' beliefs. As $\alpha \to \infty$, then $\frac{1+D}{1-F\left(h_1^*,...,h_N^*\right)} < \frac{1}{\left|\xi_\psi(x(1,\alpha))\right|}$ and financial expansion increases the leverage ratios.

From the proof of Theorem 1, $F(h_1^*, ..., h_N^*; \alpha)$ is a strictly decreasing function of α. This implies that $\frac{1+mD}{1-F\left(\overrightarrow{h_1^*},...,\overrightarrow{h_N^*}\right)}$ is strictly decreasing in α. For the Clayton, Frank, and Joe copulas, $x(m, \alpha)$ is strictly decreasing in α. This implies that $\frac{1}{\left|\xi_\psi(x(m,\alpha))\right|}$ is strictly increasing in α. For any m (particularly $m = 1$), there exists at most one cutoff value α^* such that the wealth effect equals the belief dependence effect: $\frac{1+D}{1-F\left(h_1^*,...,h_N^*\right)} = \frac{1}{\left|\xi_\psi(x(1,\alpha))\right|}$. If a cutoff value α^* exists, then financial expansion decreases the leverage ratios for $\alpha < \alpha^*$ and increases the leverage ratios for $\alpha > \alpha^*$.

A cutoff value $\alpha^* > 0$ is guaranteed to exist for the Clayton copula, since it has the property that $|\xi_\psi(x(1, \alpha))| \to \infty$ as $\alpha \to 0$. Such beliefs are referred to as 'perfectly elastic' beliefs. For the Frank and Joe copula, a cutoff value may not exist. If a cutoff value does not exist, then financial expansion increases the leverage ratios for all values of belief dependence.

The following subsection considers an example that illustrates the joint effects of belief dependence and financial expansion on leverage.

4.2.3 A simple example

This example will demonstrate, for particular parameter values, the level of belief dependence above which financial expansion increases the leverage ratios. The effects of financial expansion on leverage will be seen in the example by comparing the $m = 1$ base economy to the $m = 2$ expanded economy.

Consider an economy with $N = 2$ asset and dividends $(d_1, d_2) = (0.3, 0.2)$. Suppose the cdf of household beliefs is governed by a Clayton copula with belief dependence parameters $\alpha \in \left\{\frac{1}{2}, \frac{4}{3}, 3, 8\right\}$. This corresponds to Kendall's tau values $\tau \in \{0.2, 0.4, 0.6, 0.8\}$, respectively. Table I in Appendix C shows the leverage ratios for the two assets as a function of the belief dependence parameter and the number of replications (m) of the base economy.

The theoretical results analyzed the local effect of financial expansion at $m = 1$. For the example, I compare the leverage ratios before financial expansion $(m = 1)$ and after financial expansion $(m = 2)$. The qualitative findings hold if I were to consider different values of m to define "after financial expansion."

Financial expanion leads to lower leverage ratios for both assets when $\alpha \in \left\{\frac{1}{2}, \frac{4}{3}\right\}$. For $\alpha = 3$, financial expansion has a negligible effect on the leverage ratios (both ratios decrease by a very small amount). This indicates that the cutoff value for this economy is close to $\alpha = 3$. For $\alpha = 8$, financial expansion leads to an increase in the leverage ratios for both assets.

The example illustrates that with two competing mechanisms to explain the effects of financial expansion on leverage, the dominance of either mechanism depends on belief dependence. The wealth effect is dominant for the case of $\alpha = 8$, which is the closest economy to comonotonic beliefs. The belief dependence effect is dominant for $\alpha \in \left\{\frac{1}{2}, \frac{4}{3}\right\}$ as the beliefs become less dependent and closer to independent beliefs.

5 Perfectly Dependent and Independent Beliefs

The two limits of the general copulas are the comonotonic beliefs and the independent beliefs. Comonotonic beliefs correspond to Archimedean copulas with $\alpha \to \infty$. Independent beliefs correspond to Clayton copulas with $\alpha \to 0$, Frank copulas with $\alpha \to 0$, and Joe copulas with $\alpha = 1$. Under Clayton copulas, comonotonic beliefs correspond to Kendall's tau equal to 1 and independent beliefs correspond to Kendall's tau equal 0. I provide the main results for these two extreme cases. The cdf for comonotonic beliefs is non-differentiable, so the equilibrium equations and results are gathered in Appendix A.3.

5.1 Comonotonic beliefs

Comonotonic beliefs means that the cumulative distribution is given by

$$F_{H_1,...,H_N}(h_1,....,h_N) = \min\{h_1,...,h_N\}.$$

The household beliefs satisfy $h_i = h_1$ for all $i \in \mathbf{I}$. Assign the beliefs such that household h has beliefs $h = h_i$ for all $i \in \mathbf{I}$.[8]

Define the cutoff household h_i^* such that the household with beliefs $h = h_i^*$ is indifferent between buying and selling asset i. For comonotonic beliefs (only), the cutoff household is identical for all assets: $h_i^* = h_1^*$ for all $i \in \mathbf{I}$ (see Appendix A.3). For simplicity, define h^* such that $h_i^* = h^*$ for all $i \in \mathbf{I}$. From the first order conditions for this cutoff household:

$$p_i = h^* + (1 - h^*)\,d_i. \tag{11}$$

All borrowing contracts can be traded, but the only ones actually traded in equilibrium are the following: contract $(1, d_1)$ with collateral of 1 unit of asset 1 and promise to repay d_1 units, contract $(2, d_2)$ with collateral of 1 unit of asset 2 and promise to repay d_2 units, and so forth for all assets $i \in \mathbf{I}$.

To illustrate the effects of asset heterogeneity, consider an economy with $N = 2$ assets. The leverage ratios for varying combinations of (d_1, d_2) are displayed in Table II in Appendix C. The leverage ratios are symmetric, meaning that if the dividend values are interchanged, then the leverage ratios are interchanged as well. From Table II, I conclude that lev_i is strictly increasing in d_i and strictly decreasing in d_j, where $j \neq i$. Further, an increase in $d_1 + d_2$, while holding the ratio $\frac{d_1}{d_2}$ fixed, leads to an increase in both leverage ratios. These results are consistent with the fact that the wealth effects are the dominant mechanism, namely that an increase in dividends means an increase in the wealth per asset market and an increase in the asset prices.

The effects of expanding the financial markets are captured using the concept of replica economies exactly as in Section 4. A replica economy is such that the number of assets N and the total dividends D are both scaled up by the factor $m \geq 1$ to create a new

[8] Since the marginal distributions of $F_{H_1,...,H_N}(h_1,....,h_N)$ are uniform, then h must be drawn from a $Unif[0,1]$ distribution, which is the same distribution used for the household indices.

economy with mN assets and mD total dividends. The beliefs after replication, represented by an $mN-$dimensional cdf, are comonotonic just as the original beliefs, represented by an $N-$dimensional cdf, are comonotonic.

Claim 4 *With comonotonic beliefs, the leverage ratios for all assets are a strictly increasing function of m, the number of replications of the financial side of the economy.*

Proof. See Appendix A.3. ■

This result is consistent with Theorem 3 as comonotonic beliefs imply perfectly inelastic beliefs such that $\frac{1+D}{1-F\left(h_1^*,...,h_N^*\right)} < \frac{1}{\left|\xi_\psi(x(1,\alpha))\right|}$.

5.2 Independent beliefs

Independent beliefs means that $(h_1, ..., h_N)$ are iid random variables, all drawn from a $Unif[0,1]$ distribution. Specifically, the cumulative distribution $F_{H_1,...,H_N}(h_1, ..., h_N) = h_1 h_2 \cdot \cdot \cdot h_N$.

Consider an economy with $N = 2$ assets. The leverage ratios for varying combinations of (d_1, d_2) are displayed in Table III in Appendix C. The leverage ratios are symmetric, meaning that if the dividend values are interchanged, then the leverage ratios are interchanged as well. From Table III, we conclude that lev_i is strictly increasing in d_i and strictly decreasing in d_j, where $j \neq i$. Further, an increase in $d_1 + d_2$, while holding the ratio $\frac{d_1}{d_2}$ fixed, leads to an increase in both leverage ratios.

Comparing Tables II and III, the leverage ratios are smaller under independent beliefs (compared to comonotonic beliefs). Further, the effects of a change in dividends are smaller under independent beliefs (compared to comonotonic beliefs). The size of the wealth effects is a function of the belief dependence and is smaller for independent beliefs (compared to comonotonic beliefs).

The cdf is differentiable, meaning that equilibrium characterization is identical to the case with copulas in Section 4. Financial expansion is defined exactly as in Section 4.

Claim 5 *With independent beliefs, the leverage ratios for all assets are a strictly decreasing function of m, the number of replications of the financial side of the economy.*

Proof. Independent beliefs are defined as the limit of the Clayton copula as $\alpha \to 0$. Since the Clayton copulas has the property that $|\xi_\psi(x(1,\alpha))| \to \infty$ as $\alpha \to 0$, then $\frac{1+D}{1-F\left(h_1^*,...,h_N^*\right)} > \frac{1}{\left|\xi_\psi(x(1,\alpha))\right|}$ and Theorem 3 implies that financial expansion decreases all leverage ratios. ■

6 Concluding Remarks

This paper examined how dependent beliefs across a portfolio of financial assets impact endogenous leverage. Increasing the dependence between beliefs leads to an increase in the leverage ratios for all assets. Financial expansion may or may not lead to an increase in the leverage ratios and this relation is governed by the size of the belief elasticity. When beliefs are perfectly inelastic, financial expansion always increases the leverage ratios. When beliefs are perfectly elastic, financial expansion always decreases the leverage ratios. The effects of financial expansion on leverage are characterized by a cutoff value for belief dependence, above which financial expansion increases leverage ratios and below which financial expansion decreases leverage ratios.

If policymakers are intent on regulating leverage, understanding the mechanisms that affect equilibrium leverage is crucial. This paper considered two possible economic primitives to explain movements in leverage: belief dependence and the number of financial markets. Future work seeks to quantify the size of these effects in a production economy with a longer time horizon that allows for the possibility of leverage cycles.

