Stationary Inflation and Pareto Efficiency with Incomplete Markets and a Large Open Economy

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Abstract

In a pure exchange economy with heterogeneous households and incomplete markets, this paper characterizes necessary conditions under which monetary policies are compatible with Pareto efficiency.

Keywords incomplete markets – asset span – monetary policy

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1 Introduction

Can government debt serve as effective instruments for risk-sharing in a large open economy? This paper characterizes the conditions under which coordinated domestic and foreign debt management is able to support a Pareto efficient allocation of resources.

The paper uses two classical ideas in general equilibrium to analyze a modern policy problem. First, with incomplete markets and a fixed asset structure, Pareto efficiency can be supported if the asset span contains the excess demand vectors for all households (Magill

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and Quinzii, 1996).[1] Second, with incomplete markets, monetary policy can have real effects by changing the asset span (Magill and Quinzii, 1992).

Though many applications of the policy-induced asset span may be considered, this paper focuses on the role of monetary policy in mitigating financial frictions caused by incomplete markets. Debt contracts are typically written with nominally risk-free payouts. Monetary policy anchors the domestic price level and thus the real payouts of the debt contracts. Governments set monetary policy by targeting the domestic nominal interest rates and transacting in both domestic and foreign debt. Policy is helpful in mitigating the incomplete markets friction if it enables households to use the financial markets to reduce their exposure to risk.

While the present paper analyzes Pareto efficiency, Peiris and Polemarchakis (2013), Magill and Quinzii (2014a,b), and Adão et al. (2014) analyze the effects of monetary policy on determinacy under incomplete markets. Araújo et al. (2013) analyzes monetary policy that targets the collateral constraints, whereas monetary policy in the present paper targets the asset span.

The related works of Koenig (2013) and Sheedy (2014) analyze how policies of nominal GDP targeting, or monetary growth rate targeting, can mitigate the friction of incomplete markets. Koenig (2013) considers a stylized 2-period setting and Sheedy (2014) assumes a special structure on the preferences to ensure a constant wealth distribution. Targeting policies cannot generically support Pareto efficiency (Hoelle, 2014) meaning that more general policies are required. This paper characterizes necessary conditions under which such policies are able to support Pareto efficiency.

2 The Model

The model describes a large open economy with $N > 1$ countries $i \in \mathbf{I} = \{1, ..., N\}$, each containing a monetary authority and a distinct currency.

Time is discrete and infinite with time periods $t \in \{0, 1, ...\}$. The filtration of uncertainty follows a one-period Markov process with finite state space $\mathbf{S} = \{1, ..., S\}$ and Markov transition matrix Γ. Define the history of realizations up to and including the realization s_t in period t as the state $s^t = (s_0, s_1, ..., s_t)$.

Since there will be N assets (one for each country), the incomplete markets assumption requires $N < S$.

[1] In a finite-horizon model with a single commodity, this asset span condition is satisfied under identical utility of the CARA or CRRA form when the endowment vector of each household is contained in the asset span.

In each state, and in each country, a single commodity is traded.

In each country, a unit mass of infinite-lived and homogeneous households reside. Households in country $i \in \mathbf{I}$ receive the endowments $\{y_i(s^t)\}$. The endowments are only in terms of the domestic commodity. The endowments are stationary, meaning there exists a mapping $\mathbf{y}_i : \mathbf{S} \to \mathbb{R}_{++}$ such that $y_i(s^t) = \mathbf{y}_i(s_t)$ for all states and all countries $i \in \mathbf{I}$. Denote the aggregate endowment as $\mathbf{Y} : \mathbf{S} \to \mathbb{R}_{++}$ such that $\mathbf{Y}(s) = \sum_{i \in \mathbf{I}} \mathbf{y}_i(s) \; \forall s \in \mathbf{S}$.

Assumption 1 (independent endowments) For all countries, $(\mathbf{y}_i(s))_{s \in \mathbf{S}}$ is linearly independent from $(\mathbf{Y}(s))_{s \in \mathbf{S}}$.

Households in country i consume commodities sold in all countries. The consumption of country j commodity by country i households in state s^t is denoted $c_{i,j}(s^t) \in \mathbb{R}_+$. The commodities are assumed to be perfect substitutes, meaning that households only care about the total consumption $c_i(s^t) = \sum_{j \in \mathbf{I}} c_{i,j}(s^t)$. Household expected utility is given by

$$E_0 \sum_{t=0}^{\infty} \beta^t U\left(c_i\left(s^t\right)\right). \tag{1}$$

Assumption 2 (preferences) The discount factor $\beta \in (0, 1)$ and $U : \mathbb{R}_+ \to \mathbb{R}$ is C^1, strictly increasing, strictly concave, homothetic, and satisfies the Inada condition $(\lim_{c \to 0} U_c(c) = \infty)$.

In state s^t, the money supply in country $i \in \mathbf{I}$ is $M_i(s^t) > 0$ and the nominal price level is $p_i(s^t) > 0$. Let $\xi_i(s^t)$ be the nominal exchange rate for country i, relative to country 1. Specifically, $\xi_i(s^t)$ is the number of units of country 1 currency for each one unit of country i currency.

