Stationary Inflation and Pareto Efficiency with Incomplete Markets and a Large Open Economy

Matthew Hoelle*
Purdue University
Department of Economics
403 W. State Street
West Lafayette, IN 47907 (USA)

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Abstract

In a large open economy with pure exchange and incomplete markets, this paper characterizes necessary conditions under which monetary policies are compatible with Pareto efficiency.

Keywords incomplete markets – asset span – monetary policy

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1 Introduction

Can government debt serve as effective instruments for risk-sharing in a large open economy? This paper characterizes the conditions under which coordinated domestic and foreign debt management is able to support a Pareto efficient allocation of resources.

The paper uses two classical ideas in general equilibrium to analyze a modern policy problem. First, with incomplete markets and a fixed asset structure, Pareto efficiency can be supported if the asset span contains the excess demand vectors for all households (Magill

* Corresponding author: M. Hoelle, mhoelle@purdue.edu, (+1)765-496-2737.

and Quinzii, 1996).[1] Second, with incomplete markets, monetary policy can have real effects by changing the asset span (Magill and Quinzii, 1992).

Though many applications of the policy-induced asset span may be considered, this paper focuses on the role of monetary policy in mitigating financial frictions caused by incomplete markets. Debt contracts are typically written with nominally risk-free payouts. Monetary policy anchors the domestic price level and thus the real payouts of the debt contracts. Governments set monetary policy by targeting the domestic nominal interest rates and transacting in both domestic and foreign debt. Policy is helpful in mitigating the incomplete markets friction if it enables households to use the financial markets to reduce their exposure to risk.

While the present paper analyzes Pareto efficiency, Peiris and Polemarchakis (2013), Magill and Quinzii (2014a,b), and Adão et al. (2014) analyze the effects of monetary policy on determinacy under incomplete markets. Araújo et al. (2013) analyzes monetary policy that targets the collateral constraints, whereas monetary policy in the present paper targets the asset span.

The related works of Koenig (2013) and Sheedy (2014) analyze how policies of nominal GDP targeting, or monetary growth rate targeting, can mitigate the friction of incomplete markets. Koenig (2013) considers a stylized 2-period setting and Sheedy (2014) assumes a special structure on the preferences to ensure a constant wealth distribution. Targeting policies cannot generically support Pareto efficiency (Hoelle, 2014) meaning that more general policies are required. This paper characterizes necessary conditions under which such policies are able to support Pareto efficiency.

2 The Model

The model describes a large open economy with $N > 1$ countries $i \in \mathbf{I} = \{1, ..., N\}$, each containing a monetary authority and a distinct currency.

Time is discrete and infinite with time periods $t \in \{0, 1, ...\}$. The filtration of uncertainty follows a one-period Markov process with finite state space $\mathbf{S} = \{1, ..., S\}$ and Markov transition matrix Γ. Define the history of realizations up to and including the realization s_t in period t as the state $s^t = (s_0, s_1, ..., s_t)$.

Since there will be N assets (one for each country), the incomplete markets assumption requires $N < S$.

[1] In a finite-horizon model with a single commodity, this asset span condition is satisfied under identical utility of the CARA or CRRA form when the endowment vector of each household is contained in the asset span.

In each state, and in each country, a single commodity is traded.

In each country, a unit mass of infinite-lived and homogeneous households reside. Households in country $i \in \mathbf{I}$ receive the endowments $\{y_i(s^t)\}$. The endowments are only in terms of the domestic commodity. The endowments are stationary, meaning there exists a mapping $\mathbf{y}_i : \mathbf{S} \to \mathbb{R}_{++}$ such that $y_i(s^t) = \mathbf{y}_i(s_t)$ for all states and all countries $i \in \mathbf{I}$. Denote the aggregate endowment as $\mathbf{Y} : \mathbf{S} \to \mathbb{R}_{++}$ such that $\mathbf{Y}(s) = \sum_{i \in \mathbf{I}} \mathbf{y}_i(s)\ \forall s \in \mathbf{S}$.

Assumption 1 (independent endowments) For all countries, $(\mathbf{y}_i(s))_{s \in \mathbf{S}}$ is linearly independent from $(\mathbf{y}_j(s))_{s \in \mathbf{S}}$ for any $i \neq j$.

Households in country i consume commodities sold in all countries. The consumption of country j commodity by country i households in state s^t is denoted $c_{i,j}(s^t) \in \mathbb{R}_+$. The commodities are assumed to be perfect substitutes, meaning that households only care about the total consumption $c_i(s^t) = \sum_{j \in \mathbf{I}} c_{i,j}(s^t)$. Household expected utility is given by

$$E_0 \sum_{t=0}^{\infty} \beta^t U\left(c_i\left(s^t\right)\right). \tag{1}$$

Assumption 2 (preferences) The discount factor $\beta \in (0,1)$ and $U : \mathbb{R}_+ \to \mathbb{R}$ satisfies either CRRA, i.e., $U(c) = \left\{\begin{matrix} \frac{c^{1-\eta}}{1-\eta} & \eta \neq 1 \\ \ln(c) & \eta = 1 \end{matrix}\right\}$ for relative risk aversion parameter η, or CARA, i.e., $U(c) = \left\{\begin{matrix} \frac{1-\exp(-\chi c)}{\chi} & \chi \neq 0 \\ c & \chi = 0 \end{matrix}\right\}$ for absolute risk aversion parameter χ.

In state s^t, the money supply in country $i \in \mathbf{I}$ is $M_i(s^t) > 0$ and the nominal price level is $p_i(s^t) > 0$. Let $\xi_i(s^t)$ be the nominal exchange rate for country i, relative to country 1. Specifically, $\xi_i(s^t)$ is the number of units of country 1 currency for each one unit of country i currency.