References

[1] Araujo, Aloisio, Felix Kubler, and Susan Schommer. 2012. "Regulating collateral-requirements when markets are incomplete." *Journal of Economic Theory* 147(2): 450-476.

[2] Clayton, D.G. 1978. "A model for association in bivariate life tables and its application in epidemiological studies of family tendency in chronic disease incidence." *Biometrika* 65(1): 141-151.

[3] Embrechts, Paul, Alexander McNeil, and Daniel Straumann. 1999. "Correlation and dependence in risk management: Properties and pitfalls." Mimeo, ETH Zurich RiskLab, ftp://ftp.sam.math.ethz.ch/pub/risklab/papers/CorrelationPitfalls.pdf.

[4] Embrechts, Paul, Rüdiger Frey, and Alexander J. McNeil. 2005. *Quantitative Risk Management: Concepts, Techniques and Tools.* Princeton, NJ: Princeton University Press.

[5] Eling, Martin, and Denis Toplek. 2009. "Modeling and management of nonlinear dependencies: Copulas in dynamic financial analysis." *The Journal of Risk and Insurance* 76(3): 651-681.

[6] Fostel, Ana, and John Geanakoplos. 2008. "Leverage cycle and the anxious economy." *American Economic Review* 98(4): 1211-1244.

[7] Fostel, Ana, and John Geanakoplos. 2012. "Why does bad news increase volatility and decrease leverage?" *Journal of Economic Theory* 147(2): 501-525.

[8] Fostel, Ana, and John Geanakoplos. 2015. "Leverage and default in binomial economies: A complete characterization." *Econometrica*, forthcoming.

[9] Fredericks, Gregory A., and Roger B. Nelson. 2007. "On the relationship between Spearman's rho and Kendall's tau for pairs of continuous random variables." *Journal of Statistical Planning and Inference* 137: 2143-2150.

[10] Geanakoplos, John. 1997. "Promises, promises." In: W.B. Arthur, S. Durlauf, and D. Lane (eds.) *The Economy as an Evolving Complex System II.* Reading, MA: Addison-Wesley (pp. 285-320).

[11] Geanakoplos, John. 2003. "Liquidity, default and crashes: Endogenous contracts in general equilibrium." In: *Advances in Economics and Econometrics: Theory and Applications, Eighth World Conference, Volume II.* Econometric Society Monographs (pp. 170-205).

[12] Geanakoplos, John. 2010. "The leverage cycle." In: Daron Acemoglu, Kenneth Rogoff and Michael Woodford (eds.) *NBER Macroeconomics Annual 2009, Volume 24.* Chicago: University of Chicago Press (pp. 1-45).

[13] Geanakoplos, John, and William R. Zame. 2014. "Collateral equilibrium I: A basic framework." *Economic Theory* 56(3): 443-492.

[14] Geerolf, Francois. 2015. "Leverage and disagreement." Working paper, UCLA, http://www.econ.ucla.edu/fgeerolf/geerolf-leverage.pdf.

[15] Gumbel, E.J. 1960. "Bivariate exponential distributions." *Journal of the American Statistical Association* 55: 698-707.

[16] Nelsen, Roger B. 2006. *An Introduction to Copulas.* New York: Springer.

[17] Phelan, Gregory. 2017. "Collateralized borrowing and increasing risk." *Economic Theory* 63(2): 471-502.

[18] Simsek, Alp. 2013. "Belief disagreement and collateral constraints." *Econometrica* 81(1): 1-53.

[19] Sklar, A. 1959. "Fonctions de répartition à n dimensions et leurs marges." *Publ. Inst. Statist. Univ. Paris* 8, 229-231.

A Technical Appendix

A.1 Statistics

For simplicity, order the dividends such that $d_1 \geq ... \geq d_N$. Claim 2 implies $h_1^* \geq ... \geq h_N^*$.

Consider an economy with N assets and focus on households with $(h_i, h_j) \geq \left(h_i^*, h_j^*\right)$ and $h_k < h_k^* \ \forall k \in \mathbf{I} \backslash \{i, j\}$, where $i < j$. From Claim 1, the expected payout from only holding asset i equals $\frac{h_i}{h_i^*}$ and the expected payout from only holding asset j equals $\frac{h_j}{h_j^*}$. A household with $(h_i, h_j) \geq \left(h_i^*, h_j^*\right)$ therefore holds asset i only when $\frac{h_i}{h_i^*} > \frac{h_j}{h_j^*}$ and asset j only when $\frac{h_i}{h_i^*} < \frac{h_j}{h_j^*}$. The measure zero subset of households with $\frac{h_i}{h_i^*} = \frac{h_j}{h_j^*}$ that are indifferent between assets i and j are irrelevant when we add the budget constraints over all households and apply the market clearing conditions.

Define the random variable $Z_{ij} = \frac{h_j}{h_i}$. The cdf $F_{Z_{ij}}(x_{ij}) = P(Z_{ij} < x_{ij}) = P\left(\frac{h_j}{h_i} < x_{ij}\right)$. Remember that the marginal distribution for h_i is the uniform distribution between h_i^* and 1 and the marginal distribution for h_j is the uniform distribution between h_j^* and 1. Therefore,

$$P\left(\frac{h_j}{h_i} < x_{ij}\right) = \int_{h_i^*}^{1} P\left(\frac{h_j}{h_i} < x_{ij} | h_i = y\right) dy = \int_{h_i^*}^{1} P(h_j < x_{ij} y) \frac{dy}{1 - h_i^*}.$$

Observe that $P(h_j < x_{ij} y) = 1$ if $x_{ij} y > 1$. Without loss of generality, we focus on $x_{ij} \leq 1$, meaning that $P(h_j < x_{ij} y) = \frac{x_{ij} y - h_j^*}{1 - h_j^*}$. This means that for $x_{ij} \leq 1:$

$$P\left(\frac{h_j}{h_i} < x_{ij}\right) = \int_{h_i^*}^{1} \frac{x_{ij} y - h_j^*}{(1 - h_i^*)\left(1 - h_j^*\right)} dy = \frac{1}{1 - h_j^*}\left(\frac{x_{ij}}{2}(1 + h_i^*) - h_j^*\right).$$

Returning to the problem, the probability that households with $(h_i, h_j) \geq \left(h_i^*, h_j^*\right)$ only hold asset i is found by setting $x_{ij} = \frac{h_j^*}{h_i^*}:$

$$P\left(\frac{h_j}{h_i} < \frac{h_j^*}{h_i^*}\right) = \frac{1}{2} \frac{h_j^*}{h_i^*} \frac{(1 - h_i^*)}{\left(1 - h_j^*\right)} \leq \frac{1}{2}.$$

The probability that households with $(h_i, h_j) \geq \left(h_i^*, h_j^*\right)$ only hold asset j is equal to $1 - \frac{1}{2} \frac{h_j^*}{h_i^*} \frac{(1 - h_i^*)}{\left(1 - h_j^*\right)} \geq \frac{1}{2}$. Using the law of large numbers, these probabilities are equal to fractions of households.

For any $i < j$, define $\xi^{ij} = \frac{h_j^*}{h_i^*}\frac{\left(1-h_i^*\right)}{\left(1-h_j^*\right)}$. By definition, $\xi^{ij} \leq 1$. For the case where the choice is between only $m = 2$ assets, define $\left(\Delta_i^{ij}, \Delta_j^{ij}\right) \in \left\{\mathbb{R}^2 : \Delta_i^{ij} + \Delta_j^{ij} = 0\right\}$ such that the fraction of households with $(h_i, h_j) \geq \left(h_i^*, h_j^*\right)$ that only hold asset i equals $\frac{1}{2}\left(1 + \Delta_i^{ij}\right)$ and the fraction that only hold asset j equals $\frac{1}{2}\left(1 + \Delta_j^{ij}\right)$. The specific values are $\Delta_i^{ij} = \xi^{ij} - 1$ and $\Delta_j^{ij} = 1 - \xi^{ij}$.

Now focus on the choice between $m = 3$ assets, specifically households with $(h_i, h_j, h_k) \geq \left(h_i^*, h_j^*, h_k^*\right)$ and $h_l < h_l^*$ $\forall l \in \mathbf{I} \backslash \{i, j, k\}$, where $i < j < k$. These households only hold asset i when $\frac{h_i}{h_i^*} > \max\left\{\frac{h_j}{h_j^*}, \frac{h_k}{h_k^*}\right\}$. Proceeding as above:

$$P\left(\frac{h_j}{h_i} < x_{ij} \;\&\; \frac{h_k}{h_i} < x_{ik}\right) = \int_{h_i^*}^{1} P\left(\frac{h_j}{h_i} < x_{ij} \;\&\; \frac{h_k}{h_i} < x_{ik} | h_i = y\right) dy$$
$$= \int_{h_i^*}^{1} P\left(h_j < x_{ij} y\right) P\left(h_k < x_{ik} y\right) \frac{dy}{1 - h_i^*}.$$

We only consider $x_{ij} \leq 1$ and $x_{ik} \leq 1$. The calculus simplifies to:

$$P\left(\frac{h_j}{h_i} < x_{ij} \;\&\; \frac{h_k}{h_i} < x_{ik}\right) = \frac{\frac{x_{ij}x_{ik}}{3}\left(1 + h_i^* + (h_i^*)^2\right) - \frac{x_{ij}h_k^* + x_{ik}h_j^*}{2}\left(1 + h_i^*\right) + h_j^* h_k^*}{\left(1 - h_j^*\right)\left(1 - h_k^*\right)}.$$

The fraction of households that only hold asset i is found by setting $x_{ij} = \frac{h_j^*}{h_i^*}$ and $x_{ik} = \frac{h_k^*}{h_i^*}$:

$$P\left(\frac{h_j}{h_i} < \frac{h_j^*}{h_i^*} \;\&\; \frac{h_k}{h_i} < \frac{h_k^*}{h_i^*}\right) = \frac{1}{3}\left(\frac{h_j^*}{h_i^*}\frac{\left(1 - h_i^*\right)}{\left(1 - h_j^*\right)}\right)\left(\frac{h_k^*}{h_i^*}\frac{\left(1 - h_i^*\right)}{\left(1 - h_k^*\right)}\right).$$