In state s^t, country $i \in \mathbf{I}$ issues a short-term (one-period) nominally risk-free bond with price $q_i(s^t)$ and a nominal payout of 1 in states $(s^t, \sigma)_{\sigma \in \mathbf{S}}$.

Money is valued via cash-in-advance constraints (Lucas and Stokey, 1987). Each state is divided into two subperiods. In the first subperiod, the money markets and bond markets open. Denote $\hat{m}_i(s^t) = (\hat{m}_{i,j}(s^t))_{j \in \mathbf{I}} \in \mathbb{R}^N_+$ as the vector of money holdings (across all currencies) and $b_i(s^t) = (b_{i,j}(s^t))_{j \in \mathbf{I}} \in \mathbb{R}^N$ as the vector of bond holdings held by country i households at the close of the first subperiod. Denote $\omega_i(s^t) \in \mathbb{R}$ as the nominal wealth brought into state s^t by a country i household. The budget constraint in the first subperiod is given by:

$$\frac{1}{\xi_i\left(s^t\right)}\left(\sum_{j\in\mathbf{I}}\xi_j\left(s^t\right)\left(\hat{m}_{i,j}\left(s^t\right)+q_j\left(s^t\right)b_{i,j}\left(s^t\right)\right)\right)\leq\omega_i\left(s^t\right). \tag{2}$$

In the second subperiod, the commodity markets open. The purchase of the commodities is subject to the cash-in-advance constraints:

$$p_j\left(s^t\right)c_{i,j}\left(s^t\right)\leq\hat{m}_{i,j}\left(s^t\right)\quad\forall\left(i,j\right)\in\mathbf{I}^2. \tag{3}$$

At the same time that consumption is being purchased on the commodity markets, households receive income from selling their endowments. Denote $m_i\left(s^t\right)=\left(m_{i,j}\left(s^t\right)\right)_{j\in\mathbf{I}}\in\mathbb{R}_+^N$ as the money holding of the country i households at the end of the second subperiod where

$$\begin{aligned} m_{i,j}\left(s^t\right) &= \hat{m}_{i,j}\left(s^t\right)+p_i\left(s^t\right)\mathbf{y}_i\left(s_t\right)-p_j\left(s^t\right)c_{i,j}\left(s^t\right)\text{ for } i=j. \\ m_{i,j}\left(s^t\right) &= \hat{m}_{i,j}\left(s^t\right)-p_j\left(s^t\right)c_{i,j}\left(s^t\right)\text{ for } i\neq j. \end{aligned} \tag{4}$$

Given the money definition (4), the cash-in-advance constraints (3) can be rewritten as:

$$\begin{aligned} m_{i,j}\left(s^t\right) &\geq p_i\left(s^t\right)\mathbf{y}_i\left(s_t\right)\text{ for } i=j. \\ m_{i,j}\left(s^t\right) &\geq 0\text{ for } i\neq j. \end{aligned} \tag{5}$$

The nominal wealth of country i households entering into state $\left(s^t,\sigma\right)$ is equal to the money holding plus the portfolio payout:

$$\omega_i\left(s^t,\sigma\right)=\frac{1}{\xi_i\left(s^t,\sigma\right)}\left(\sum_{j\in\mathbf{I}}\xi_j\left(s^t,\sigma\right)\left(m_{i,j}\left(s^t\right)+b_{i,j}\left(s^t\right)\right)\right). \tag{6}$$

The optimization problem includes an implicit debt constraint:

$$\begin{array}{ll} \underset{\{c_i(s^t),m_i(s^t),b_i(s^t)\}}{max} & E_0\sum_{t=0}^{\infty}\beta^tU\left(c_i\left(s^t\right)\right) \\ subj.\ to & \text{budget constraint (2) with (4) and (6) }\forall t,s^t \\ & \text{cash-in-advance constraint (5) }\forall t,s^t \\ & \liminf_{t,s^t}\left(\sum_{j\in\mathbf{I}}q_j\left(s^t\right)b_{i,j}\left(s^t\right)\right)>-\infty. \end{array} \tag{7}$$

Monetary authorities are responsible for choosing the domestic money supply, the domestic nominal interest rate, and the exchange rates.[2] The country i monetary authority

[2] In reality, the fiscal authority issues the domestic debt and the monetary authority can buy or sell this

chooses the portfolio of net debt positions $B_i(s^t) = (B_{i,j}(s^t))_{j\in\mathbf{I}} \in \mathbb{R}^N$ and the money supply $M_i(s^t) \geq 0$ in each state s^t to satisfy the following constraints:

$$M_i\left(s^{t-1}\right) + \frac{\sum_{j\in\mathbf{I}} \xi_j(s^t) B_{i,j}(s^{t-1})}{\xi_i(s^t)} = M_i\left(s^t\right) + \frac{\sum_{j\in\mathbf{I}} \xi_j(s^t) q_j(s^t) B_{i,j}(s^t)}{\xi_i(s^t)}. \tag{8}$$

Seigniorage revenue is included in the constraints and allows a monetary authority to reduces its current net debt positions. These constraints imply the equilibrium condition requiring that current debt obligations equal the discounted expected seigniorage revenues.