In state s^t, country $i \in \mathbf{I}$ issues a short-term (one-period) nominally risk-free bond with price $q_i(s^t)$ and a nominal payout of 1 in states $(s^t, \sigma)_{\sigma \in \mathbf{S}}$, where both price and payout are in units of currency i.

Denote $\omega_i(s^t) \in \mathbb{R}$ as the nominal wealth brought into state s^t by a country i household (in units of currency i). The initial period value $\omega_i(s_0)$ is a parameter of the model. Each period is divided into two subperiods. In the first subperiod, the money markets and bond markets open. Denote $\hat{m}_i(s^t) = (\hat{m}_{i,j}(s^t))_{j \in \mathbf{I}} \in \mathbb{R}_+^N$ as the vector of money holdings (across all currencies) and $b_i(s^t) = (b_{i,j}(s^t))_{j \in \mathbf{I}} \in \mathbb{R}^N$ as the vector of bond holdings held by country

i households at the close of the first subperiod. The budget constraint in the first subperiod is given by:

$$\frac{1}{\xi_i\left(s^t\right)}\left(\sum_{j\in\mathbf{I}}\xi_j\left(s^t\right)\left(\hat{m}_{i,j}\left(s^t\right)+q_j\left(s^t\right)b_{i,j}\left(s^t\right)\right)\right)\leq\omega_i\left(s^t\right). \tag{2}$$

In the second subperiod, the commodity markets open. The purchase of the commodities is subject to the cash-in-advance constraints (Lucas and Stokey, 1987):

$$p_j\left(s^t\right)c_{i,j}\left(s^t\right)\leq\hat{m}_{i,j}\left(s^t\right)\quad\forall\left(i,j\right)\in\mathbf{I}^2. \tag{3}$$

At the same time that consumption is being purchased on the commodity markets, households receive domestic income from selling their endowments. Denote $m_i\left(s^t\right)=\left(m_{i,j}\left(s^t\right)\right)_{j\in\mathbf{I}}\in\mathbb{R}_+^N$ as the money holding of the country i households at the end of the second subperiod:

$$\begin{aligned} m_{i,j}\left(s^t\right) &= \hat{m}_{i,j}\left(s^t\right)+p_i\left(s^t\right)\mathbf{y}_i\left(s_t\right)-p_j\left(s^t\right)c_{i,j}\left(s^t\right)\ \text{ for } i=j. \\ m_{i,j}\left(s^t\right) &= \hat{m}_{i,j}\left(s^t\right)-p_j\left(s^t\right)c_{i,j}\left(s^t\right)\ \text{ for } i\neq j. \end{aligned} \tag{4}$$

Given the money definition (4), the cash-in-advance constraints (3) can be rewritten as:

$$\begin{aligned} m_{i,j}\left(s^t\right) &\geq p_i\left(s^t\right)\mathbf{y}_i\left(s_t\right)\ \text{ for } i=j. \\ m_{i,j}\left(s^t\right) &\geq 0 \text{ for } i\neq j. \end{aligned} \tag{5}$$

The nominal wealth of country i households entering into state $\left(s^t,\sigma\right)$ is equal to the money holding plus the portfolio payout:

$$\omega_i\left(s^t,\sigma\right)=\frac{1}{\xi_i\left(s^t,\sigma\right)}\left(\sum_{j\in\mathbf{I}}\xi_j\left(s^t,\sigma\right)\left(m_{i,j}\left(s^t\right)+b_{i,j}\left(s^t\right)\right)\right). \tag{6}$$

An implicit debt constraint must be included in the household optimization problem to rule out Ponzi schemes. The household problem is given by:

$$\begin{array}{ll} \underset{\{c_i(s^t),m_i(s^t),b_i(s^t)\}}{max} & E_0\sum_{t=0}^{\infty}\beta^tU\left(c_i\left(s^t\right)\right) \\ subj.\ to & \text{budget constraint (2) with (4) and (6) } \forall t,s^t \\ & \text{cash-in-advance constraint (5) } \forall t,s^t \\ & \liminf_{t,s^t}\left(\sum_{j\in\mathbf{I}}q_j\left(s^t\right)b_{i,j}\left(s^t\right)\right)>-\infty. \end{array} \tag{7}$$

Monetary authorities are responsible for choosing the domestic money supply, the domestic nominal interest rate, and the exchange rates.[2] The country i monetary authority chooses the money supply $M_i(s^t) \geq 0$ and the portfolio of net debt positions $B_i(s^t) = (B_{i,j}(s^t))_{j\in\mathbf{I}} \in \mathbb{R}^N$ in each state s^t, where $B_{i,j}(s^t)$ is the net position of country j debt held by the country i monetary authority. The domestic debt position must be nonnegative ($B_{i,i}(s^t) \geq 0$), while the foreign debt positions can either be positive or negative.[3]

In the initial period s_0, the country i monetary authority has the nominal obligation $W_i(s_0)$. The country i monetary authority constraint in the initial period s_0 is given by:

$$W_i(s_0) = M_i(s_0) + \frac{\sum_{j\in\mathbf{I}} \xi_j(s_0)\, q_j(s_0)\, B_{i,j}(s_0)}{\xi_i(s_0)}. \tag{8}$$

In any date-event s^t with $t > 0$, the country i monetary authority constraint is given by:

$$M_i\left(s^{t-1}\right) + \frac{\sum_{j\in\mathbf{I}} \xi_j(s^t)\, B_{i,j}(s^{t-1})}{\xi_i(s^t)} = M_i\left(s^t\right) + \frac{\sum_{j\in\mathbf{I}} \xi_j(s^t)\, q_j(s^t)\, B_{i,j}(s^t)}{\xi_i(s^t)}. \tag{9}$$

Seigniorage revenue is included in the constraints and allows a monetary authority to reduces its current net debt positions. These constraints imply the equilibrium condition equating current debt obligations with the discounted expected seigniorage revenues.