Using the definition for ξ^{ij} and ξ^{ik}, then $P\left(\frac{h_j}{h_i} < \frac{h_j^*}{h_i^*} \;\&\; \frac{h_k}{h_i} < \frac{h_k^*}{h_i^*}\right) = \frac{1}{3}\xi^{ij}\xi^{ik}$. For the case where the choice is between only $m = 3$ assets, define $\left(\Delta_i^{ijk}, \Delta_j^{ijk}, \Delta_k^{ijk}\right) \in \left\{\mathbb{R}^3 : \Delta_i^{ijk} + \Delta_j^{ijk} + \Delta_k^{ijk} = 0\right\}$ such that the fraction of households that only hold asset $p \in \{i, j, k\}$ equals $\frac{1}{3}\left(1 + \Delta_p^{ijk}\right)$. Since $P\left(\frac{h_j}{h_i} < \frac{h_j^*}{h_i^*} \;\&\; \frac{h_k}{h_i} < \frac{h_k^*}{h_i^*}\right) = \frac{1}{3}\xi^{ij}\xi^{ik}$, then $\Delta_i^{ijk} = \xi^{ij}\xi^{ik} - 1$. The remaining fraction $\left(1 - \frac{1}{3}\left(1 + \Delta_i^{ijk}\right)\right)$ of households either hold asset j or asset k. This reduces to the case with a choice between $m = 2$ assets, meaning that $\frac{1}{2}\left(1 + \Delta_j^{jk}\right)$ of the fraction $\left(1 - \frac{1}{3}\left(1 + \Delta_i^{ijk}\right)\right)$ only hold asset j. Setting $\frac{1}{3}\left(1 + \Delta_j^{ijk}\right)$ equal to the fraction that only hold asset j, I find $\Delta_j^{ijk} = \Delta_j^{jk} - \frac{1}{2}\Delta_i^{ijk} - \frac{1}{2}\Delta_j^{jk}\Delta_i^{ijk}$. The remaining households hold asset k, implying $\Delta_k^{ijk} = -\Delta_i^{ijk} - \Delta_j^{ijk}$.

The process continues by induction. Consider any $4 \leq m \leq N$ and the $m-$dimensional index $ijk....l$. Consider households such that $h_p \geq h_p^*$ for all assets p in the index $ijk....l$

and $h_p < h_p^*$ for all other assets. The variable $\Delta_p^{ijk....l}$ is defined such that the fraction of these households that only choose asset p equals $\frac{1}{m}\left(1+\Delta_p^{ijk....l}\right)$. The construction of $\Delta_p^{ijk....l}$ proceeds by induction. The order in the index is such that $i < j < k < ... < l$. For the asset i, $\Delta_i^{ijk....l} = (\xi_{ij}) \cdots (\xi_{il}) - 1$. There now remain $(m-1)$ terms in the index $jk.....l$. Of the remaining fraction of households $\left(1 - \frac{1}{m}\left(1+\Delta_i^{ijk....l}\right)\right)$, the fraction $\frac{1}{m-1}\left(1+\Delta_j^{jk....l}\right)$ hold asset j, so $\Delta_j^{ijk....l}$ is defined such that:

$$\frac{1}{m}\left(1+\Delta_j^{ijk....l}\right) = \left(1 - \frac{1}{m}\left(1+\Delta_i^{ijk....l}\right)\right)\frac{1}{m-1}\left(1+\Delta_j^{jk....l}\right).$$

There now remain $(m-2)$ terms in the index and the process continues until values for all variables $\Delta_p^{ijk....l}$ are specified. By construction, $-1 < \Delta_p^{ijk....l} < 1$ for all Δ terms.

A.2 Risk Aversion

A.2.1 One asset case

Consider risk-averse households. For simplicity, consider the single-asset $(N = 1)$ case. With risk-aversion, the cutoff household h^* has beliefs $h = h^*$ and is indifferent between buying and selling the asset. From the first order conditions for this cutoff household:

$$pu'\left(c^h(0)\right) = h^* u'\left(c^h(H)\right) + (1-h^*)\, u'\left(c^h(L)\right) d, \tag{12}$$

where $\mathbf{S} = \{H, L\}$, the two dividend realizations for the single asset. With indifference, suppose h^* is a lender. The lender holds $a^h = 0$ units of the asset, implying that $c^h(H) = c^h(L)$. The lender uses a combination of storage and lending to transfer resources from the initial period $t = 0$ (with income $1+p$) into states H and L (with income 0). Both storage and lending take place such that consumption is perfectly smoothed: $c^h(0) = c^h(H) = c^h(L)$. This implies that equation (12) reduces to the original:

$$p = h^* + (1-h^*)\, d.$$

With only a single asset, the only collateral is this asset. The possible borrowing contracts are $j \in \mathbb{R}_+$. The following result is identical to the Binomial No-Default Theorem of Fostel and Geanakoplos (2015) and general results by Araujo et al. (2012). The borrowing contract $j = d$ is termed the no-default borrowing contract as this is the highest promise repayment such that the borrower always repays its promised amount in all states. For all $j < d$, the payout vector is identical to the borrowing contract $j = d$. For all $j \geq 1$, the payout vector is identical to the asset itself (pays out 1 in state H and d in state L). For all $j \in (d, 1)$, the payout vector is equal to $\min\{j, 1\} = j$ in state H and $\min\{j, d\} = d$ in state L. This is simply a convex combination of the borrowing contract $j = d$ and the asset (the weight $\theta = \frac{1-j}{1-d}$ is placed on contract $j = d$ and the remaining weight $1 - \theta = \frac{j-d}{1-d}$ on the asset). For any households, including risk-averse ones, any contract $j \in \mathbb{R}_+ \backslash \{d\}$ can be traded in equilibrium, but it will be linearly dependent, meaning that the real equilibrium variables resulting from the trading of a contract $j \in \mathbb{R}_+ \backslash \{d\}$ are equivalent to the real equilibrium variables when only the no-default borrowing contract $j = d$ is traded.

Since it is innocuous (no real effects) to assume that the only contract traded is $j = d$, then I can use the budget constraints and first order conditions for $h \geq h^*$, together with market clearing conditions, to derive the remaining equilibrium equations relating p and h^*. Under risk-neutrality, the first order condition for households $h \geq h^*$ implies that they would set $c^h(0) = s^h = 0$ in order to purchase as many units of the asset as possible. This allowed me to pin down the asset holding a^h as all initial income $1 + p$ is spent for asset purchase. With risk-aversion, households $h \geq h^*$ optimally choose $c^h(0)$ and s^h to satisfy budget constraints and first order conditions.

The challenge with the equilibrium characterization under risk-aversion is two-fold: (i) more equilibrium equations (only 2 under risk-neutrality) and (ii) households $h \geq h^*$ no longer have symmetric asset holdings (as they did under risk-neutrality). With risk-neutrality, the asset holdings for $h \geq h^*$ were symmetric as they were constrained by the initial endowment. With risk-aversion, the asset holdings for $h \geq h^*$ are heterogeneous as they depend upon the belief parameter h.

Notice that this result does not extend outside the binomial economy. As shown in Araujo et al. (2012), with $S > 2$ possible dividend realizations, then there exists $S - 1$ borrowing contracts that are linearly independent. From this finite set of borrowing contracts, the optimal choice is a function of the degree of risk aversion.

A.2.2 Multiple asset case

With multiple assets ($N \geq 2$), we can still work with the binomial economy framework as each asset has binomial payouts and is an independent distribution. To do this, we have to impose the following additional assumption.

> Assumption A A borrowing contract is supported by the holding of one asset as collateral. It does not matter which of the N assets are specified by the contract, but combinations of different assets is not accepted as collateral.

Defining the unit vector $e_i = (0, ...0, 1, 0, ..., 0)$ with i^{th} element equal to 1, then the set of possible collateral is $\theta \in \{e_1, ..., e_N\}$. I will first show why Assumption A allows us to extend the results from the one asset case and then discuss why combinations of collateral $\theta \in \Delta^{N-1} \setminus \{e_1, ..., e_N\}$ are impermissible with risk aversion.

For any borrowing contract with collateral $\theta \in \{e_1, ..., e_N\}$, the results from the Binomial No-Default Theorem and the one asset case above are naturally extended and imply that either (i) in equilibrium, only the borrowing contracts $\{(e_1, d_1), ..., (e_N, d_N)\}$ are traded or (ii) any other equilibrium has identical real equilibrium variables as the equilibrium in which only $\{(e_1, d_1), ..., (e_N, d_N)\}$ are traded.

Suppose, in order to understand the sticking point in the argument, that Assumption A is not imposed. With N assets, each borrowing contract is secured with 1 total unit of asset (our normalization) and this 1 total unit can be any combination of the N available assets. Denote $\theta \in \Delta^{N-1}$ as one of the possible combinations. For each θ, the possible borrowing contracts are $j \in \mathbb{R}_+$.

Define $\#\mathbf{I}^+$ as the number of elements in the set $\mathbf{I}^+ = \{i \in \mathbf{I} : \theta_i > 0\}$. Consider any $\theta \in \Delta^{N-1} \setminus \{e_1, ..., e_N\}$, meaning that $\#\mathbf{I}^+ > 1$ for the contract under consideration. The payout in any date-event $s \in \mathbf{S}$ is equal to $\min \left\{ \sum_{i \in \mathbf{I}} \theta_i d_i(s), k \right\}$. Generically on θ, there are a total of $2^{\#\mathbf{I}^+}$ date-events with distinct values for $\sum_{i \in \mathbf{I}} \theta_i d_i(s)$. Even allowing for redundancy, with $\#\mathbf{I}^+ > 1$, the minimum number of distinct values for $\sum_{i \in \mathbf{I}} \theta_i d_i(s)$ is equal to 3 (this includes $\sum_{i \in \mathbf{I}} \theta_i d_i$ and 1 and at least one value in between). With 3 or more date-events (defined in terms of distinct payouts), we are no longer in the binomial framework. As shown in Araujo et al. (2012), with more than 2 date-events, it is no longer the case that the no-default loan is the only contract traded. With S date-events, there are $S - 1$ linearly independent contracts that can be traded in equilibrium. This result holds for all monotonic household preferences (including risk-aversion) as the result has nothing to do with preferences, but only with the geometry of the payout vectors. In fuller detail, the equilibrium in which the traded contracts are optimally chosen from the set of $S - 1$ linearly independent contracts has the same values for the real equilibrium variables as any other equilibrium.