Definition 1 *A sequential competitive equilibrium (SCE) is such that:*

1. *Households solve the household problem (7).*
2. *Monetary authorities satisfy the constraints (8).*
3. *Commodity markets clear:*

$$\sum_{i\in\mathbf{I}} c_i\left(s^t\right) = \mathbf{Y}(s_t) \text{ for every } t, s^t. \tag{9}$$

4. *Money markets clear:*

$$\sum_{i\in\mathbf{I}} m_{i,j}\left(s^t\right) = M_j\left(s^t\right) \text{ for every } j\in\mathbf{I} \text{ and for every } t, s^t. \tag{10}$$

5. *Bond markets clear:*

$$\sum_{i\in\mathbf{I}} b_{i,j}\left(s^t\right) = \sum_{i\in\mathbf{I}} B_{i,j}\left(s^t\right) \text{ for every } j\in\mathbf{I} \text{ and for every } t, s^t. \tag{11}$$

6. *Total initial obligations of monetary authorities equal total initial household wealth:*

$$\sum_{i\in\mathbf{I}} \xi_i(s_0)\,\omega_i(s_0) = \sum_{i\in\mathbf{I}} \xi_i(s_0)\,W_i(s_0). \tag{12}$$

debt. The fiscal authority can also hold foreign debt positions. Since the model does not contain fiscal policy choices of taxes or government spending, all choices of the fiscal authority are subsumed by the monetary authority. The assumption is that both authorities are acting in perfect concert.

3 Equilibrium properties

Under Assumption 2, standard existence results ensure that a SCE always exists. Given the assumption that the commodities from both countries are perfect substitutes, the equilibrium exchange rates satisfy Purchasing Power Parity:

$$\xi_i\left(s^t\right) = \frac{p_1\left(s^t\right)}{p_i\left(s^t\right)} \quad \forall i \in \mathbf{I}.$$

Equilibrium bond prices satisfy $q_i\left(s^t\right) \leq 1$. If $q_i\left(s^t\right) < 1$, the cash-in-advance constraints (5) bind and the Quantity Theory of Money holds:

$$M_i\left(s^t\right) = p_i\left(s^t\right) \mathbf{y}_i\left(s_t\right). \tag{13}$$

The Friedman rule for country i in state s^t is such that $q_i\left(s^t\right) = 1$. It is innocuous (i.e., without real effects) under the Friedman rule to set the household money holdings such that the cash-in-advance constraints (5) bind and the Quantity Theory of Money (13) holds.

For simplicity, I define the new variables $\hat{b}_{i,j}\left(s^t\right)$ and $\hat{B}_{i,j}\left(s^t\right)$ such that

$$\begin{array}{ll} \hat{b}_{i,j}\left(s^t\right) = \mathbf{y}_i\left(s_t\right) + \frac{b_{i,j}\left(s^t\right)}{p_j\left(s^t\right)} \text{ for } i = j. & \hat{B}_{i,j}\left(s^t\right) = \mathbf{y}_i\left(s_t\right) + \frac{B_{i,j}\left(s^t\right)}{p_j\left(s^t\right)} \text{ for } i = j. \\ \hat{b}_{i,j}\left(s^t\right) = \frac{b_{i,j}\left(s^t\right)}{p_j\left(s^t\right)} \text{ for } i \neq j. & \hat{B}_{i,j}\left(s^t\right) = \frac{B_{i,j}\left(s^t\right)}{p_j\left(s^t\right)} \text{ for } i \neq j. \end{array}$$

Additionally, define the inflation rate

$$\pi_i\left(s^t\right) = \frac{p_i\left(s^t\right)}{p_i\left(s^{t-1}\right)} \text{ for } i \in \mathbf{I}.$$

A policy rule for country i consists of the sequences of bond prices $\{q_i(s^t)\}$ and debt positions $\left\{\hat{B}_i\left(s^t\right)\right\}$. These choices are mutually dependent as they must be consistent with equilibrium conditions. The policy choices $q_i\left(s^t\right)$ and $\hat{B}_{i,i}\left(s^t\right)$ target the short-term nominal interest rate, while the policy choices $\left(\hat{B}_{i,j}\left(s^t\right)\right)_{j \neq i}$ target the exchange rates.

The monetary authority constraints (8) under the change of variables are given by:

$$\sum_{j \in \mathbf{I}} \frac{\hat{B}_{i,j}\left(s^{t-1}\right)}{\pi_j\left(s^t\right)} = \mathbf{y}_i\left(s_t\right)\left(1 - q_i\left(s^t\right)\right) + \sum_{j \in \mathbf{I}} q_j\left(s^t\right) \hat{B}_{i,j}\left(s^t\right) \quad \forall i \in \mathbf{I}. \tag{14}$$

Define the real wealth for a country i household entering state s^t as $\hat{\omega}_i(s^t) = \frac{\omega_i(s^t)}{p_i(s^t)}$. The equilibrium household budget constraints (2) (with (4) and (6)) under the change of variables are given by:

$$c_i\left(s^t\right) - q_i\left(s^t\right)\mathbf{y}_i\left(s_t\right) + \sum_{j\in\mathbf{I}} q_j\left(s^t\right)\hat{b}_{i,j}\left(s^t\right) = \hat{\omega}_i(s^t). \tag{15}$$