Definition 1 *A sequential competitive equilibrium (SCE) is such that:*

1. *Households solve the household problem (7).*
2. *Monetary authorities satisfy the constraints (8) and (9).*
3. *Commodity markets clear:*

$$\sum_{i\in\mathbf{I}} c_i\left(s^t\right) = \mathbf{Y}(s_t) \text{ for every } t, s^t. \tag{10}$$

4. *Money markets clear:*

$$\sum_{i\in\mathbf{I}} m_{i,j}\left(s^t\right) = M_j\left(s^t\right) \text{ for every } j \in \mathbf{I} \text{ and for every } t, s^t. \tag{11}$$

[2] In reality, the fiscal authority issues the domestic debt and the monetary authority can buy or sell this debt. The fiscal authority can also hold foreign debt positions. Since the model does not contain fiscal policy choices of taxes or government spending, all choices of the fiscal authority are subsumed by the monetary authority. The assumption is that both authorities are acting in perfect concert.

[3] The monetary authority cannot hold more domestic debt than was initially issued by the fiscal authority.

5. Bond markets clear:

$$\sum_{i\in\mathbf{I}} b_{i,j}\left(s^t\right) = \sum_{i\in\mathbf{I}} B_{i,j}\left(s^t\right) \text{ for every } j\in\mathbf{I} \text{ and for every } t, s^t. \tag{12}$$

6. Total initial obligations of monetary authorities equal total initial household wealth:

$$\sum_{i\in\mathbf{I}} \xi_i\left(s_0\right)\omega_i\left(s_0\right) = \sum_{i\in\mathbf{I}} \xi_i\left(s_0\right)W_i\left(s_0\right). \tag{13}$$

3 Equilibrium properties

Under Assumption 2, standard existence results ensure that a SCE always exists. Given the assumption that the commodities from both countries are perfect substitutes, the equilibrium exchange rates satisfy Purchasing Power Parity:

$$\xi_i\left(s^t\right) = \frac{p_1\left(s^t\right)}{p_i\left(s^t\right)} \quad \forall i\in\mathbf{I}.$$

Equilibrium bond prices satisfy $q_i\left(s^t\right)\leq 1$. If $q_i\left(s^t\right)<1$, the cash-in-advance constraints (5) bind and the Quantity Theory of Money holds:

$$M_i\left(s^t\right) = p_i\left(s^t\right)\mathbf{y}_i\left(s_t\right). \tag{14}$$

The Friedman rule for country i in state s^t is such that $q_i\left(s^t\right)=1$. It is innocuous (i.e., without real effects) under the Friedman rule to set the household money holdings such that the cash-in-advance constraints (5) bind and the Quantity Theory of Money (14) holds.

For simplicity, I define the new variables $\hat{b}_{i,j}\left(s^t\right)$ and $\hat{B}_{i,j}\left(s^t\right)$ such that

$$\begin{array}{ll} \hat{b}_{i,j}\left(s^t\right) = \mathbf{y}_i\left(s_t\right) + \frac{b_{i,j}\left(s^t\right)}{p_j\left(s^t\right)} \text{ for } i=j. & \hat{B}_{i,j}\left(s^t\right) = \mathbf{y}_i\left(s_t\right) + \frac{B_{i,j}\left(s^t\right)}{p_j\left(s^t\right)} \text{ for } i=j. \\ \hat{b}_{i,j}\left(s^t\right) = \frac{b_{i,j}\left(s^t\right)}{p_j\left(s^t\right)} \text{ for } i\neq j. & \hat{B}_{i,j}\left(s^t\right) = \frac{B_{i,j}\left(s^t\right)}{p_j\left(s^t\right)} \text{ for } i\neq j. \end{array}$$

Additionally, define the inflation rate

$$\pi_i\left(s^t\right) = \frac{p_i\left(s^t\right)}{p_i\left(s^{t-1}\right)} \text{ for } i\in\mathbf{I}.$$

A policy rule for country i consists of the sequences of bond prices $\{q_i(s^t)\}$ and debt positions $\left\{\hat{B}_i\left(s^t\right)\right\}$. These choices are mutually dependent as they must be consistent with equilibrium conditions. The policy choices $q_i\left(s^t\right)$ and $\hat{B}_{i,i}\left(s^t\right)$ target the short-term nominal interest rate, while the policy choices $\left(\hat{B}_{i,j}\left(s^t\right)\right)_{j\neq i}$ target the exchange rates.