This exposition identifies the key trade-off. With risk-averse household, Assumption A is required in order to guarantee that the equilibrium is equivalent to one in which only no-default loans are traded. With risk-averse households and without Assumption A, equilibrium real effects can arise from (i) default in equilibrium and (ii) multiple contracts traded with the same collateral requirement. Future research is required in such settings in order to fully analyze the equilibrium characterization and the belief dependence effects.

A.3 Comonotonic Beliefs

For simplicity, denote the household contract choices for (i, d_i) as b_i^h and the contract price as q_i. The payout of contract (i, d_i) is equal to d_i in all date-events in the final period (a no-default loan). For all households, the first order conditions require that:

$$q_i = d_i.$$

Claim 6 *Under comonotonic beliefs, $h_i^* = h_1^*$ for all $i \in \mathbf{I}$.*

Proof. Suppose, in order to obtain a contradiction, that $h_i^* > h_k^*$ for some i, k. The down payment for asset i in the initial period is $p_i - d_i$. For each one unit of investment (which allows for $\frac{1}{p_i - d_i}$ units of the asset to be purchased), the difference between the expected payout and the price for asset i is:

$$\frac{(h_i + (1 - h_i)\, d_i) - p_i}{p_i - d_i} = \frac{h_i}{h_i^*} - 1,$$

after using the definition of the cutoff (11). In similar fashion, for each one unit of investment on asset k, the difference between the expected payout and the price is equal to:

$$\frac{(h_k + (1 - h_k)\, d_k) - p_k}{p_k - d_k} = \frac{h_k}{h_k^*} - 1.$$

Under comonotonic beliefs, $h = h_i = h_k$. For $h \geq h_i^*$, both differences are positive. Since $h_i^* > h_k^*$, then the difference for asset k is larger, meaning households $h \geq h_i^*$ only purchase asset k.

By definition, households $h < h_i^*$ will not purchase asset i. This violates the market clearing for asset i, since some households must be willing to purchase asset i in order to satisfy the market clearing condition $\int_{h \in \mathbf{H}} a_i^h = 1$. ■

For simplicity, define $h^* = h_i^*$ for all $i \in \mathbf{I}$.

In the initial period, households with $h \geq h^*$ are indifferent about which asset to buy. This feature is unique to comonotonic beliefs. Households $h \geq h^*$ set $c^h(0) = s^h = 0$ in order to purchase as many units of the assets as possible.

On the other side, households $h < h^*$ sell all assets to the point where the short-sale constraints bind: $a_i^h = 0$ for all $i \in \mathbf{I}$.

The collateral constraint under EA is given by:

$$\max\left\{b_i^h, 0\right\} \leq a_i^h \quad \forall i \in \mathbf{I}.$$

Risk-neutral households that choose to borrow will borrow up until the point that the collateral constraint binds.

Claim 7 *If $h \geq h^*$, then $b_i^h = a_i^h$ for all $i \in \mathbf{I}$.*

Proof. This proof is a special case of the proof of Claim 1. ■

Using these facts, add up the initial period budget constraints (1) for all households $h \geq h^*$:

$$\sum_{i \in \mathbf{I}} (p_i - d_i) \int_{h^*}^{1} a_i^h dh = (1 - h^*) \left(1 + \sum_{i \in \mathbf{I}} p_i\right). \tag{13}$$

From the market clearing condition:

$$\sum_{i \in \mathbf{I}} p_i - \sum_{i \in \mathbf{I}} d_i = (1 - h^*) \left(1 + \sum_{i \in \mathbf{I}} p_i\right). \tag{14}$$

The equilibrium equations are:

$$\sum_{i \in \mathbf{I}} p_i - \sum_{i \in \mathbf{I}} d_i = (1 - h^*) \left(1 + \sum_{i \in \mathbf{I}} p_i\right) \tag{15}$$

and

$$p_i = h^* + (1 - h^*) \, d_i \quad \forall i \in \mathbf{I}.$$

Define $D = \sum_{i \in \mathbf{I}} d_i$. Plug the price equations into the first equilibrium equation to obtain a single equation in terms of the variable h^*. The single equation can be written as a quadratic equation in the form:

$$(h^*)^2 (N - D) + h^* (1 + D) - (1 + D) = 0.$$

The equilibrium equation indicates that h^* does not depend on the dividend distribution, only on N and total dividends D.

Claim 8 $h^* \in (0, 1)$.

Proof. Since $d_i < 1$ for all $i \in \mathbf{I}$, then $N > D$. Using this fact, there exists a unique strictly positive solution to the quadratic equation:

$$h^* = \frac{-(1 + D) + \sqrt{(1 + D)^2 + 4(N - D)(1 + D)}}{2(N - D)}.$$

Since $N > D$, it is straightforward to show that $h^* \in (0, 1)$. ■

Denote lev_i as the leverage ratio for asset i. The price of asset i equals p_i and the loan size is equal to the low dividend payout d_i. By definition,

$$lev_i = \frac{p_i}{p_i - d_i}. \tag{16}$$

Using the equilibrium price equations, the leverage ratios are equivalently expressed as:

$$lev_i = \frac{h^* (1 - d_i) + d_i}{h^* (1 - d_i)}.$$

Leverage lev_i is strictly decreasing in h^* and strictly increasing in d_i. This is verified via the derivatives:

$$\frac{\partial lev_i}{\partial h^*} = \frac{-d_i (1 - d_i)}{[h^* (1 - d_i)]^2} < 0.$$
$$\frac{\partial lev_i}{\partial d_i} = \frac{h^*}{[h^* (1 - d_i)]^2} > 0.$$

Theorem 4 *With comonotonic beliefs, the leverage ratios for all assets are a strictly increasing function of m, the number of replications of the financial side of the economy.*

Proof. For the replica economies, the quadratic equation is given by

$$(h^*)^2 (mN - mD) + h^* (1 + mD) - (1 + mD) = 0.$$

Notice that if the endowment was scaled up proportionately (by m), then the wealth per asset market would not change and the equilibrium (h^*, p) would not change. There exists a unique strictly positive solution to the quadratic equation:

$$h^* = \frac{-(1 + mD) + \sqrt{(1 + mD)^2 + 4m(N - D)(1 + mD)}}{2m(N - D)}.$$

As before, $h^* \in (0, 1)$. Using the quotient rule:

$$\frac{\partial h^*}{\partial m} = \frac{2(N - D)\Psi}{[2m(N - D)]^2},$$

where

$$\Psi = 1 + \frac{(1 + mD)^2 - (1 + mD) + 4m(N - D)(1 + mD) - 2m(N - D)}{\sqrt{(1 + mD)^2 + 4m(N - D)(1 + mD)}}$$
$$- \sqrt{(1 + mD)^2 + 4m(N - D)(1 + mD)}.$$

Algebraically $\Psi < 0$ iff

$$\sqrt{(1+mD)^2 + 4m(N-D)(1+mD)} < (1+mD) + 2m(N-D),$$

where the latter strict inequality holds using the same algebra used in the proof of Claim 8. Therefore, $\frac{\partial h^*}{\partial m} < 0$.

In each asset market, the dividends remain unchanged. From above, $\frac{\partial lev_i}{\partial h^*} < 0$. Therefore, $\frac{\partial lev_i}{\partial m} > 0$ for all asset markets. ■

B Proofs

B.1 Proof of Claim 1

For simplicity, define $D = \sum_{i \in \mathbf{I}} d_i$ and $P = \sum_{i \in \mathbf{I}} p_i$.

Households $h : a_i^h > 0$ set $c^h(0) = s^h = 0$ in order to purchase as many units of asset i as possible. Their objective function is equal to expected consumption across all states $s \in \mathbf{S}$. The payout in state $s \in \mathbf{S}$ is given by

$$a_i^h d_i(s) - b_i^h d_i.$$

For s such that $d_i(s) = 1$, the payout equals $a_i^h - b_i^h d_i$. For s such that $d_i(s) = d_i$, the payout equals 0. The expected consumption for households $h : a_i^h > 0$ equals

$$h_i \left(a_i^h - b_i^h d_i \right).$$

Define the fraction ϕ_i^h such that $b_i^h = \phi_i^h a_i^h$. The collateral constraint requires that $\phi_i^h \leq 1$. The expected consumption is therefore expressed as

$$h_i a_i^h \left(1 - \phi_i^h d_i \right) = h_i a_i^h (1 - d_i) + h_i a_i^h \left(1 - \phi_i^h \right) d_i.$$

From the household budget constraint in the initial period:

$$a_i^h (p_i - d_i) + a_i^h \left(1 - \phi_i^h \right) d_i = 1 + P.$$

From (5),

$$p_i - d_i = h_i^* (1 - d_i).$$

The expected consumption is equal to:

$$\frac{h_i}{h_i^*} (1 + P) + \left(h_i - \frac{h_i}{h_i^*} \right) a_i^h \left(1 - \phi_i^h \right) d_i.$$

Since $h_i < \frac{h_i}{h_i^*}$, then expected consumption is maximized at $\phi_i^h = 1$. Expected consumption is simply $\frac{h_i}{h_i^*} (1 + P)$. Households $h : a_i^h > 0$ must be such that $h_i \geq h_i^*$. The expected consumption $\frac{h_i}{h_i^*} (1 + P)$ is higher than what is possible through storage alone. The optimal choice for households $h : a_i^h > 0$ is to borrow so that the collateral constraints bind on all traded contracts.

I next verify that market clearing permits such levels of borrowing.

In equilibrium, each household $h : a_i^h > 0$ borrows the amounts $q_i b_i^h = d_i a_i^h$. The total amount borrowed equals

$$\int_{h:a_i^h>0} d_i a_i^h dh.$$

By market clearing, the total amount borrowed by households $h : a_i^h > 0$ equals d_i. Across all assets, the total amount borrowed equals D. The total amount lent must also be D.