The first order conditions with respect to the bond holdings are given by:

$$q_j\left(s^t\right) = \beta\sum_{\sigma\in\mathbf{S}}\Gamma\left(s_t,\sigma\right)\frac{U_c\left(c_i(s^t,\sigma)\right)}{U_c\left(c_i(s^t)\right)}\frac{1}{\pi_j\left(s^t,\sigma\right)}\ \forall(i,j)\in\mathbf{I}^2. \tag{16}$$

4 Necesssary condition for Pareto efficiency

4.1 Main theorem

Bond prices $\{q_i(s^t)\}$ and government debt $\left\{\hat{B}_i\left(s^t\right)\right\}$ are the two policy instruments chosen by the monetary authorities. This paper assumes all countries adopt stationary policy rules.

Definition 2 *The policy rule* $\left\{q_i(s^t), \hat{B}_i\left(s^t\right)\right\}$ *is stationary if there exists* $\left(\mathbf{q}_i\left(s\right)\right)_{s\in\mathbf{S}}\in\mathbb{R}^S_+$ *and* $\left(\hat{\mathbf{B}}_i\left(s\right)\right)_{s\in\mathbf{S}}\in\mathbb{R}^{SN}$ *such that* $q_i\left(s^t\right) = \mathbf{q}_i\left(s_t\right)$ *and* $\hat{B}_i\left(s^t\right) = \hat{\mathbf{B}}_i\left(s_t\right)$ *for all states.*

Define the $N\times N$ debt matrix $\hat{B}\left(s^t\right) = \begin{bmatrix}\hat{B}_1\left(s^t\right) & ... & \hat{B}_N\left(s^t\right)\end{bmatrix}$. The monetary authority constraints (14) are written in matrix notation as

$$\left(\frac{1}{\pi_i\left(s^t\right)}\right)^T_{i\in\mathbf{I}}\left(\hat{B}\left(s^{t-1}\right)\right) = \left(\mathbf{y}_i\left(s_t\right)\left(1-q_i\left(s^t\right)\right)\right)^T_{i\in\mathbf{I}} + \left(q_i\left(s^t\right)\right)^T_{i\in\mathbf{I}}\left(\hat{B}\left(s^t\right)\right).$$

Provided that the policies are such that $\hat{B}\left(s^{t-1}\right)$ is a full rank matrix, inflation rates are uniquely determined as functions of the bond prices and government debt:

$$\left(\frac{1}{\pi_i\left(s^t\right)}\right)^T_{i\in\mathbf{I}} = \left[\hat{B}\left(s^{t-1}\right)\right]^{-1}\left[\left(\mathbf{y}_i\left(s_t\right)\left(1-q_i\left(s^t\right)\right)\right)^T_{i\in\mathbf{I}} + \left(q_i\left(s^t\right)\right)^T_{i\in\mathbf{I}}\left(\hat{B}\left(s^t\right)\right)\right].$$

With stationary policy rules, the inflation rates must be stationary as well.

Definition 3 *The inflation rates* $\{\pi_i\left(s^t,\sigma\right)\}$ *are stationary if there exists* $\left(\boldsymbol{\pi}_i\left(\sigma\right)\right)_{\sigma\in\mathbf{S}}$ *such that* $\pi_i\left(s^t,\sigma\right) = \boldsymbol{\pi}_i\left(\sigma\right)$ *for all states.*

For bonds obtained in state s^t, the payout matrix for all states $\left(s^t,\sigma\right)_{\sigma\in\mathbf{S}}$ is given by:

$$R\left(s^t\right) = \begin{bmatrix} r_1\left(s^t,1\right) & .. & r_N\left(s^t,1\right)\\ : & .. & :\\ r_1\left(s^t,S\right) & .. & r_N\left(s^t,S\right)\end{bmatrix} = \begin{bmatrix}\frac{1}{\pi_1(s^t,1)} & .. & \frac{1}{\pi_N(s^t,1)}\\ : & .. & :\\ \frac{1}{\pi_1(s^t,S)} & .. & \frac{1}{\pi_N(s^t,S)}\end{bmatrix}.$$

Under stationary inflation, $R(s^t) = R$ for all states. Define

$$span(R) = \left\{z : \exists b \in \mathbb{R}^N : z = Rb\right\}.$$

The span is a $(rank(R))$ $-$dimensional linear subspace of $\mathbb{R}^S$.

Rather than proving a generic result, it is instructive to see what restrictions to policy are sufficient for a general result.

Assumption 3 (three restrictions on stationary policy)

1. Stationary policy is such that no country adopts the Friedman rule in all periods ($\forall i \in \mathbf{I}$, $\exists s$ such that $\mathbf{q}_i(s) < 1$).
2. Stationary policy is such that $rank(R) = N$.
3. Stationary policy is such that $rank\left[..., (\mathbf{q}_i(s)\,\mathbf{y}_i(s))_{s\in\mathbf{S}}, ...\right] = N$.