The monetary authority constraints (9) under the change of variables are given by:

$$\sum_{j\in\mathbf{I}}\frac{\hat{B}_{i,j}\left(s^{t-1}\right)}{\pi_j\left(s^t\right)}=\mathbf{y}_i\left(s_t\right)\left(1-q_i\left(s^t\right)\right)+\sum_{j\in\mathbf{I}}q_j\left(s^t\right)\hat{B}_{i,j}\left(s^t\right)\quad\forall i\in\mathbf{I}. \tag{15}$$

Define the real wealth for a country i household entering state s^t as $\hat{\omega}_i(s^t)=\frac{\omega_i(s^t)}{p_i(s^t)}$. The equilibrium household budget constraints (2) (with (4) and (6)) under the change of variables are given by:

$$c_i\left(s^t\right)-q_i\left(s^t\right)\mathbf{y}_i\left(s_t\right)+\sum_{j\in\mathbf{I}}q_j\left(s^t\right)\hat{b}_{i,j}\left(s^t\right)=\hat{\omega}_i(s^t). \tag{16}$$

The first order conditions with respect to the bond holdings are given by:

$$q_j\left(s^t\right)=\beta\sum_{\sigma\in\mathbf{S}}\Gamma\left(s_t,\sigma\right)\frac{U_c\left(c_i(s^t,\sigma)\right)}{U_c\left(c_i(s^t)\right)}\frac{1}{\pi_j\left(s^t,\sigma\right)}\ \forall(i,j)\in\mathbf{I}^2. \tag{17}$$

4 Necesssary conditions for Pareto efficiency

4.1 Pareto efficiency

The normative criterion assumed in this paper is the Pareto criterion. To specify this criterion mathematically, I endow each monetary authority with an objective function. Each monetary authority maximizes the domestic welfare, meaning the monetary authority in country i adopts monetary policy to maximize $E_0\sum_{t=0}^{\infty}\beta^tU\left(c_i\left(s^t\right)\right)$, where $\left\{c_i\left(s^t\right)\right\}$ is the sequence of consumption for the households in country i.

Definition 2 *The allocation* $\left\{\left(c_i\left(s^t\right)\right)_{i\in\mathbf{I}}\right\}$ *is feasible if* $\sum_{i\in\mathbf{I}}c_i\left(s^t\right)=\mathbf{Y}\left(s_t\right)$ *for every* t,s^t.

Definition 3 *The allocation* $\left\{\left(c_i\left(s^t\right)\right)_{i\in\mathbf{I}}\right\}$ *is Pareto efficient if it is feasible and there does not exist another feasible allocation* $\left\{\left(\hat{c}_i\left(s^t\right)\right)_{i\in\mathbf{I}}\right\}$ *such that* $E_0\sum_{t=0}^{\infty}\beta^tU\left(\hat{c}_i\left(s^t\right)\right)\geq E_0\sum_{t=0}^{\infty}\beta^tU\left(c_i\left(s^t\right)\right)$ *for all countries, with strict inequality for at least 1.*

One immediate implication of Pareto efficiency under Assumption 2 is constant consumption fractions.

Lemma 1 *For any Pareto efficient allocation, there exists* $\left(\theta_i\right)_{i\in\mathbf{I}}$ *such that* $c_i\left(s^{t+k}\right)=\theta_i\mathbf{Y}\left(s_{t+k}\right)$ *for all* $i\in\mathbf{I}$ *and for every* t,s^t.

Proof. See Magill and Quinzii (96). ■

The research objective is to determine the conditions required for an equilibrium allocation to be Pareto efficient. Policy is adopted to support equilibrium and the nature of this policy is considered in the next subsection.

4.2 Stationary policy

Bond prices $\{q_i(s^t)\}$ and government debt $\left\{\hat{B}_i\left(s^t\right)\right\}$ are the two policy instruments chosen by the monetary authorities. This paper assumes all countries adopt stationary policy rules.

Definition 4 *The policy rule* $\left\{q_i(s^t), \hat{B}_i\left(s^t\right)\right\}$ *is stationary if there exists* $\left(\mathbf{q}_i\left(s\right)\right)_{s\in\mathbf{S}} \in \mathbb{R}^S_+$ *and* $\left(\hat{\mathbf{B}}_i\left(s\right)\right)_{s\in\mathbf{S}} \in \mathbb{R}^{SN}$ *such that* $q_i\left(s^t\right) = \mathbf{q}_i\left(s_t\right)$ *and* $\hat{B}_i\left(s^t\right) = \hat{\mathbf{B}}_i\left(s_t\right)$ *for all states.*

Define the $N \times N$ debt matrix $\hat{B}\left(s^t\right) = \left[\hat{B}_1\left(s^t\right) \quad ... \quad \hat{B}_N\left(s^t\right)\right]$. The monetary authority constraints (15) are written in matrix notation as

$$\left(\frac{1}{\pi_i\left(s^t\right)}\right)^T_{i\in\mathbf{I}} \left(\hat{B}\left(s^{t-1}\right)\right) = \left(\mathbf{y}_i\left(s_t\right)\left(1-q_i\left(s^t\right)\right)\right)^T_{i\in\mathbf{I}} + \left(q_i\left(s^t\right)\right)^T_{i\in\mathbf{I}}\left(\hat{B}\left(s^t\right)\right).$$

Provided that the policies are such that $\hat{B}\left(s^{t-1}\right)$ is a full rank matrix, inflation rates are uniquely determined as functions of the bond prices and government debt:

$$\left(\frac{1}{\pi_i\left(s^t\right)}\right)^T_{i\in\mathbf{I}} = \left[\hat{B}\left(s^{t-1}\right)\right]^{-1}\left[\left(\mathbf{y}_i\left(s_t\right)\left(1-q_i\left(s^t\right)\right)\right)^T_{i\in\mathbf{I}} + \left(q_i\left(s^t\right)\right)^T_{i\in\mathbf{I}}\left(\hat{B}\left(s^t\right)\right)\right].$$

With stationary policy rules, the inflation rates must be stationary as well.