Recall that households with beliefs $(h_1, ..., h_N) \leq (h_1^*, ..., h_N^*)$ sell all assets and become lenders. This means that the loan size for each lender is equal to $\frac{D}{F_{H_1,...,H_N}\left(h_1^*,....,h_N^*\right)}$. The income that lenders have available to lend has value $1 + P$. Lenders have sufficient resources to lend provided that

$$(1 + P) F_{H_1,...,H_N} (h_1^*, ..., h_N^*) > D. \tag{17}$$

Add together equation (8) over all $i \in \mathbf{I}$. The resulting equation is:

$$P - D = \left(\sum_{i \in \mathbf{I}} Fr \left\{ a_i^h > 0 \right\} \right) (1 + P) . \tag{18}$$

From (6),

$$\sum_{i \in \mathbf{I}} Fr \left\{ a_i^h > 0 \right\} = 1 - F_{H_1,...,H_N} (h_1^*, ..., h_N^*) .$$

Using this fact, equation (18) can be rewritten:

$$(1 + P) F_{H_1,...,H_N} (h_1^*, ..., h_N^*) = 1 + D. \tag{19}$$

Equation (19) implies that inequality (17) is satisfied.

As the market for loans is slack, all households $h : a_i^h > 0$ will borrow until the collateral constraint binds.

B.2 Proof of Claim 2

From (5) and (8),

$$\frac{h_i^*(1 - d_i)}{Fr \left\{ a_i^h > 0 \right\}} = 1 + D + \sum_{i \in \mathbf{I}} h_i^*(1 - d_i).$$

This implies that for any i, j :

$$\frac{h_i^*(1 - d_i)}{h_j^*(1 - d_j)} = \frac{Fr \left\{ a_i^h > 0 \right\}}{Fr \left\{ a_j^h > 0 \right\}}. \tag{20}$$

Suppose, to construct a proof by contraposition, that $h_i^* \leq h_j^*$. From Appendix A.1, the fraction of households that strictly prefer asset i is larger than the fraction of households that strictly prefer asset $j : Fr\left\{a_i^h > 0\right\} \geq Fr\left\{a_j^h > 0\right\}$. From equation (20), this implies:

$$h_i^*(1 - d_i) \geq h_j^*(1 - d_j).$$

This holds only if $(1 - d_i) \geq (1 - d_j)$, which is equivalent to $d_i \leq d_j$.

B.3 Proof of Claim 3

Consider any asset $\theta \in \Delta^{N-1}$. Define contract $j^*(\theta) = \sum_{i \in \mathbf{I}} \theta_i d_i$. This corresponds to the convex combination of the asset values when all assets have a low dividend realization. I will compare the borrowing contracts $(\theta, j^*(\theta))$ and (θ, k) for any $k \in \mathbb{R}_+ \backslash \{j^*(\theta)\}$.

First, let $k < j^*(\theta)$. The returns for contracts $(\theta, j^*(\theta))$ and (θ, k) are equal in all date-events $s \in \mathbf{S}$, but borrowers are required to hold more collateral per unit borrowed under contract (θ, k). For this reason, they strictly prefer $(\theta, j^*(\theta))$.

Second, let $k > j^*(\theta)$. The contract $(\theta, j^*(\theta))$ has risk-free payouts in all date-events $s \in \mathbf{S}$. The household first order conditions under risk-neutrality and no discounting implies that the return equals 1 in all date-events $s \in \mathbf{S}$.

The expected payout for household h from contract (θ, k) is denoted γ^h and is defined by

$$\gamma^h = \sum_{s \in \mathbf{S}} \pi^h(s) \min\left\{\sum_{i \in \mathbf{I}} \theta_i d_i(s), k\right\},$$

where $d_i(s)$ is the dividend for asset i in date-event s.

By definition, the cutoff household h_i^* is such that

$$p_i = h_i^* + (1 - h_i^*) d_i.$$

Denote $\mathbf{I}^+ = \{i \in \mathbf{I} : \theta_i > 0\}$. Consider a household with beliefs $h_i = h_i^*$ for all $i \in \mathbf{I}^+$. This household is indifferent between buying and selling the assets $i \in \mathbf{I}^+$ and is therefore indifferent between borrowing and lending using the contract (θ, k). The asset price $q_k(\theta)$ is defined as the expected payout γ^h for the household with beliefs $h_i = h_i^*$ for all $i \in \mathbf{I}^+$. This household has expected return equal to 1, but the returns vary across date-events $s \in \mathbf{S}$.

Under risk-neutrality, maximizing household utility is equivalent to maximizing expected return for lenders and minimizing expected return for borrowers. Given that households have the option of contract $(\theta, j^*(\theta))$ with return equal to 1, households with expected payout $\gamma^h < q_k(\theta)$ would only be willing to borrow using contract (θ, k) and households with expected payout $\gamma^h > q_k(\theta)$ would only be willing to lend using contract (θ, k). Consider two households (h, h') such that h finds it optimal to borrow and h' finds it optimal to lend. By definition, $\gamma^h < \gamma^{h'}$. Household h has beliefs $(h_1, ..., h_N)$ and household h' has beliefs $(h'_1, ..., h'_N)$. By definition, there exists some $i \in \mathbf{I}^+$ such that $h_i < h'_i$.

To borrow, household h must hold a portfolio of collateral with θ_i units of each asset i. It is only optimal to purchase asset i if $h_i \geq h_i^*$, so this requires that $h_i \geq h_i^* \ \forall i \in \mathbf{I}^+$. Combining the previous two facts, there exists some i such that $h'_i > h_i^*$. As a lender, it is optimal for household h' to not hold any units of asset. This contradicts that $h'_i > h_i^*$. Thus, households with expected payout $\gamma^h < q_k(\theta)$ do not borrow on contract (θ, k) and households with expected payout $\gamma^h > q_k(\theta)$ do not lend using contract (θ, k). The contract (θ, k) is not traded. Out of all borrowing contracts $(\theta, k)_{k \in \mathbb{R}_+}$, the only contract traded is $(\theta, j^*(\theta))$.

Consider the set of contracts $\{(1, d_1), ..., (N, d_N)\}$ and consider any contract

$$(\theta, k) \in (\theta, j^*(\theta))_{\theta \in \Delta^{N-1}} \setminus \{(1, d_1), ..., (N, d_N)\}.$$

The contract $(\theta, j^*(\theta))$ has the identical return to all contracts $\{(1, d_1), ..., (N, d_N)\}$ since the payouts in all states $s \in \mathbf{S}$ are equal to $j^*(\theta)$, a risk-free payout. The cost of acquiring such a portfolio of collateral is identical to the cost of acquiring a portfolio with θ_i units of each asset i. Therefore, out of the set $(\theta, j^*(\theta))_{\theta \in \Delta^{N-1}}$, there are only N linearly independent contracts. If we specify those contracts as the unit contracts $\{(1, d_1), ..., (N, d_N)\}$, then any real equilibrium variables must be equivalent to the real variables in an equilibrium in which only the contracts $\{(1, d_1), ..., (N, d_N)\}$ are traded.

B.4 Proof of Theorem 1

The proof is broken into two parts:

1. An increase in $h_{-i}^* = \left(h_1^*, .., h_{i-1}^*, h_{i+1}^*, ..., h_N^*\right)$ leads to an increase in h_i^*.
2. Holding h_{-i}^* fixed, an increase in the parameter α leads to an increase h_i^*.

B.4.1 Part 1

Consider an economy with $N \geq 2$ assets. Recall that $D = \sum_{p \in \mathbf{I}} d_p$. Consider the equilibrium equation (9) for asset i :

$$h_i^*(1 - d_i) = Fr\left\{a_i^h > 0\right\}\left(1 + D + \sum_{p \in \mathbf{I}} h_p^*(1 - d_p)\right). \tag{21}$$

Define $\omega^1 = F(1, h_2^*, ..., h_N^*) - F(h_1^*, h_2^*, ..., h_N^*)$ and similar for $(\omega^i)_{i \in \mathbf{I} \setminus \{1\}}$. Define $\omega^{12} = F(1, 1, h_3^*, ..., h_N^*) - F(h_1^*, h_2^*, h_3^*, ..., h_N^*)$ and similar for $(\omega^{ij})_{(i,j) \in \mathbf{I}^2 \setminus \{(1,2)\}}$. The order for the superscripts does not matter, so $\omega^{12} = \omega^{21}$. Continue to define all terms until $\omega^{12...N} = F(1, ..., 1) - F(h_1^*, h_2^*, ..., h_N^*)$. For any subset $S \subseteq \mathcal{P}(\mathbf{I})$, where $\mathcal{P}(\mathbf{I})$ is the 2^N-dimensional power set of $\mathbf{I}$, the fraction ω^S indicates the fraction of households that will purchase an asset from the subset S.

The fraction ω^S is the fraction of households such that expected payout exceeds price for at least one asset in S (and not for any assets in $S_{\mathbf{I}}^C$). I define the fraction $\hat{\omega}^S$ as the fraction of households such that the expected payout exceeds price for all assets in S (and not for any assets in $S_{\mathbf{I}}^C$). If S contains $\#S$ assets, then the fraction $\hat{\omega}^S$ is divided into $\#S$ groups and each household optimally chooses to hold just one of the S assets.[9]

Using combinatorics, the fractions $\hat{\omega}$ are defined as follows:

1. For all $i \in \mathbf{I} : \hat{\omega}^i = \omega^i$.
2. For all $(i, j) \in \mathbf{I}^2 : \hat{\omega}^{ij} = \omega^{ij} - (\hat{\omega}^i + \hat{\omega}^j)$.
3. For all $(i, j, k) \in \mathbf{I}^3 : \hat{\omega}^{ijk} = \omega^{ijk} - \left(\hat{\omega}^i + \hat{\omega}^j + \hat{\omega}^k\right) - \left(\hat{\omega}^{ij} + \hat{\omega}^{ik} + \hat{\omega}^{jk}\right)$.
4. For all $(i, j, k, l) \in \mathbf{I}^4 : \hat{\omega}^{ijkl} = \omega^{ijkl} - \left(\hat{\omega}^i + \hat{\omega}^j + \hat{\omega}^k + \hat{\omega}^l\right)$
   $- \left(\hat{\omega}^{ij} + \hat{\omega}^{ik} + \hat{\omega}^{il} + \hat{\omega}^{jk} + \hat{\omega}^{jl} + \hat{\omega}^{kl}\right) - \left(\hat{\omega}^{ijk} + \hat{\omega}^{ijl} + \hat{\omega}^{ikl} + \hat{\omega}^{jkl}\right)$.
5. The pattern continues...