Hoelle (2014) proves that any country adopting the Friedman rule is inconsistent with Pareto efficiency (generically), so Assumption 3.1 is innocuous. Assumptions 3.2 and 3.3, both rank conditions, hold generically over the set of policies. Hoelle (2014) proves that a rank deficient R is inconsistent with the maximization of a social welfare function (generically), so Assumption 3.2 is innocuous.

Consider any state s^t. Multiply the budgets constraints in states $(s^t, \sigma)_{\sigma\in\mathbf{S}}$ by $\beta\left(\frac{c_i(s^t,\sigma)}{c_i(s^t)}\right)^{-\rho}$, take the conditional expectation, and simplify using the Euler equation (16):

$$E_t\left[\beta\frac{U_c(c_i(s^t,\sigma))}{U_c(c_i(s^t))}\left(c_i\left(s^t,\sigma\right) - \mathbf{q}_i(\sigma)\,\mathbf{y}_i(\sigma) + \sum_{j\in\mathbf{I}}\mathbf{q}_j(\sigma)\,\hat{b}_{i,j}(s^t,\sigma)\right)\right] = \sum_{j\in\mathbf{I}}\mathbf{q}_j(s_t)\hat{b}_{i,j}(s^t). \tag{17}$$

Iterate this expression forward and cite the transversality condition:

$$\hat{\omega}_i(s^t) = \sum_{k=0}^{\infty}\beta^k E_t\left[\frac{U_c(c_i(s^t,\sigma))}{U_c(c_i(s^t))}\left(c_i\left(s^{t+k}\right) - \mathbf{q}_i(s_{t+k})\mathbf{y}_i(s_{t+k})\right)\right]. \tag{18}$$

If the equilibrium allocation is Pareto efficient, there exists θ_i such that $c_i\left(s^{t+k}\right) = \theta_i\mathbf{Y}(s_{t+k})$. Equation (18) is updated as:

$$\hat{\omega}_i(s^t) = \sum_{k=0}^{\infty}\beta^k E_t\left[\frac{U_c(\mathbf{Y}(s_{t+k}))}{U_c(\mathbf{Y}(s_t))}\left(\theta_i\mathbf{Y}(s_{t+k}) - \mathbf{q}_i(s_{t+k})\mathbf{y}_i(s_{t+k})\right)\right]. \tag{19}$$

If the allocation is Pareto efficient, the real wealth vectors $\hat{\omega}_i(s^t)$ are stationary, meaning that there exists $(\hat{\boldsymbol{\omega}}_i(s))_{(i,s)\in\mathbf{I}\times\mathbf{S}}$ such that $\hat{\omega}_i(s^t) = \hat{\boldsymbol{\omega}}_i(s_t)$ for all i and all states.

The initial period discounted present value equation for households is:

$$\frac{\omega_i(s_0)}{p_i(s_0)} = \sum_{k=0}^{\infty} \beta^k E_0 \left[\frac{U_c(\mathbf{Y}(s_k))}{U_c(\mathbf{Y}(s_0))} (\theta_i \mathbf{Y}(s_k) - \mathbf{q}_i(s_k)\, \mathbf{y}_i(s_k)) \right]. \tag{20}$$

The initial period discounted present value equation for monetary authorities is:

$$\frac{W_i(s_0)}{p_i(s_0)} = \sum_{k=0}^{\infty} \beta^k E_0 \left[\frac{U_c(\mathbf{Y}(s_k))}{U_c(\mathbf{Y}(s_0))} \mathbf{y}_i(s_k)(1 - \mathbf{q}_i(s_k)) \right]. \tag{21}$$

Under Pareto efficiency, the Euler equations (16) are updated:

$$\mathbf{q}_i(s) = \beta \sum_{\sigma\in\mathbf{S}} \Gamma(s,\sigma) \frac{U_c(\mathbf{Y}(\sigma))}{U_c(\mathbf{Y}(s))} \frac{1}{\boldsymbol{\pi}_i(\sigma)} \quad \forall (i,s) \in \mathbf{I}\times\mathbf{S}. \tag{22}$$

With stationary wealth and inflation rates, the bond holdings are constant. Define the constant bond holdings $(\mathbf{b}_{i,j})_{(i,j)\in\mathbf{I}^2}$ and constant debt holdings $(\mathbf{B}_{i,j})_{(i,j)\in\mathbf{I}^2}$ such that $\hat{b}_{i,j}(s^t) = \mathbf{b}_{i,j}$ and $\hat{B}_{i,j}(s^t) = \mathbf{B}_{i,j}$ for all (i,j) and for all states.