Definition 5 *The inflation rates* $\left\{\pi_i\left(s^t,\sigma\right)\right\}$ *are stationary if there exists* $\left(\boldsymbol{\pi}_i\left(\sigma\right)\right)_{\sigma\in\mathbf{S}}$ *such that* $\pi_i\left(s^t,\sigma\right) = \boldsymbol{\pi}_i\left(\sigma\right)$ *for all states.*

For bonds obtained in state s^t, the payout matrix for all states $\left(s^t,\sigma\right)_{\sigma\in\mathbf{S}}$ is given by:

$$R\left(s^t\right) = \begin{bmatrix} r_1\left(s^t,1\right) & .. & r_N\left(s^t,1\right) \\ : & .. & : \\ r_1\left(s^t,S\right) & .. & r_N\left(s^t,S\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{\pi_1(s^t,1)} & .. & \frac{1}{\pi_N(s^t,1)} \\ : & .. & : \\ \frac{1}{\pi_1(s^t,S)} & .. & \frac{1}{\pi_N(s^t,S)} \end{bmatrix}.$$

Under stationary inflation, $R\left(s^t\right) = R$ for all states. Define

$$span\left(R\right) = \left\{z : \exists b \in \mathbb{R}^N : z = Rb\right\}.$$

The span is a $(rank\,(R))$ −dimensional linear subspace of $\mathbb{R}^S$. This paper restricts attention to full rank policies such that $rank\,(R) = N$. If Pareto efficiency cannot be supported with a full rank policy, then it cannot be supported with a rank deficient policy.

Consider any state s^t. Multiply the budgets constraints in states $(s^t, \sigma)_{\sigma \in \mathbf{S}}$ by $\beta \frac{U_c\left(c_i\left(s^t,\sigma\right)\right)}{U_c(c_i(s^t))}$, take the conditional expectation, and simplify using the Euler equation (17):

$$E_t\left[\beta\frac{U_c\left(c_i\left(s^t,\sigma\right)\right)}{U_c\left(c_i\left(s^t\right)\right)}\left(c_i\left(s^t,\sigma\right) - \mathbf{q}_i\left(\sigma\right)\mathbf{y}_i\left(\sigma\right) + \sum_{j\in\mathbf{I}}\mathbf{q}_j\left(\sigma\right)\hat{b}_{i,j}(s^t,\sigma)\right)\right] = \sum_{j\in\mathbf{I}}\mathbf{q}_j(s_t)\hat{b}_{i,j}(s^t). \tag{18}$$

Iterate this expression forward and cite the transversality condition:

$$\hat{\omega}_i(s^t) = \sum_{k=0}^{\infty}\beta^k E_t\left[\frac{U_c\left(c_i\left(s^t,\sigma\right)\right)}{U_c\left(c_i\left(s^t\right)\right)}\left(c_i\left(s^{t+k}\right) - \mathbf{q}_i(s_{t+k})\mathbf{y}_i\left(s_{t+k}\right)\right)\right]. \tag{19}$$

Using Lemma 1, equation (19) is updated as:

$$\hat{\omega}_i(s^t) = \sum_{k=0}^{\infty}\beta^k E_t\left[\frac{U_c\left(\mathbf{Y}\left(s_{t+k}\right)\right)}{U_c\left(\mathbf{Y}\left(s_t\right)\right)}\left(\theta_i\mathbf{Y}\left(s_{t+k}\right) - \mathbf{q}_i(s_{t+k})\mathbf{y}_i\left(s_{t+k}\right)\right)\right]. \tag{20}$$

If the allocation is Pareto efficient, equation (20) implies that the real wealth vectors $\hat{\omega}_i\left(s^t\right)$ are stationary, meaning that there exists $\left(\hat{\boldsymbol{\omega}}_i\left(s\right)\right)_{(i,s)\in\mathbf{I}\times\mathbf{S}}$ such that $\hat{\omega}_i\left(s^t\right) = \hat{\boldsymbol{\omega}}_i\left(s_t\right)$ for all i and all states.

4.3 Equations for Pareto efficient equilibrium

Applying equation (20) to the initial period, the initial period discounted present value equations for households are:

$$\frac{\omega_i\left(s_0\right)}{p_i\left(s_0\right)} = \sum_{k=0}^{\infty}\beta^k E_0\left[\frac{U_c\left(\mathbf{Y}(s_k)\right)}{U_c\left(\mathbf{Y}(s_0)\right)}\left(\theta_i\mathbf{Y}(s_k) - \mathbf{q}_i\left(s_k\right)\mathbf{y}_i\left(s_k\right)\right)\right]. \tag{21}$$

Writing the discounted present value constraints (21) in recursive notation and applying matrix algebra yields the useful equations:

$$\frac{\omega_i\left(s_0\right)}{p_i\left(s_0\right)} = \left(I_S - \hat{\Gamma}\right)^{-1}_{(s_0)}\left(\theta_i\mathbf{Y}(s) - \mathbf{q}_i(s)\mathbf{y}_i(s)\right)_{s\in\mathbf{S}}, \tag{22}$$

where I_S is the $S\times S$ identity matrix, $\hat{\Gamma}$ is the $S\times S$ stochastic discount factor matrix with elements $\hat{\Gamma}\left(s,\sigma\right) = \beta\Gamma\left(s,\sigma\right)\frac{U_c(\mathbf{Y}(\sigma))}{U_c(\mathbf{Y}(s))}$, and $\left(I_S - \hat{\Gamma}\right)^{-1}_{(s_0)}$ is row s_0 of the matrix $\left(I_S - \hat{\Gamma}\right)^{-1}$.