It is straightforward to express the definitions for $\hat{\omega}$ only in terms of ω :

1. For all $i \in \mathbf{I} : \hat{\omega}^i = \omega^i$.
2. For all $(i, j) \in \mathbf{I}^2 : \hat{\omega}^{ij} = \omega^{ij} - (\omega^i + \omega^j)$.
3. For all $(i, j, k) \in \mathbf{I}^3 : \hat{\omega}^{ijk} = \omega^{ijk} - \left(\omega^{ij} + \omega^{ik} + \omega^{jk}\right) + \left(\omega^i + \omega^j + \omega^k\right)$.
4. For all $(i, j, k, l) \in \mathbf{I}^4 : \hat{\omega}^{ijkl} = \omega^{ijkl} - \left(\omega^{ijk} + \omega^{ijl} + \omega^{ikl} + \omega^{jkl}\right)$
   $+ \left(\omega^{ij} + \omega^{ik} + \omega^{il} + \omega^{jk} + \omega^{jl} + \omega^{kl}\right) - \left(\omega^i + \omega^j + \omega^k + \omega^l\right)$.
5. The pattern continues...

[9] As above, the fraction of households that are indifferent between multiple assets will have no weight when the fractions of asset holders are computed.

Using the analysis from Appendix A.1, for any $i \in \mathbf{I}$,

$$Fr\left\{a_i^h > 0\right\} = \hat{\omega}^i + \frac{1}{2}\left(\sum_{j\in\mathbf{I}\backslash\{i\}}\left(1+\Delta_i^{ij}\right)\hat{\omega}^{ij}\right) + \frac{1}{3}\left(\sum_{j\in\mathbf{I}\backslash\{i\}}\sum_{k\in\mathbf{I}\backslash\{i,j\}}\left(1+\Delta_i^{ijk}\right)\hat{\omega}^{ijk}\right)$$
$$+\frac{1}{4}\left(\sum_{j\in\mathbf{I}\backslash\{i\}}\sum_{k\in\mathbf{I}\backslash\{i,j\}}\sum_{l\in\mathbf{I}\backslash\{i,j,k\}}\left(1+\Delta_i^{ijkl}\right)\hat{\omega}^{ijkl}\right) + ... + \frac{1}{N}\left(1+\Delta_i^{12....N}\right)\hat{\omega}^{12...N}.$$

Using the expressions for $\hat{\omega}$ in terms of ω, the equation for $Fr\left\{a_i^h > 0\right\}$ can be simplified:

$$Fr\left\{a_i^h > 0\right\} = \Phi_i^i\omega^i + \left(\sum_{j\in\mathbf{I}\backslash\{i\}}\Phi_i^{ij}\left(\omega^{ij}-\omega^j\right)\right) + \left(\sum_{j\in\mathbf{I}\backslash\{i\}}\sum_{k\in\mathbf{I}\backslash\{i,j\}}\Phi_i^{ijk}\left(\omega^{ijk}-\omega^{jk}\right)\right)$$
$$+\left(\sum_{j\in\mathbf{I}\backslash\{i\}}\sum_{k\in\mathbf{I}\backslash\{i,j\}}\sum_{l\in\mathbf{I}\backslash\{i,j,k\}}\Phi_i^{ijkl}\left(\omega^{ijkl}-\omega^{jkl}\right)\right) + ... + \Phi_i^{12...N}\omega^{12...N}.$$

for coefficients $\left(\Phi_i^i, \left(\Phi_i^{ij}\right)_{j\in\mathbf{I}\backslash\{i\}}, \left(\Phi_i^{ijk}\right)_{j,k\in\mathbf{I}\backslash\{i\}\times\mathbf{I}\backslash\{i,j\}}, ..., \Phi_i^{12...N}\right)$. The coefficients are given by:

$$\begin{aligned}
\Phi_i^i &= 1 - \frac{1}{2}\sum_{j\in\mathbf{I}\backslash\{i\}}\left(1+\Delta_i^{ij}\right) + \frac{1}{3}\sum_{j\in\mathbf{I}\backslash\{i\}}\sum_{k\in\mathbf{I}\backslash\{i,j\}}\left(1+\Delta_i^{ijk}\right) \\
&\quad - \frac{1}{4}\sum_{j\in\mathbf{I}\backslash\{i\}}\sum_{k\in\mathbf{I}\backslash\{i,j\}}\sum_{l\in\mathbf{I}\backslash\{i,j,k\}}\left(1+\Delta_i^{ijkl}\right) + ... \\
\Phi_i^{ij} &= \frac{1}{2}\left(1+\Delta_i^{ij}\right) - \frac{1}{3}\sum_{k\in\mathbf{I}\backslash\{i,j\}}\left(1+\Delta_i^{ijk}\right) \\
&\quad + \frac{1}{4}\sum_{k\in\mathbf{I}\backslash\{i,j\}}\sum_{l\in\mathbf{I}\backslash\{i,j,k\}}\left(1+\Delta_i^{ijkl}\right) - ... \\
&\vdots \\
\Phi_i^{12...N} &= \frac{1}{N}\left(1+\Delta_i^{12....N}\right).
\end{aligned}$$

I will show that $\left(\Phi_i^i, \left(\Phi_i^{ij}\right)_{j\in\mathbf{I}\backslash\{i\}}, \left(\Phi_i^{ijk}\right)_{j,k\in\mathbf{I}\backslash\{i\}\times\mathbf{I}\backslash\{i,j\}}, ..., \Phi_i^{12...N}\right) >> 0$.

By definition,

$$\Phi_i^{12...N} = \frac{1}{N}\left(1+\Delta_i^{12....N}\right) > 0,$$

since $\frac{1}{N}\left(1+\Delta_i^{12....N}\right)$ is the fraction of households with $(h_1, ..., h_N) \geq (h_1^*, ..., h_N^*)$ that only purchase asset i.

In general, consider any set $S \subseteq \mathcal{P}(\mathbf{I})$ such that $i \in S$. Define $m = \#S$ as the number of elements in S, where $1 \leq m < N$. The term Φ^S contains $(N - m + 1)$ terms:

$$\Phi_i^S = \frac{1}{m}\left(1 + \Delta_i^S\right) - ... + (-1)^{N-m}\frac{1}{N}\left(1 + \Delta_i^{12....N}\right).$$

The first term is positive and includes the term $\frac{1}{m}\left(1 + \Delta_i^S\right)$ associated with the only $m-$dimensional set that contains S. The second term is negative and equals the summation over the $\begin{pmatrix} N-m \\ 1 \end{pmatrix} = N - m$ terms $\frac{1}{m+1}\left(1 + \Delta_i^{S^{+1}}\right)$ associated with the $(m+1)-$dimensional subsets S^{+1} that contain S. The third time is positive and equals the summation over the $\begin{pmatrix} N-m \\ 2 \end{pmatrix} = \frac{(N-m)(N-m-1)}{2}$ terms $\frac{1}{m+2}\left(1 + \Delta_i^{S^{+2}}\right)$ associated with the $(m+2)-$dimensional subsets S^{+2} that contain S. The process continues. The final term is $(-1)^{N-m}\frac{1}{N}\left(1 + \Delta_i^{12....N}\right)$, which is positive if $(N - m + 1)$ is odd and negative if $(N - m + 1)$ is even, and this represents the only $N-$dimensional subset that contains S. Including all $(N - m + 1)$ terms and the summations within those terms, there are an equal number of positive terms and negative terms for any $1 \leq m < N$.

When all terms $\Delta_i = 0$, then

$$\Phi_i^S = \sum_{p=0}^{N-m} \frac{(-1)^{N-m}(-1)^{N-m-p}\begin{pmatrix} N-m \\ p \end{pmatrix}}{m+p} = \frac{(N-m)!}{m(m+1)....(N-1)(N)} > 0. \qquad (22)$$

The terms Δ_i are small and centered around 0. Up to a first-order approximation, the products $\Delta_i^S \Delta_i^{S'} \approx 0$. Additionally, I define $\mathcal{S}_{m+p}^S = \{S' \subseteq \mathcal{P}(\mathbf{I}) : \#S' = m + p \text{ and } S \subseteq S'\}$. Including the terms Δ_i :

$$\Phi_i^S = \frac{(N-m)!}{m(m+1)....(N-1)(N)} + \sum_{p=0}^{N-m} \frac{(-1)^{N-m}(-1)^{N-m-p} \sum_{S' \in \mathcal{S}_{m+p}^S} \Delta_i^{S'}}{m+p}. \qquad (23)$$

From (22), the first term in (23) can be written as an infinite sum. Comparing terms in the infinite sums of the updated expression for (23), a selected term in each summation would have a value 1 in the first summation and a value $\Delta_i^{S'}$ in the second. From Appendix A.1, $\left|\Delta_i^{S'}\right| < 1$. Therefore, regardless of the sign for the Δ_i terms, $\Phi_i^S > 0$.