The household budget constraints (15) are:

$$\theta_i \mathbf{Y}(s) - \mathbf{q}_i(s)\, \mathbf{y}_i(s) + \sum_{j\in\mathbf{I}} \mathbf{q}_j(s)\, \mathbf{b}_{i,j} = \sum_{j\in\mathbf{I}} \frac{\mathbf{b}_{i,j}}{\boldsymbol{\pi}_j(s)} \quad \forall (i,s) \in \mathbf{I}\times\mathbf{S}. \tag{23}$$

The monetary authority constraints (14) are:

$$\mathbf{y}_i(s)(1 - \mathbf{q}_i(s)) + \sum_{j\in\mathbf{I}} \mathbf{q}_j(s)\, \mathbf{B}_{i,j} = \sum_{j\in\mathbf{I}} \frac{\mathbf{B}_{i,j}}{\boldsymbol{\pi}_j(s)} \quad \forall (i,s) \in \mathbf{I}\times\mathbf{S}. \tag{24}$$

Equations (20)-(24) provide a total of $2N - 1 + SN + S(2N-1)$ equations. The equilibrium variables number $2N - 1 + 2SN + N(2N-1)$ and are given in the following table:

initial prices	$(p_i(s_0))_{i\in\mathbf{I}}$	N variables
consumption fractions	$(\theta_i)_{i\in\mathbf{I}}$	$N-1$ variables
bond prices	$(\mathbf{q}_i(s))_{(i,s)\in\mathbf{I}\times\mathbf{S}}$	SN variables
inflation rates	$(\boldsymbol{\pi}_i(s))_{(i,s)\in\mathbf{I}\times\mathbf{S}}$	SN variables
bond holdings	$(\mathbf{b}_{i,j})_{(i,j)\in\mathbf{I}\times\mathbf{I}}$	N^2 variables
debt positions	$(\mathbf{B}_{i,j})_{(i,j)\in\mathbf{I}\setminus\{N\}\times\mathbf{I}}$	$N(N-1)$ variables

.

Table 1: Variables for stationary inflation policies

The number of variables exceeds the number of equations when $S \leq 2N + \frac{N}{N-1}$.

Theorem 1 *If $S > 2N + \frac{N}{N-1}$ and Assumptions 1-3 hold, there does not exist a stationary policy rule under which the equilibrium allocation is Pareto efficient.*

4.2 Proof of Theorem 1

Define the vector of stationary real household wealth as

$$\hat{\boldsymbol{\omega}}_i = \left(\hat{\boldsymbol{\omega}}_i(s)\right)_{s \in \mathbf{S}}.$$

Lemma 1 *The stationary real wealth vectors have full rank, namely rank $\left(\left(\hat{\boldsymbol{\omega}}_i\right)_{i \in \mathbf{I}}\right) = N$.*

Proof. See Subsection A below. ■

Under Assumption 3.2, $span(R)$ is an $N-$dimensional linear subspace of $\mathbb{R}^S$. The set of all $N-$dimensional linear subspaces of $\mathbb{R}^S$ is the Grassmanian manifold $Gr(N, S)$, which is an $N(S-N)-$dimensional set. From Lemma 1, there exists a unique $span(R)$ in this $N(S-N)-$dimensional set such that $\hat{\boldsymbol{\omega}}_i \in span(R)$ $\forall i \in \mathbf{I}$. The variables $\left(\boldsymbol{\pi}_i\left(s\right)\right)_{(i,s) \in \mathbf{I} \times \mathbf{S}}$, $\left(\theta_i\right)_{i \in \mathbf{I}}$, and $\left(p_i\left(s_0\right)\right)_{i \in \mathbf{I}}$ are uniquely determined from (20)-(22) in terms of bond prices $\left(\mathbf{q}_i\left(s\right)\right)_{(i,s) \in \mathbf{I} \times \mathbf{S}}$.

Define $\hat{\Gamma}$ as the stochastic discount factor matrix with elements $\hat{\Gamma}\left(s, \sigma\right) = \beta \Gamma\left(s, \sigma\right) \frac{U_c(\mathbf{Y}(\sigma))}{U_c(\mathbf{Y}(s))}$. Using the Euler equation (22), the household budget constraints (23) can be written using the span definition:

$$\left(I_S - \hat{\Gamma}\right)^{-1} \left(\theta_i \mathbf{Y}(s) - \mathbf{q}_i\left(s\right) \mathbf{y}_i(s)\right)_{s \in \mathbf{S}} \in span(R). \tag{25}$$

Since $span(R)$ is unique, it must be equal to $span\left(\left(\theta_i \mathbf{Y}(s) - \mathbf{q}_i\left(s\right) \mathbf{y}_i(s)\right)_{s,i \in \mathbf{S} \times \mathbf{I}}\right)$.

A similar span condition must hold for the monetary authorities. From Walras' Law, only constraints for $N-1$ monetary authorities are considered, since the constraint for country $i = N$ always holds. The constraints are expressed using the span definition:

$$\left(I_S - \hat{\Gamma}\right)^{-1} \left(\mathbf{y}_i(s)\left(1 - \mathbf{q}_i\left(s\right)\right)\right)_{s \in \mathbf{S}} \in span(R). \tag{26}$$

For each $i \in \mathbf{I} \backslash \{N\}$, Assumption 1 and 3.3 imply that (26) imposes $(S-N)$ restrictions on the policy choices $\left(\mathbf{q}_i\left(s\right)\right)_{s \in \mathbf{S}}$. The variables that change the asset span have dimension $S + N(N-1)$. The first S choices are $\left(\mathbf{q}_N\left(s\right)\right)_{s \in \mathbf{S}}$ and the final $N(N-1)$ choices are the N free policy variables for each $i \in \mathbf{I} \backslash \{N\}$. These variables must be sufficient in number in order to support any asset span in the set $Gr(N,S)$, since only 1 asset span in that set is consistent with (25). This requires that

$$S + N(N-1) \geq N(S-N).$$

Algebraically, this inequality reduces to

$$S \leq 2N + \frac{N}{N-1}.$$

If this condition is violated, there does not exist an asset span such that (25) and (26) are satisfied.