In a manner similar to how equation (20) was derived for households, we can derive discounted present value constraints for the monetary authorities. The initial period discounted present value equations for monetary authorities are:

$$\frac{W_i\left(s_0\right)}{p_i\left(s_0\right)}=\sum_{k=0}^{\infty}\beta^k E_0\left[\frac{U_c\left(\mathbf{Y}(s_k)\right)}{U_c\left(\mathbf{Y}(s_0)\right)}\mathbf{y}_i\left(s_k\right)\left(1-\mathbf{q}_i\left(s_k\right)\right)\right]. \tag{23}$$

Writing the discounted present value constraints (23) in recursive notation and applying matrix algebra yields the useful equations:

$$\frac{W_i\left(s_0\right)}{p_i\left(s_0\right)}=\left(I_S-\hat{\Gamma}\right)^{-1}_{(s_0)}\left(\mathbf{y}_i(s)\left(1-\mathbf{q}_i(s)\right)\right)_{s\in\mathbf{S}}, \tag{24}$$

Solving equations (24) and (22) for the consumption fractions $\left(\theta_i\right)_{i\in\mathbf{I}\setminus\{N\}}$ (the fraction $\theta_N=1-\sum_{i\in\mathbf{I}\setminus\{N\}}\theta_i$, by definition) yields:

$$\theta_i=\frac{\frac{\omega_i(s_0)}{W_i(s_0)}\left(I_S-\hat{\Gamma}\right)^{-1}_{(s_0)}\left(\mathbf{y}_i(s)\right)_{s\in\mathbf{S}}+\left(1-\frac{\omega_i(s_0)}{W_i(s_0)}\right)\left(I_S-\hat{\Gamma}\right)^{-1}_{(s_0)}\left(\mathbf{q}_i(s)\mathbf{y}_i(s)\right)_{s\in\mathbf{S}}}{\left(I_S-\hat{\Gamma}\right)^{-1}_{(s_0)}\left(\mathbf{Y}(s)\right)_{s\in\mathbf{S}}}. \tag{25}$$

Under Pareto efficiency, the Euler equations (17) are updated:

$$\mathbf{q}_i\left(s\right)=\beta\sum_{\sigma\in\mathbf{S}}\Gamma\left(s,\sigma\right)\frac{U_c\left(\mathbf{Y}(\sigma)\right)}{U_c\left(\mathbf{Y}(s)\right)}\frac{1}{\boldsymbol{\pi}_i\left(\sigma\right)}\quad\forall\left(i,s\right)\in\mathbf{I}\times\mathbf{S}. \tag{26}$$

With stationary wealth and inflation rates, the bond holdings are constant. Define the constant bond holdings $\left(\mathbf{b}_{i,j}\right)_{(i,j)\in\mathbf{I}^2}$ and constant debt holdings $\left(\mathbf{B}_{i,j}\right)_{(i,j)\in\mathbf{I}^2}$ such that $\hat{b}_{i,j}\left(s^t\right)=\mathbf{b}_{i,j}$ and $\hat{B}_{i,j}\left(s^t\right)=\mathbf{B}_{i,j}$ for all (i,j) and for all states. The household budget constraints (16) are:

$$\theta_i\mathbf{Y}(s)-\mathbf{q}_i\left(s\right)\mathbf{y}_i(s)+\sum_{j\in\mathbf{I}}\mathbf{q}_j\left(s\right)\mathbf{b}_{i,j}=\sum_{j\in\mathbf{I}}\frac{\mathbf{b}_{i,j}}{\boldsymbol{\pi}_j\left(s\right)}\quad\forall\left(i,s\right)\in\mathbf{I}\times\mathbf{S}. \tag{27}$$

The monetary authority constraints (15) are:

$$\mathbf{y}_i(s)\left(1-\mathbf{q}_i\left(s\right)\right)+\sum_{j\in\mathbf{I}}\mathbf{q}_j\left(s\right)\mathbf{B}_{i,j}=\sum_{j\in\mathbf{I}}\frac{\mathbf{B}_{i,j}}{\boldsymbol{\pi}_j\left(s\right)}\quad\forall\left(i,s\right)\in\mathbf{I}\times\mathbf{S}. \tag{28}$$

Equations (24)-(28) provide a total of $2N-1+SN+S\left(2N-1\right)$ equations. The number of variables equals $2N-1+2SN+N(2N-1)$, which are listed in Table 2:

initial prices	$(p_i(s_0))_{i\in\mathbf{I}}$	N variables
consumption fractions	$(\theta_i)_{i\in\mathbf{I}}$	$N-1$ variables
bond prices	$(\mathbf{q}_i(s))_{(i,s)\in\mathbf{I}\times\mathbf{S}}$	SN variables
inflation rates	$(\boldsymbol{\pi}_i(s))_{(i,s)\in\mathbf{I}\times\mathbf{S}}$	SN variables
bond holdings	$(\mathbf{b}_{i,j})_{(i,j)\in\mathbf{I}\times\mathbf{I}}$	N^2 variables
debt positions	$(\mathbf{B}_{i,j})_{(i,j)\in\mathbf{I}\setminus\{N\}\times\mathbf{I}}$	$N(N-1)$ variables

.

Table 1: Variables for stationary inflation policies

The number of variables exceeds the number of equations when $S\leq 2N+\frac{N}{N-1}$. This upper bound on S from the algebraic approach is identical to the upper bound found below from the geometric approach.

4.4 Main result

Theorem 1 *If $S>2N+\frac{N}{N-1}$, Assumptions 1-2 hold, and $rank(R)=N$, then over a generic subset of household endowments, there does not exist a stationary policy rule under which the equilibrium allocation is Pareto efficient.*

The remainder of this section will prove this generic Pareto inefficiency result.

The variables $(\boldsymbol{\pi}_i(s))_{(i,s)\in\mathbf{I}\times\mathbf{S}}$, $(\theta_i)_{i\in\mathbf{I}}$, and $(p_i(s_0))_{i\in\mathbf{I}}$ are uniquely determined from (24)-(26) in terms of bond prices $(\mathbf{q}_i(s))_{(i,s)\in\mathbf{I}\times\mathbf{S}}$.