By definition, the derivative $\frac{\partial Fr\left\{a_i^h>0\right\}}{\partial h_z^*}$ is equal to:

$$\frac{\partial Fr\left\{a_i^h>0\right\}}{\partial h_z^*}=\Phi_i^i\frac{\partial\omega^i}{\partial h_z^*}+\left(\sum_{j\in\mathbf{I}\backslash\{i\}}\Phi_i^{ij}\frac{\partial\left(\omega^{ij}-\omega^j\right)}{\partial h_z^*}\right)+\left(\sum_{j\in\mathbf{I}\backslash\{i\}}\sum_{k\in\mathbf{I}\backslash\{i,j\}}\Phi_i^{ijk}\frac{\partial\left(\omega^{ijk}-\omega^{jk}\right)}{\partial h_z^*}\right)$$
$$+\left(\sum_{j\in\mathbf{I}\backslash\{i\}}\sum_{k\in\mathbf{I}\backslash\{i,j\}}\sum_{l\in\mathbf{I}\backslash\{i,j,k\}}\Phi_i^{ijkl}\frac{\partial\left(\omega^{ijkl}-\omega^{jkl}\right)}{\partial h_z^*}\right)+.....$$

I just showed that all coefficients Φ_i are strictly positive. I will now show that $\frac{\partial\omega^i}{\partial h_z^*}>0$, $\sum_{j\in\mathbf{I}\backslash\{i\}}\frac{\partial\left(\omega^{ij}-\omega^j\right)}{\partial h_z^*}>0$, and so forth.

Consider the partial derivative $\frac{\partial\omega^i}{\partial h_z^*}$ for any $z\in\mathbf{I}\backslash\{i\}$:

$$\frac{\partial\omega^i}{\partial h_z^*}=F_{h_z}\left(h_1^*,...,h_{i-1}^*,1,h_{i+1}^*,...,h_N^*\right)-F_{h_z}\left(h_1^*,...,h_{i-1}^*,h_i^*,h_{i+1}^*,...,h_N^*\right).$$

By the definition of an Archimedean copula,

$$F_{h_z}\left(h_1,...,h_N\right)=\psi'\left(\sum_{p\in\mathbf{I}}\psi^{-1}\left(h_p\right)\right)\left(\psi^{-1}\right)'\left(h_z\right)=\frac{\psi'\left(\sum_{p\in\mathbf{I}}\psi^{-1}\left(h_p\right)\right)}{\psi'\left(\psi^{-1}\left(h_z\right)\right)},\tag{24}$$

where the latter expression uses the Inverse Function Theorem. Since $\psi'(\cdot)<0$ and ψ' is strictly increasing, then $F_{h_z}\left(h_1,...,h_N\right)\in(0,1]$. By definition, $\psi(0)=1$, so $\psi^{-1}(1)=0$. This implies $\sum_{p\in\mathbf{I}}\psi^{-1}\left(h_p^*\right)>\sum_{p\in\mathbf{I}\backslash\{i\}}\psi^{-1}\left(h_p^*\right)$. Since ψ' is strictly increasing, then $\psi'\left(\sum_{p\in\mathbf{I}}\psi^{-1}\left(h_p^*\right)\right)>\psi'\left(\sum_{p\in\mathbf{I}\backslash\{i\}}\psi^{-1}\left(h_p^*\right)\right)$. Since $\psi'(\cdot)<0$, then $\frac{\psi'\left(\sum_{p\in\mathbf{I}}\psi^{-1}\left(h_p^*\right)\right)}{\psi'\left(\psi^{-1}\left(h_z^*\right)\right)}<\frac{\psi'\left(\sum_{p\in\mathbf{I}\backslash\{i\}}\psi^{-1}\left(h_p^*\right)\right)}{\psi'\left(\psi^{-1}\left(h_z^*\right)\right)}$. By definition (24),

$$F_{h_z}\left(h_1^*,...,h_{i-1}^*,h_i^*,h_{i+1}^*,...,h_N^*\right)<F_{h_z}\left(h_1^*,...,h_{i-1}^*,1,h_{i+1}^*,...,h_N^*\right).$$

This implies $\frac{\partial\omega^i}{\partial h_z^*}>0$ for any $z\in\mathbf{I}\backslash\{i\}$.

Consider the partial derivative $\frac{\partial\left(\omega^{ij}-\omega^j\right)}{\partial h_z^*}$ for any $z\in\mathbf{I}\backslash\{i\}$. By definition,

$$\omega^{ij}-\omega^j=F\left(h_1^*,..,h_{i-1}^*,1,h_{i+1}^*,..,h_{j-1}^*,1,h_{j+1}^*,..,h_N^*\right)$$
$$-F\left(h_1^*,..,h_{i-1}^*,h_i^*,h_{i+1}^*,..,h_{j-1}^*,1,h_{j+1}^*,..,h_N^*\right).$$

The derivative

$$\frac{\partial\left(\omega^{ij}-\omega^{j}\right)}{\partial h_z^*} = F_{h_z}\left(h_1^*,..,h_{i-1}^*,1,h_{i+1}^*,..,h_{j-1}^*,1,h_{j+1}^*,..,h_N^*\right) - F_{h_z}\left(h_1^*,..,h_{i-1}^*,h_i^*,h_{i+1}^*,..,h_{j-1}^*,1,h_{j+1}^*,..,h_N^*\right).$$

If $z=j$, then $\frac{\partial\left(\omega^{ij}-\omega^{j}\right)}{\partial h_z^*}=0$ as both $F_{h_z}\left(h_1^*,..,h_{i-1}^*,1,h_{i+1}^*,..,h_{j-1}^*,1,h_{j+1}^*,..,h_N^*\right)=0$ and $F_{h_z}\left(h_1^*,..,h_{i-1}^*,h_i^*,h_{i+1}^*,..,h_{j-1}^*,1,h_{j+1}^*,..,h_N^*\right)=0$. If $z\neq j$, follow the same steps as before: (i) $\sum\limits_{p\in\mathbf{I}\backslash\{j\}}\psi^{-1}\left(h_p^*\right) > \sum\limits_{p\in\mathbf{I}\backslash\{i,j\}}\psi^{-1}\left(h_p^*\right)$, (ii) $\psi'\left(\sum\limits_{p\in\mathbf{I}\backslash\{j\}}\psi^{-1}\left(h_p^*\right)\right) > \psi'\left(\sum\limits_{p\in\mathbf{I}\backslash\{i,j\}}\psi^{-1}\left(h_p^*\right)\right)$ (since ψ' is strictly increasing), (iii) $\frac{\psi'\left(\sum\limits_{p\in\mathbf{I}\backslash\{j\}}\psi^{-1}\left(h_p^*\right)\right)}{\psi'\left(\psi^{-1}\left(h_z^*\right)\right)} < \frac{\psi'\left(\sum\limits_{p\in\mathbf{I}\backslash\{i,j\}}\psi^{-1}\left(h_p^*\right)\right)}{\psi'\left(\psi^{-1}\left(h_z^*\right)\right)}$ (since $\psi'(\cdot)<0$), and (iv)

$$F_{h_z}\left(h_1^*,..,h_{i-1}^*,h_i^*,h_{i+1}^*,..,h_{j-1}^*,1,h_{j+1}^*,..,h_N^*\right) < F_{h_z}\left(h_1^*,..,h_{i-1}^*,1,h_{i+1}^*,..,h_{j-1}^*,1,h_{j+1}^*,..,h_N^*\right)$$

by definition (24). This implies $\frac{\partial\left(\omega^{ij}-\omega^{j}\right)}{\partial h_z^*}>0$ for any $z\in\mathbf{I}\backslash\{i,j\}$.

The pattern continues. Therefore, the derivative $\frac{\partial Fr\left\{a_i^h>0\right\}}{\partial h_z^*}>0$ for any $z\in\mathbf{I}\backslash\{i\}$.

Since $Fr\left\{a_i^h>0\right\}<1$, if $\left(h_z^*\right)_{z\in\mathbf{I}\backslash\{i\}}$ increase, meaning all terms weakly increase with at least one term that strictly increases, then (21) will only continue to be satisfied if h_i^* strictly increases. Therefore, an increase in $h_{-i}^*=\left(h_1^*,..,h_{i-1}^*,h_{i+1}^*,...,h_N^*\right)$ (all terms at least as large and at least one strictly larger) leads to a strict increase in h_i^*.

B.4.2 Part 2

Defining $D=\sum\limits_{p\in\mathbf{I}}d_p$ as before, add up the equilibrium equations (21) over all assets and use fact (6):

$$\left(1+D+\sum_{p\in\mathbf{I}}h_p^*\left(1-d_p\right)\right)F\left(h_1^*,...,h_N^*;\alpha\right)=1+D. \tag{25}$$

Write the generator function ψ both in terms of the variable $x=\psi^{-1}\left(h_1^*\right)+....+\psi^{-1}\left(h_N^*\right)$ and in terms of the parameter α. By the definition of an Archimedean copula,

$$F\left(h_1^*,...,h_N^*;\alpha\right)=\psi\left(\psi^{-1}\left(h_1^*;\alpha\right)+....+\psi^{-1}\left(h_N^*;\alpha\right);\alpha\right).$$

The function $\psi(x;\alpha)$ has previously documented properties: (i) $\psi(x;\alpha)\geq 0$, (ii) $\psi_x(x;\alpha)<0$, and (iii) $\psi_{x,x}^2(x;\alpha)\geq 0$. For the Clayton copula, $\psi_\alpha(x;\alpha)>0$ and$\left(\psi^{-1}\right)_\alpha(x;\alpha)>0$. For the Frank and Joe copulas, $\psi_{\alpha}(x;\alpha)<0$ and$(\psi^{-1})_{\alpha}(x;\alpha)<0$.

Define the function

$$G(h_1^*,...,h_N^*;\alpha)=\left(1+D+\sum_{p\in\mathbf{I}}h_p^*(1-d_p)\right)F(h_1^*,...,h_N^*;\alpha)-(1+D)$$

such that $G(h_1^*,...,h_N^*;\alpha)=0$ iff equation (25) is satisfied.

