Incomplete Markets and the Distribution of Risk in a Large Open Economy under Targeting Rules*

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Abstract

In stochastic models of large open economies with heterogeneous households and incomplete markets, the monetary policies of central banks have real effects. This paper considers a pure exchange economy with heterogeneous households and incomplete markets and central bank policy that targets both the short-term nominal interest rate and the exchange rates. The paper proves that inflation rate targeting and nominal GDP targeting policies are incompatible with Pareto efficiency (generically), and characterizes necessary conditions under which more general policies are compatible with Pareto efficiency.

Keywords inflation rate targeting – nominal GDP targeting – Pareto inefficiency

JEL Classification **E44, E52, F41, G15**

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1 Introduction

The justification for monetary policy lies not in its ability to achieve nominal objectives, but in its ability to achieve real objectives. The nominal targets are but a means to an end. Policies of inflation stabilization are optimal in some settings because stable inflation mitigates the friction of nominal rigidities that may distort relative prices in the short-run and lead to an inefficient allocation of resources. Stable inflation in these settings promotes the efficient operation of markets.

Advocating for a monetary policy requires an understanding of the market frictions that such policy seeks to mitigate and a model mechanism that describes how the policy interacts with the friction. This paper focuses on the role of monetary policy in mitigating financial frictions caused by incomplete markets. Policy is helpful in mitigating this friction if it enables households to use the financial markets to reduce their exposure to risk.

Debt contracts are typically written with nominal payouts that are not contingent on future realizations of uncertainty. Some country's currency serves as the unit of account for these debt contracts. Monetary policy in that country anchors the domestic price level in all future realizations and thus the real payouts of the debt contracts. It is not only the expected inflation rates that matter for policy effects, but the vector of inflation rates for all realizations of uncertainty in the future. If the inflation rates are state-contingent, then so too are the real payouts of the debt contracts.

To illustrate the policy mechanism for international financial markets, consider a stochastic model of a large open economy with two countries and two time periods. Financial markets are complete within countries, but incomplete across countries. The only international assets are nominally risk-free bonds, one for each country. Suppose both countries adopt policies such that domestic nominal income is constant across all realizations of uncertainty. Households respond to this policy by holding a portfolio that involves lending with the foreign bond and borrowing with the domestic bond. The real payouts of such a portfolio are positive when domestic output is relatively low (as low output implies a high nominal price level and low real payouts) and negative when domestic output is relatively high (relative to the other country). This allows for consumption smoothing even under incomplete markets.

With an infinite time horizon, the same mechanism allows households to smooth expenditures, where expenditures include spending on consumption plus the portfolio of bonds to carry into the next period. The interest rates vary based upon the current output realizations, so expenditure smoothing is not equivalent to consumption smoothing. The spanning effects of international financial markets determine how close the set of budget-feasible consumption vectors lie to an ideal vector where consumption is smoothed efficiently (according to the Pareto criterion).

Monetary policy determines the real payouts of the assets and therefore determines which consumption vectors are budget-feasible. Given the ability to hold domestic and foreign bonds, a household's feasible choice set is determined by both domestic and foreign monetary policy.

Governments set monetary policy by targeting the domestic nominal interest rates and transacting in both domestic and foreign debt. The cumulative actions of governments and households must be such that the total net public debt is equal to the total net private savings. The nominal interest rates affect the net portfolio expenditures of households. While monetary policy supporting constant domestic nominal incomes allows households to smooth expenditures, it also leads to large differences in the interest rates across realizations of uncertainty. This draws expenditure smoothing further away from consumption smoothing, where the latter is the goal of monetary policy in this model.

Monetary policy could be conducted independently by each country only adopting domestic policies of open market operations, but this would require large fluctuations in the domestic debt positions every period. Rather, by incorporating foreign debt into the portfolio, governments can achieve the same real objectives with a stationary portfolio.

1.1 Overview of the model and the main results

This paper considers a stochastic monetary model of a large open economy with N countries. Each country in the model issues a distinct currency, issues short-term debt in the form of a 1-period nominally risk-free bond, and contains a unit mass of homogeneous households. Households receive endowments according to a Markov process with S possible realizations.