5 Two examples

Theorem 1 provides a necessary condition for Pareto efficiency, namely $S \leq 2N + \frac{N}{N-1}$. The condition is necessary, but not sufficient, for Pareto efficiency since an equilibrium requires that bond prices and interest rates are nonnegative:

$$\mathbf{q}_i(s) \in [0,1] \text{ for all } (i,s) \in \mathbf{I} \times \mathbf{S}. \tag{27}$$

This fact is illustrated with two examples.

5.1 Example 1

Consider an economy with $S = 3$ states and $N = 2$ countries. Suppose that the economy does not contain aggregate risk and the total endowment is normalized to 1: $\mathbf{Y}(s) = 1$ $\forall s \in \{1,2,3\}$. Country 1 household endowment is $(\mathbf{y}_1(s))_{s\in\{1,2,3\}} = (0.3, 0.6, 0.89)$, with the rest of the endowment held by country 2 households. The discount factor is $\beta = 0.9$.[3] The shocks are iid with equal weights.

The initial wealth ratio for country 1 is $\frac{\omega_1(s_0)}{W_1(s_0)} = 1.34$. Country 2 adopts a fixed stationary monetary policy given by $(\boldsymbol{\pi}_2(s))_{s\in\{1,2,3\}} = (1.02, 1.03, 1.04)$. Country 2 chooses this policy independently and without regard for Pareto efficiency. Country 2 presumably chooses this policy to maximize its domestic welfare, but this policy is simply held fixed (for tractability).

Country 1 chooses a monetary policy $(\boldsymbol{\pi}_1(s))_{s\in\{1,2,3\}}$ such that the equilibrium allocation is Pareto efficient. With iid shocks and no aggregate risk, the bond prices for country 1 and

[3] Without aggregate risk, the stationary equilibrium supporting Pareto efficiency is independent of the utility function.

country 2 are independent of the shocks. This doesn't limit the ability of country 1 to achieve Pareto efficiency as they choose a state-contingent vector of inflation rates $(\boldsymbol{\pi}_1(s))_{s\in\{1,2,3\}}$.

With country 2 policy held fixed, we lose 3 degrees of freedom with our choice of the optimal policy. The condition $S \leq 2N + \frac{N}{N-1}$ when $N = 2$ specifies that $S \leq 6$. Losing 3 degrees of freedom, the necessary condition becomes $S \leq 3$. For the economy described above, there exists a unique stationary policy rule such that Pareto efficiency is supported. The following table displays the equilibrium result.

$\theta_1 = 0.6$	
$(\mathbf{q}_1(s))_{s\in\{1,2,3\}} = (0.83, 0.83, 0.83)$	$(\mathbf{q}_2(s))_{s\in\{1,2,3\}} = (0.87, 0.87, 0.87)$
$(\boldsymbol{\pi}_1(s))_{s\in\{1,2,3\}} = (1.02, 1.08, 1.146)$	$(\boldsymbol{\pi}_2(s))_{s\in\{1,2,3\}} = (1.02, 1.03, 1.04)$
$\mathbf{b}_1 = (5.29, -4.00)$	$\mathbf{b}_2 = (-5.88, 6.11)$
$\mathbf{B}_1 = (-1.31, 2.28)$	$\mathbf{B}_2 = (0.72, -0.17)$

.

Table 2: Existence of equilibrium

If we ignore the constraints on the bond prices, the condition $S \leq 2N + \frac{N}{N-1}$ is necessary and sufficient for Pareto efficiency as it is always possible to construct variables to satisfy all equilibrium equations. The question is whether the constructed variables satisfy $\mathbf{q}_i(s) \in [0, 1]$ for all $(i, s) \in \mathbf{I} \times \mathbf{S}$. The following section shows when a problem may arise.

5.2 Example 2

Consider an economy similar to Example 1, except $(\mathbf{y}_1(s))_{s\in\{1,2,3\}} = (0.1, 0.2, 0.3)$ and $\frac{\omega_1(s_0)}{W_1(s_0)} = 2.53$. Country 2 continues to choose $(\boldsymbol{\pi}_2(s))_{s\in\{1,2,3\}} = (1.02, 1.03, 1.04)$.

The only variables that solve the system of equations are given in the table below. This is not an equilibrium as $\mathbf{q}_1(s) > 1$ violates the zero lower bound on the nominal interest rate. The necessary condition $S \leq 3$ does not provide us any extra degrees of freedom with which to find an equilibrium satisfying all economic constraints. Thus, for this particular economy, there does not exist a stationary policy rule under which Pareto efficiency is supported.