Select any asset $i\in\mathbf{I}$. The derivative

$$\frac{\partial G}{\partial h_i^*}=(1-d_i)F(h_1^*,...,h_N^*;\alpha)+\left(1+D+\sum_{p\in\mathbf{I}}h_p^*(1-d_p)\right)F_{h_i}(h_1^*,...,h_N^*;\alpha).$$

By definition (24), $F_{h_i}(h_1^*,...,h_N^*;\alpha)\in(0,1]$, implying $\frac{\partial G}{\partial h_i^*}>0$. The derivative

$$\frac{\partial G}{\partial\alpha}=\left(1+D+\sum_{p\in\mathbf{I}}h_p^*(1-d_p)\right)F_{\alpha}(h_1^*,...,h_N^*;\alpha),$$

where

$$F_{\alpha}(h_1^*,...,h_N^*;\alpha)=\psi_{\alpha}\left(\psi^{-1}(h_1^*;\alpha)+....+\psi^{-1}(h_N^*;\alpha);\alpha\right)\cdot\left(\left(\psi^{-1}\right)_{\alpha}(h_1^*)+...+\left(\psi^{-1}\right)_{\alpha}(h_N^*)\right).$$

Under the Clayton, Frank, and Joe copulas, $F_{\alpha}(h_1^*,...,h_N^*)>0$, implying $\frac{\partial G}{\partial\alpha}>0$.

From the Implicit Function Thoerem

$$\frac{\partial h_i^*}{\partial\alpha}=-\frac{\frac{\partial G}{\partial\alpha}}{\frac{\partial G}{\partial h_i^*}}=-\frac{(+)}{(+)}<0.$$

Due to symmetry and Part 1, an increase in α leads to a strict decrease in $(h_1^*,...,h_N^*)$. Since $\frac{\partial lev_i}{\partial h_i^*}<0$ for all $i\in\mathbf{I}$, then an increase in α will lead to a strict increase in the leverage ratios for all assets.

B.5 Proof of Theorem 2

Consider equation (25) updated to account for replica economies:

$$\left(1+mD+m\sum_{p\in\mathbf{I}}h_p^*(1-d_p)\right)F\left(\overrightarrow{h_1^*},...,\overrightarrow{h_N^*}\right)=1+mD. \tag{26}$$

By assumption that beliefs are comonotonic across markets with identical dividends:

$$F\left(\overrightarrow{h_1^*}, ..., \overrightarrow{h_N^*}\right) = F\left(h_1^*, ..., h_N^*\right).$$

Define

$$G(h_1^*, ..., h_N^*, m) = \left(1 + mD + m\sum_{p\in\mathbf{I}} h_p^*\left(1 - d_p\right)\right) F\left(h_1^*, ..., h_N^*\right) - (1 + mD)$$

such that $G(h_1^*, ..., h_N^*, m) = 0$ iff equation (26) is satisfied. By the definition of an Archimedean copula,

$$F\left(h_1^*, ..., h_N^*\right) = \psi\left(\psi^{-1}\left(h_1^*\right) + ... + \psi^{-1}\left(h_N^*\right)\right).$$

The derivative

$$\frac{\partial G}{\partial h_i^*} = m h_i^*\left(1 - d_i\right) F\left(h_1^*, ..., h_N^*\right) + \left(1 + mD + m\sum_{p\in\mathbf{I}} h_p^*\left(1 - d_p\right)\right) F_{h_i}\left(h_1^*, ..., h_N^*\right),$$

where $F_{h_i}\left(h_1^*, ..., h_N^*\right) = \frac{\psi'\left(\psi^{-1}\left(h_1^*\right)+....+\psi^{-1}\left(h_N^*\right)\right)}{\psi'\left(\psi^{-1}\left(h_i^*\right)\right)} \in (0, 1]$ as previously derived. This implies $\frac{\partial G}{\partial h_i^*} > 0$.

The derivative

$$\frac{\partial G}{\partial m} = \left(D + \sum_{p\in\mathbf{I}} h_p^*\left(1 - d_p\right)\right) F\left(h_1^*, ..., h_N^*\right) - D.$$

In equilibrium, $G(h_1^*, ..., h_N^*, m) = 0$ iff

$$\left(D + \sum_{p\in\mathbf{I}} h_p^*\left(1 - d_p\right)\right) F\left(h_1^*, ..., h_N^*\right) - D = \frac{1}{m}\left(1 - F\left(h_1^*, ..., h_N^*\right)\right) > 0.$$

Therefore, $\frac{\partial G}{\partial m} > 0$.

From the Implicit Function Theorem,

$$\frac{\partial h_i^*}{\partial m} = -\frac{\frac{\partial G}{\partial m}}{\frac{\partial G}{\partial h_i^*}} < 0.$$

The same effect holds for all assets $i \in \mathbf{I}$. An increase in m leads to a decrease in $\left(h_1^*, ..., h_N^*\right)$. Since $\frac{\partial lev_i}{\partial h_i^*} < 0$ for all $i \in \mathbf{I}$, then an increase in m will lead to an increase in the leverage ratios for all assets.

B.6 Proof of Theorem 3

Define

$$G(h_1^*,...,h_N^*,m) = \left(1 + mD + m\sum_{p\in\mathbf{I}} h_p^*(1-d_p)\right) F\left(\overrightarrow{h_1^*},...,\overrightarrow{h_N^*}\right) - (1+mD)$$

such that $G(h_1^*,...,h_N^*,m) = 0$ iff equation (26) is satisfied. By the definition of an Archimedean copula,

$$F\left(\overrightarrow{h_1^*},...,\overrightarrow{h_N^*}\right) = \psi\left(m\psi^{-1}(h_1^*) + ... + m\psi^{-1}(h_N^*)\right).$$

The derivative

$$\frac{\partial G}{\partial h_i^*} = mh_i^*(1-d_i) F\left(\overrightarrow{h_1^*},...,\overrightarrow{h_N^*}\right) + \left(1 + mD + m\sum_{p\in\mathbf{I}} h_p^*(1-d_p)\right) F_{h_i}\left(\overrightarrow{h_1^*},...,\overrightarrow{h_N^*}\right),$$

where $F_{h_i}\left(\overrightarrow{h_1^*},...,\overrightarrow{h_N^*}\right) = \frac{\psi'\left(m\psi^{-1}(h_1^*)+....+m\psi^{-1}(h_N^*)\right)}{\psi'\left(m\psi^{-1}(h_i^*)\right)} \in (0,1]$ as previously derived. This implies $\frac{\partial G}{\partial h_i^*} > 0$.

Define $x = m\psi^{-1}(h_1^*) + ... + m\psi^{-1}(h_N^*)$. The derivative

$$\frac{\partial G}{\partial m} = \left(D + \sum_{p\in\mathbf{I}} h_p^*(1-d_p)\right)\psi(x) + \left(D + \sum_{p\in\mathbf{I}} h_p^*(1-d_p)\right)\psi'(x)\,x + \frac{1}{m}\psi'(x)\,x - D.$$

Using the equilibrium relation $G(h_1^*,...,h_N^*,m) = 0$:

$$\frac{\partial G}{\partial m} = \frac{1}{m}(1-\psi(x)) + \frac{1}{m}(1+mD)\frac{\psi'(x)\,x}{\psi(x)}.$$

From the Implicit Function Theorem,

$$\frac{\partial h_i^*}{\partial m} = -\frac{\frac{\partial G}{\partial m}}{\frac{\partial G}{\partial h_i^*}}.$$

The derivative $\frac{\partial h_i^*}{\partial m} < 0$ iff $\frac{\partial G}{\partial m} > 0$, which holds iff (using the definition of elasticity):

$$(1-\psi(x)) + (1+mD)\,\xi_\psi(x) > 0.$$

The derivative $\frac{\partial h_i^*}{\partial m} < 0$ iff

$$\frac{1+mD}{1-\psi(x)} < \frac{1}{|\xi_\psi(x)|}. \tag{27}$$

Since $\frac{\partial lev_i}{\partial h_i^*} < 0$ for all $i \in \mathbf{I}$, then $\frac{\partial lev_i}{\partial m} > 0$ for all $i \in \mathbf{I}$ iff (27) holds.

C Tables and Figures

Leverage ratio are displayed for dividends $(d_1, d_2) = (0.3, 0.2)$ and the Clayton copula over a range of belief dependence values.

Belief dependence	$\alpha = \frac{1}{2}$	$\alpha = \frac{4}{3}$	$\alpha = 3$	$\alpha = 8$
Kendall's tau	$\tau = 0.2$	$\tau = 0.4$	$\tau = 0.6$	$\tau = 0.8$
Asset 1				
$m = 1$	1.570	1.585	1.609	1.647
$m = 2$	1.533	1.559	1.606	1.688
$(m = 2) - (m = 1)$	-0.037	-0.026	-0.003	0.041
Asset 2				
$m = 1$	1.340	1.348	1.362	1.383
$m = 2$	1.316	1.332	1.360	1.409
$(m = 2) - (m = 1)$	-0.024	-0.016	-0.002	0.026

Table I: Effects of financial expansion with $N = 2$

Leverage ratio for asset 1 is displayed on the left and the leverage ratio for asset 2 is displayed on the right of each cell in the following table.

	$d_2 = 0.1$	$d_2 = 0.2$	$d_2 = 0.3$	$d_2 = 0.4$	$d_2 = 0.5$
$d_1 = 0.1$	1.203 1.203				
$d_1 = 0.2$	1.437 1.194	1.420 1.420			
$d_1 = 0.3$	1.720 1.187	1.693 1.405	1.669 1.669		
$d_1 = 0.4$	2.079 1.180	2.040 1.390	2.005 1.646	1.972 1.972	
$d_1 = 0.5$	2.561 1.173	2.507 1.377	2.457 1.625	2.410 1.940	2.366 2.366

Table II: Effects of dividends under comonotonic beliefs with $N = 2$

Leverage ratio for asset 1 is displayed on the left and the leverage ratio for asset 2 is displayed on the right of each cell in the following table.

	$d_2 = 0.1$	$d_2 = 0.2$	$d_2 = 0.3$	$d_2 = 0.4$	$d_2 = 0.5$
$d_1 = 0.1$	1.159 1.159				
$d_1 = 0.2$	1.344 1.157	1.339 1.339			
$d_1 = 0.3$	1.568 1.154	1.561 1.335	1.555 1.555		
$d_1 = 0.4$	1.850 1.153	1.842 1.331	1.834 1.549	1.827 1.827	
$d_1 = 0.5$	2.228 1.151	2.218 1.328	2.208 1.545	2.200 1.820	2.191 2.191

Table III: Effects of dividends under independent beliefs with $N = 2$