Using the Euler equation (26), the household budget constraints (27) can be written using the span definition:

$$\left(I_S-\hat{\Gamma}\right)^{-1}\left(\theta_i\mathbf{Y}(s)-\mathbf{q}_i(s)\,\mathbf{y}_i(s)\right)_{s\in\mathbf{S}}\in span(R). \tag{29}$$

Lemma 2 *Over a generic subset of household endowments,*

$$rank\left(\left(\theta_i\mathbf{Y}(s)-\mathbf{q}_i(s)\,\mathbf{y}_i(s)\right)_{(i,s)\in\mathbf{I}\times\mathbf{S}}\right)=N.$$

Proof. See Hoelle (2017). ■

The set of all $N-$dimensional linear subspaces of $\mathbb{R}^S$ is the Grassmanian manifold $Gr(N,S)$, which is an $N(S-N)-$dimensional set. Given Lemma 2, there exists a unique $span(R)$ in this $N(S-N)-$dimensional set satisfying the span condition (29) $\forall i\in\mathbf{I}$.

A similar span condition must hold for the monetary authorities. From Walras' Law, only constraints for $N-1$ monetary authorities are considered. The constraints are expressed using the span definition:

$$\left(I_S-\hat{\Gamma}\right)^{-1}\left(\mathbf{y}_i(s)\left(1-\mathbf{q}_i\left(s\right)\right)\right)_{s\in\mathbf{S}}\in span(R). \tag{30}$$

For each $i\in\mathbf{I}\backslash\{N\}$, since $span(R)$ is $N-$dimensional, equation (30) imposes $(S-N)$ restrictions on the policy choices $\left(\mathbf{q}_i\left(s\right)\right)_{s\in\mathbf{S}}$. The variables that change the asset span have dimension $S+N(N-1)$. The first S choices are $\left(\mathbf{q}_N\left(s\right)\right)_{s\in\mathbf{S}}$ and the final $N(N-1)$ choices are the free policy variables for each $i\in\mathbf{I}\backslash\{N\}$. These variables must be sufficient in number in order to support any asset span in the set $Gr(N,S)$, since only 1 asset span in that set is consistent with (29). This requires that

$$S+N(N-1)\geq N(S-N).$$

Algebraically, this inequality reduces to

$$S\leq 2N+\frac{N}{N-1}.$$

If this condition is violated, i.e., $S>2N+\frac{N}{N-1}$, there does not exist an asset span such that (29) and (30) are satisfied.

5 Examples

Theorem 1 provides a necessary condition for Pareto efficiency, namely $S\leq 2N+\frac{N}{N-1}$. The condition is necessary, but not sufficient, for Pareto efficiency since an equilibrium requires that (i) bond prices and interest rates are nonnegative, i.e., $\mathbf{q}_i\left(s\right)\in[0,1]$ for all $(i,s)\in\mathbf{I}\times\mathbf{S}$, and (ii) domestic monetary authority debt is nonnegative, i.e., $\mathbf{B}_{i,i}\geq 0$ for all $i\in\mathbf{I}$. This fact is illustrated with two examples.

5.1 Example 1

Consider an economy with $S=6$ states and $N=2$ countries. Suppose that the economy does not contain aggregate risk and the total endowment is normalized to 1: $\mathbf{Y}(s)=1\ \forall s\in\mathbf{S}$. The assumption of no aggregate risk simplifies the calculation as it implies that the policy rule is independent of the utility function. The normalization has no effect on the policy rule. Country 1 household endowment is $\left(\mathbf{y}_1\left(s\right)\right)_{s\in\mathbf{S}}=(0.95,0.8.0.75,0.7,0.65,0.6)$, with the rest of the endowment held by country 2 households. The discount factor is $\beta = 0.9$. The initial state is $s_0 = 1$ and the initial wealth ratio for country 1 is $\frac{\omega_1(s_0)}{W_1(s_0)} = 1.2$. The ratio for country 2, $\frac{\omega_2(s_0)}{W_2(s_0)}$, must be such that total initial obligations of monetary authorities equal total initial household wealth (an equilibrium requirement):

$$\sum_{i \in \mathbf{I}} \xi_i(s_0)\,\omega_i(s_0) = \sum_{i \in \mathbf{I}} \xi_i(s_0)\,W_i(s_0). \tag{31}$$

By independently specifying $\omega_1(s_0)$ and $W_1(s_0)$, the initial price $p_1(s_0)$ can be determined (and similarly for country 2). This step is omitted.

Country 1 chooses a monetary policy $(\boldsymbol{\pi}_1(s))_{s \in \mathbf{S}}$ and country 2 chooses a monetary policy $(\boldsymbol{\pi}_2(s))_{s \in \mathbf{S}}$ such that the equilibrium allocation is Pareto efficient.

Without aggregate risk, the stationary equilibrium supporting Pareto efficiency is independent of the utility function. With iid shocks and no aggregate risk, the bond prices for country 1 and country 2 are independent of the shocks. This doesn't limit the ability of the countries to achieve Pareto efficiency as they choose state-contingent vectors of inflation rates, $(\boldsymbol{\pi}_1(s))_{s \in \mathbf{S}}$ and $(\boldsymbol{\pi}_2(s))_{s \in \mathbf{S}}$, respectively. With $S = 6$ states and $N = 2$ countries, the system of equations contains 33 nonlinear equations. The algorithm works best when the Markov process is close to the special case of iid, in which case the bond prices are nearly independent of the shocks. For more general Markov processes, a more efficient algorithm is required. The Markov process used for this example is such that $\Gamma(s, \sigma) = \begin{Bmatrix} 0.175 & \text{if } s = \sigma \\ 0.165 & \text{otherwise} \end{Bmatrix}$.