Each country contains a commodity market, in which the domestic households' endowments are traded, and bond and money markets, in which the domestic bond and currency are traded. Each of these markets specifies prices in terms of the domestic currency.

Money is valued via cash-in-advance constraints. Households choose a vector of consumption (N different commodities, 1 from each country), a vector of bond holdings (using both domestic and foreign short-term bonds), and a vector of money holdings (N currencies). To focus on the effects of incomplete markets, I shut down the channel for real exchange rate variation (i.e., purchasing power parity will hold). Namely, I assume that the household preferences are such that commodities, whether purchased in domestic or foreign markets, are perfect substitutes.

Each country contains a monetary-fiscal authority. In domestic markets, the monetary-fiscal authority targets the short-term nominal interest rate, or equivalently the domestic debt position. In international markets, the monetary-fiscal authority can affect the exchange rates by holding foreign debt positions (either saving or borrowing). The model only considers nominal policy choices, including nominal interest rate and exchange rate determination, and does not include real policy choices, such as government spending and taxation.

A policy rule is a sequence of policy choices that each monetary-fiscal authority commits to. When all monetary-fiscal authorities adopt targeting rules, then over a generic subset of household endowments and initial wealth, the equilibrium allocation is not Pareto efficient. A targeting rule requires a constant growth rate for the variable being targeted. Examples include inflation rate targeting and nominal GDP targeting. A larger set of monetary policy tools are required for Pareto efficiency. I characterize necessary conditions for Pareto efficiency that illustrate exactly the types of monetary policies that outperform (in a welfare sense) targeting rules.

1.2 Literature review

The present paper is most closely related to the works of Koenig (2013) and Sheedy (2014) that consider the extent to which policies of nominal GDP targeting, or monetary growth rate targeting, can mitigate the friction of incomplete markets. In the closed economy model in Koenig (2013) and the open economy model in Sheedy (2014), policies of nominal GDP targeting support a Pareto efficient allocation. Koenig (2013) considers a stylized 2-period setting. Sheedy (2014) only considers aggregate risk and assumes a special structure on the preferences in order to ensure a constant wealth distribution. In a standard 2-period incomplete markets model, policies of nominal GDP targeting can be chosen to support a Pareto efficient equilibrium allocation. However, the present paper proves that this result is no longer true in an infinite time horizon model with time-invariant preferences.[1] The model in the present paper allows for both aggregate and idiosyncratic risk, but the Pareto inefficiency result holds true for economies with only aggregate risk and also for economies with only idiosyncratic risk.

In international economics, the typical friction studied in the literature is nominal rigidities. Models with such frictions belong to the New Keynesian tradition. Notable exceptions include Baxter and Crucini (1995) and Corsetti et al. (2008a). Baxter and Crucini (1995) analyze the effects of incomplete markets on international business cycles. They find that if the idiosyncratic output risk is small, the presence of incomplete markets does not have

[1] In the sequel, I provide the Pareto efficiency argument used for 2-period models and show precisely why it fails to hold in infinite time horizon models.

large real effects. Corsetti et al. (2008a) extends this result by showing that terms of trade volatility magnifies the effects from Baxter and Crucini (1995), meaning that exchange rate fluctuations (caused by monetary policy) allow incomplete markets to play a greater role in business cycle fluctuations.

For the class of New Keynesian models in which the friction is nominal rigidities, the key references are Benigno and Benigno (2003), Corsetti and Pesenti (2005), Devereaux and Sutherland (2008), Corsetti et al. (2008b), and Corsetti et al. (2010). The setting in this class of models is one of complete markets, or at least a stylized setting in Corsetti and Pesenti (2005) in which the real shocks do not affect the equilibrium consumption and labor choices. Firms operate in a setting of monopolistic competition. Nominal rigidities serve as the friction in these settings, where the policy predictions depend upon whether the price rigidities occur in terms of the producer's currency or in terms of the consumer's currency. Under the former (labeled producer currency pricing, or PCP), Corsetti et al. (2010) show that inward rules, or monetary policies that would be optimal in a closed economy (specifically, domestic price stabilization), may not be optimal in an open economy. Optimal policies require policy makers to trade off exchange rate stabilization (to account for the nominal rigidities being imported from foreign firms) with domestic price stabilization (to simultaneously minimize the domestic output gap and the effect of domestic nominal distortions).

