Final Exam
Thursday, October 13, 2016

Directions

- This is a 90-minute exam worth 100 points.
- This exam contains 1 question with 8 parts. The parts are equally weighted (each worth 12.5 points).

  In the Qualifying Examination, more weight is applied to harder questions, so the point distribution would be (in order that the questions are listed): 0.8, 0.2, 0.4, 0.2, 0.2, 0.8, 0.4, 1.0 (total of 4.0 points).
- You may use a calculator for this exam. You are not permitted to access user-generated files in the calculator.
- Justify your answers by showing all work.
- If you feel that additional assumptions are required to answer the question, state these assumptions clearly.

1. **Neoclassical growth model with nonlinear taxation**

   The economy consists of an infinite time horizon with homogeneous consumers and homogeneous firms. The preferences for consumers are

   $$\sum_{t=0}^{\infty} \beta^t u\left(c_t, n_t\right),$$

   where c_t is the level of consumption at time t and n_t is the labor supplied at time t.

   Consumers own capital stock. Capital depreciates at the rate of $\delta \in [0,1]$ each period. The law of motion for capital stock is given by:

   $$k_{t+1} = i_t + (1-\delta)\, k_t.$$

   Consumers earn a rate of return on capital and a wage rate, denoted in time t by R_t and w_t, respectively. Consumers are subject to a nonlinear income tax. Define $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ as the tax function. The taxable income for the consumers equals

   $$(R_t - \delta)\, k_t + w_t n_t.$$

   If a consumer has taxable income $(R_t - \delta)\, k_t + w_t n_t$, then the tax obligation for the consumer equals

   $$\tau\left((R_t - \delta)\, k_t + w_t n_t\right).$$

   Naturally, the tax function satisfies the properties: $\tau(0) = 0$ and $\tau((R_t - \delta)\, k_t + w_t n_t) \leq (R_t - \delta)\, k_t + w_t n_t$. If τ is differentiable, it must also satisfy $\tau'\left((R_t - \delta)\, k_t + w_t n_t\right) \leq 1$.

   Firms produce output using capital and labor as inputs. The production function is a standard Cobb-Douglas production function:

   $$f\left(K_t, N_t\right) = \left(K_t\right)^{\theta} \left(N_t\right)^{1-\theta} \text{ for } \theta \in (0,1).$$

   The government collects the tax revenue and distributes the proceeds lump-sum to the consumers. The government balances its budget in each and every period.

   (a) Define a Recursive Competitive Equilibrium.

   (b) What assumptions on the model parameters are required to guarantee that the Bellman equation is well-defined?

   (c) A progressive tax is a tax in which the marginal tax rate increases as the amount of taxable income increases. A regressive tax is a tax in which the marginal tax rate decreases as the amount of taxable income increases. If we want a well-defined Bellman equation, can the tax be progressive? can the tax be regressive? If we want a value function V that is both concave and differentiable, can the tax be progressive? can the tax be regressive? Justify your responses.

   (d) State the Contraction Mapping Theorem.

   (e) The space $CB(\mathbf{A})$ of continuous and bounded functions on compact domain $\mathbf{A} = \left[0, \bar{k}\right]$ is a complete metric space. Define an appropriate mapping $T : CB(\mathbf{A}) \to CB(\mathbf{A})$. You will show in (f) that it is correct to write $T : CB(\mathbf{A}) \to CB(\mathbf{A})$ and will show in (g) that T is a contraction.

(f) State and use Berge's Maximum Theorem to prove that $T : CB(\mathbf{A}) \to CB(\mathbf{A})$.

(g) State and use Blackwell's sufficient conditions to verify that T is a contraction.

(h) Assume that u is concave and differentiable. State and use the Benveniste-Scheinkman Theorem to prove that V is concave and differentiable.