The condition $S \leq 2N + \frac{N}{N-1}$ when $S = 6$ and $N = 2$ holds with equality.This means that there are zero degrees of freedom for the equilibrium variables. Either there exists a unique stationary policy rule such that Pareto efficiency is supported or no such policy rule can be found. The following policy rules support a Pareto efficient allocation:

$$\begin{aligned} (\boldsymbol{\pi}_1(s) - 1)_{s \in \mathbf{S}} &= (-0.73\%, 1.62\%, 2.42\%, 3.24\%, 4.08\%, 4.93\%). \\ (\boldsymbol{\pi}_2(s) - 1)_{s \in \mathbf{S}} &= (3.56\%, 2.86\%, 2.63\%, 2.40\%, 2.17\%, 1.95\%). \end{aligned}$$

From the Euler equations, the bond prices are such that $\mathbf{q}_1(s) \simeq 0.878$ and $\mathbf{q}_2(s) \simeq 0.877$ $\forall s \in \mathbf{S}$ (bond prices vary across states, but by at most a factor of 10^{-5}). From equation (24), the consumption fraction is $\theta_1 = 0.781$ and $\theta_2 = 1 - \theta_1 = 0.219$. We have left to verify the budget constraints for both households (in all 6 states) and the constraint for the country 1 monetary authority (in all 6 states). If we do this, then Walras' Law implies that the constraints for the country 2 monetary authority are automatically satisfied. The bond holdings are

$$\mathbf{b}_1 = (-4.17, 5.51) \quad \mathbf{b}_2 = (4.44, -4.52) \quad \mathbf{B}_1 = (0.82, 0.11) \ .$$

By market clearing, $\mathbf{B}_2 = \mathbf{b}_1 + \mathbf{b}_2 - \mathbf{B}_1 = (-0.55, 0.88)$. The household budget constraints and monetary authority constraints are given by:

$$\begin{array}{lccc} & \text{Left-hand side} & & \text{Right-hand side} \\ \text{Household} & \theta_i \mathbf{Y}(s) - \mathbf{q}_i(s)\,\mathbf{y}_i(s) + \sum_{j\in\mathbf{I}} \mathbf{q}_j(s)\,\mathbf{b}_{i,j} & = & \sum_{j\in\mathbf{I}} \frac{\mathbf{b}_{i,j}}{\boldsymbol{\pi}_j(s)} \\ \text{Mon Auth} & \mathbf{y}_i(s)\,(1-\mathbf{q}_i(s)) + \sum_{j\in\mathbf{I}} \mathbf{q}_j(s)\,\mathbf{B}_{i,j} & = & \sum_{j\in\mathbf{I}} \frac{\mathbf{B}_{i,j}}{\boldsymbol{\pi}_j(s)} \end{array} .$$

Table 2 verifies that the 18 constraints are satisfied by showing that the left-hand side (LHS) of each constraint equals the right-hand side (RHS):

	Household 1		Household 2		Mon Auth 1	
State	LHS	RHS	LHS	RHS	LHS	RHS
1	1.120	1.120	0.105	0.105	0.934	0.934
2	1.253	1.253	-0.028	-0.028	0.915	0.915
3	1.297	1.297	-0.072	-0.072	0.909	0.909
4	1.341	1.341	-0.116	-0.116	0.903	0.903
5	1.386	1.386	-0.161	-0.161	0.897	0.897
6	1.430	1.430	-0.205	-0.205	0.891	0.891

Table 2: Verification of equilibrium budget constraints

The initial endowment for the country 2 household is only 0.05. In order to support the Pareto efficient consumption of 0.219, the household has negative real wealth in most states. The borrowing used to support the negative real wealth positions is stationary and certainly permissible under equilibrium.

If we ignore the constraints on the bond prices and debt positions, the condition $S \leq 2N + \frac{N}{N-1}$ is necessary and sufficient for Pareto efficiency as it is always possible to construct variables to satisfy all equilibrium equations. When I say that no stationary policy rule can be found such that the equilibrium allocation is Pareto efficient, the issue is that the variables constructed to support Pareto efficiency violate one of the constraints, meaning either $\mathbf{q}_i(s) > 1$ for some (i, s) or $\mathbf{B}_{i,i} < 0$ for some i.

5.2 Example 2

Consider an economy similar to Example 1, with the only exception that $s_0 = 6$. In this updated version of the economy, the ratio of household endowment is $\frac{\mathbf{y}_1(6)}{\mathbf{y}_2(6)} = 1.5$, compared to $\frac{\mathbf{y}_1(6)}{\mathbf{y}_2(6)} = 20$. This changes the Pareto efficient allocation (more consumption for country 2 household) and the bond portfolios and debt positions required to support Pareto efficiency. The problem is that the supporting debt positions must be such that $\mathbf{B}_1 = (-2.34, 3.31)$ and $\mathbf{B}_2 = (2.85, -2.53)$.

This is not an equilibrium as $\mathbf{B}_{1,1} < 0$ and $\mathbf{B}_{2,2} < 0$ violate the requirement that the monetary authority cannot hold more domestic debt than was initially issued by the fiscal authority. The necessary condition $S \leq 2N + \frac{N}{N-1}$ holds with equality, so there are zero degrees of freedom with which to find an equilibrium satisfying all economic constraints. Thus, for this particular economy, there does not exist a stationary policy rule under which Pareto efficiency is supported.

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