The two trade-offs mentioned (domestic price stability and exchange rate stability) do not tell the whole story, as models need to account for a third trade-off arising from incomplete markets. From the concluding remarks of Corsetti et al. (2010):

> "Key lessons for monetary policy analysis can be learnt from models in which asset markets do not support the efficient allocation" and the "design of monetary policy in models with explicit financial distortions [serve] as a complement."
>
> -Corsetti et al. (2010), pg. 928

For the class of New Keynesian models with incomplete markets, Corsetti et al. (2008b) adds nominal rigidities to the exercise carried out in the incomplete markets setting of Corsetti et al. (2008a). Additionally, Devereaux and Sutherland (2008) show that price stability is the optimal policy to mitigate the effects of financial shocks to the interest rate rule. To capture portfolio effects, Devereaux and Sutherland (2008) perform a second-order approximation and consider an asset structure (labeled NB for "nominal bonds") that is identical to the asset structure in the present paper. Without any real shocks, however, zero inflation monetary policy removes the friction caused by the nominal rigidities, while simultaneously nullifying the interest rate shocks (a coincidence that, as with most of the New Keynesian literature, is less "divine" and more engineered).

Unlike the cited models in the New Keynesian tradition, one of the important contributions of the present paper is to consider the effects of real shocks in which there is a trade-off between the real effects of inflation and the spanning effects of the endogenous asset structure.

The related subfield of international finance (see Gourinchas and Rey (2007)) does not explicitly model policy choice, but does stress that shortfalls in current account balances can be remedied with exchange rate movements. The natural connection to the present paper is that monetary policy can have real effects through movements in the exchange rate.

The remainder of the paper is organized as follows. Section 2 introduces the model. Section 3 verifies important equilibrium properties. Section 4 considers nominal GDP targeting. Section 5 characterizes necessary conditions under which more general policies are compatible with Pareto efficiency. Section 6 provides concluding remarks and the Appendix contains the proofs of the main results.

2 The Model

The model describes a large open economy with N countries $i \in \mathbf{I} = \{1, ..., N\}$, each containing a monetary authority and a distinct currency.

Time is discrete and infinite with time periods $t \in \{0, 1, ...\}$. The filtration of uncertainty follows a one-period Markov process with finite state space $\mathbf{S} = \{1, ..., S\}$. The realized state of uncertainty in any period t, denoted s_t, is a function only of the realized state in the previous period $t-1$, denoted s_{t-1}. This random process is characterized by a transition matrix $\Gamma \in \mathbb{R}^{S,S}$ whose elements are $\Gamma(s, s')$ for row s and column s'.

This paper focuses on incomplete markets and since there will be N assets (1 for each country), the number of states must be strictly larger than the number of countries.

Assumption 1 (incomplete markets) $N < S$.

The history of all realizations up to and including the current realization completely characterizes the date-event and is required to uniquely identify the markets, household decisions, and policy choices. Define the history of realizations up to and including the realization s_t in period t as $s^t = (s_0, s_1, ..., s_t)$.

In each date-event, and in each country, a single commodity is traded.

2.1 Households

In each country, a unit mass of infinite-lived and homogeneous households reside. Households in country $i \in \mathbf{I}$ receive the endowments $\{y_i(s^t)\}$. The endowments are only in terms of the domestic commodity. Households receive zero endowments of any foreign commodities. The endowments are stationary, meaning there exists a mapping $\mathbf{y}_i : \mathbf{S} \to \mathbb{R}_{++}$ such that $y_i(s^t) = \mathbf{y}_i(s_t)$ for all date-events and all countries $i \in \mathbf{I}$. Denote the aggregate endowment as $\mathbf{Y} : \mathbf{S} \to \mathbb{R}_{++}$ such that $\mathbf{Y}(s) = \sum_{i \in \mathbf{I}} \mathbf{y}_i(s)\ \forall s \in \mathbf{S}$. The model permits aggregate risk, i.e., $\mathbf{Y}(s) \neq \mathbf{Y}(\sigma)$ for some $s, \sigma \in \mathbf{S}$.

Assumption 2 (idiosyncratic endowments) The endowment vectors $\left\{..., (\mathbf{y}_i(s))_{s \in \mathbf{S}}, ...\right\}$ are linearly independent.