$\theta_1 = 0.18$	
$(\mathbf{q}_1(s))_{s\in\{1,2,3\}} = (1.03, 1.03, 1.03)$	$(\mathbf{q}_2(s))_{s\in\{1,2,3\}} = (0.87, 0.87, 0.87)$
$(\boldsymbol{\pi}_1(s))_{s\in\{1,2,3\}} = (0.85, 0.87, 0.89)$	$(\boldsymbol{\pi}_2(s))_{s\in\{1,2,3\}} = (1.02, 1.03, 1.04)$
$\mathbf{b}_1 = (6.72, -8.24)$	$\mathbf{b}_2 = (-6.29, 8.69)$
$\mathbf{B}_1 = (0.26, -0.38)$	$\mathbf{B}_2 = (0.17, 0.84)$

.

Table 3: Variables inconsistent with equilibrium

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A Proof of Lemma 1

Recalling the definition of $\hat{\Gamma}$, household real wealth can be expressed as:

$$\hat{\boldsymbol{\omega}}_i = \left(I_S - \beta\hat{\Gamma}\right)^{-1} \left(\theta_i \mathbf{Y}(s) - \mathbf{q}_i\left(s\right)\mathbf{y}_i\left(s\right)\right)_{s\in\mathbf{S}}.$$

Define $\mathbf{z}_i = \left(\theta_i \mathbf{Y}(s) - \mathbf{q}_i\left(s\right)\mathbf{y}_i\left(s\right)\right)_{s\in\mathbf{S}}$. I claim that $rank\left(\left(\mathbf{z}_i\right)_{i\in\mathbf{I}}\right) = N$. Consider $\left(\left(\mathbf{z}_i\right)_{i\in\mathbf{I}}\right) v = 0$ for any vector $v \in \mathbb{R}^N$. This implies that

$$\left(\mathbf{Y}\left(s\right)\sum_{i\in\mathbf{I}} v_i\theta_i - \sum_{i\in\mathbf{I}} v_i\mathbf{q}_i\left(s\right)\mathbf{y}_i\left(s\right)\right)_{s\in\mathbf{S}} = 0. \tag{28}$$

Case A: If $\sum_{i\in\mathbf{I}} v_i\theta_i = 0$, Assumption 3.3 implies that $v = 0$ and $rank\left(\left(\mathbf{z}_i\right)_{i\in\mathbf{I}}\right) = N$.

Case B: Consider $\sum_{i\in\mathbf{I}} v_i\theta_i \neq 0$. Define $\mu_i = \frac{v_i}{\sum_{i\in\mathbf{I}} v_i\theta_i}$. The system of equations (28) is given by:

$$\left(\sum_{i\in\mathbf{I}} \mathbf{y}_i\left(s\right) - \sum_{i\in\mathbf{I}} \mu_i\mathbf{q}_i\left(s\right)\mathbf{y}_i\left(s\right)\right)_{s\in\mathbf{S}} = 0.$$

By Assumption 3.3, there exists $\mathbf{S}^* \subseteq \mathbf{S}$ with $\#\mathbf{S}^* = N$ such that $\left[..., \left(\mathbf{q}_i\left(s\right)\mathbf{y}_i\left(s\right)\right)_{s\in\mathbf{S}^*}, ...\right]$ is an invertible matrix. Consider only the states $s \in \mathbf{S}^*$. In matrix notation:

$$\left[..., \left(\mathbf{y}_i\left(s\right)\right)_{s\in\mathbf{S}^*}, ...\right]\overrightarrow{1} = \left[..., \left(\mathbf{q}_i\left(s\right)\mathbf{y}_i\left(s\right)\right)_{s\in\mathbf{S}^*}, ...\right]\left(\mu_i\right)_{i\in\mathbf{I}}.$$

Since $\left[..., \left(\mathbf{q}_i\left(s\right)\mathbf{y}_i\left(s\right)\right)_{s\in\mathbf{S}^*}, ...\right]$ is invertible, the unknown coefficients $\left(\mu_i\right)_{i\in\mathbf{I}}$ must satisfy:

$$\left(\mu_i\right)_{i\in\mathbf{I}} = \left[..., \left(\mathbf{q}_i\left(s\right)\mathbf{y}_i\left(s\right)\right)_{s\in\mathbf{S}^*}, ...\right]^{-1}\left[..., \left(\mathbf{y}_i\left(s\right)\right)_{s\in\mathbf{S}^*}, ...\right]\overrightarrow{1}.$$

Since $\left(\mathbf{q}_i\left(s\right)\right)_{s\in\mathbf{S}} \leq \overrightarrow{1}$ and $\left(\mathbf{q}_i\left(s\right)\right)_{s\in\mathbf{S}} \neq \overrightarrow{1}$ (Assumption 3.1) $\forall i \in \mathbf{I}$, then $\mu_i \geq 1$ $\forall i \in \mathbf{I}$, with strict inequality for at least one i. By definition, this means $v_i \geq \sum_{i\in\mathbf{I}} v_i\theta_i$ $\forall i \in \mathbf{I}$, with strict inequality for at least one i. Multiplying both sides by θ_i and summing:

$$\sum_{i\in\mathbf{I}} v_i\theta_i > \left(\sum_{i\in\mathbf{I}} \theta_i\right)\sum_{i\in\mathbf{I}} v_i\theta_i.$$

This contradicts that $\sum_{i\in\mathbf{I}} \theta_i = 1$. Therefore, only Case A is possible, and $rank\left(\left(\mathbf{z}_i\right)_{i\in\mathbf{I}}\right) = N$.