Households in country i consume commodities sold in all countries. The consumption of country j commodity by country i households in date-event s^t is denoted $c_{i,j}(s^t) \in \mathbb{R}_+$. The commodities are assumed to be perfect substitutes, meaning that households only care about the total consumption $c_i(s^t) = \sum_{j \in \mathbf{I}} c_{i,j}(s^t)$. The sequence of consumption for country i households is denoted $\{c_i(s^t)\}$.

The household preferences are assumed to be identical and satisfy constant relative risk aversion:

$$E_0 \sum_{t=0}^{\infty} \beta^t u\left(c\left(s^t\right)\right). \tag{1}$$

Assumption 3 (CRRA preferences) The discount factor $\beta \in (0,1)$ and $u(c) = \frac{c^{1-\rho}}{1-\rho}$ for $\rho > 0$ and $\rho \neq 1$ and $u(c) = \ln(c)$ for $\rho = 1$.

In each date-event s^t, the money supply in country $i \in \mathbf{I}$ is $M_i(s^t) > 0$ and the nominal price level is $p_i(s^t) > 0$. Let $\xi_i(s^t)$ be the nominal exchange rate for country i, relative to country 1. Specifically, $\xi_i(s^t)$ is the number of units of country 1 currency for each one unit of country i currency. As the numeraire country, $\xi_1(s^t) = 1$.

Each country issues a short-term (one-period) nominally risk-free bond (government debt). The nominal payout of the short-term bond issued by country i in date-event s^t equals 1 unit of country i currency in date-events $(s^t, \sigma)_{\sigma \in \mathbf{S}}$ and 0 otherwise. The asset price for a country $i \in \mathbf{I}$ bond in date-event s^t (in units of the country i currency) is $q_i(s^t)$.

Each date-event is divided into two subperiods. In the initial subperiod, the money markets and bond markets open. Denote $(\hat{m}_{i,j}(s^t))_{j \in \mathbf{I}}$ as the money holdings by country i households by the close of the money markets in date-event s^t (in terms of each currency $j \in$

I). By definition, the money holdings are nonnegative. The entire vector of money holdings is denoted $\hat{m}_i(s^t) = (\hat{m}_{i,j}(s^t))_{j\in\mathbf{I}} \in \mathbb{R}_+^N$. Denote $(b_{i,j}(s^t))_{j\in\mathbf{I}}$ as the bond holdings by country i households by the close of the bond markets in date-event s^t. Each bond can either be held long or short by the household. Denote the entire portfolio as $b_i(s^t) = (b_{i,j}(s^t))_{j\in\mathbf{I}} \in \mathbb{R}^N$.

Denote $\omega_i(s^t) \in \mathbb{R}$ as the nominal wealth held by the country i households for use in the date-event s^t. The wealth is specified in units of the country i currency. The initial period value $\omega_i(s_0)$ is a parameter of the model. The budget constraint, at the close of the money markets and bond markets in date-event s^t, is given by:

$$\frac{1}{\xi_i(s^t)}\left(\sum_{j\in\mathbf{I}} \xi_j(s^t)\left(\hat{m}_{i,j}(s^t) + q_j(s^t)\, b_{i,j}(s^t)\right)\right) \le \omega_i(s^t). \tag{2}$$

The budget constraint is specified in units of the country i currency.

In the second subperiod of each date-event, the commodity markets open. The purchase of the commodities is subject to the cash-in-advance constraints:

$$p_j(s^t)\, c_{i,j}(s^t) \le \hat{m}_{i,j}(s^t) \quad \forall (i,j) \in \mathbf{I}^2. \tag{3}$$

At the same time that consumption is being purchased on the commodity markets, the households receive income from selling their endowments. Recall that households only receive endowments of the domestic commodity. Denote $m_i(s^t) = (m_{i,j}(s^t))_{j\in\mathbf{I}} \in \mathbb{R}_+^N$ as the money holding of the country i households by the close of the commodity markets in date-event s^t, where

$$\begin{aligned} m_{i,j}(s^t) &= \hat{m}_{i,j}(s^t) + p_i(s^t)\,\mathbf{y}_i(s_t) - p_j(s^t)\, c_{i,j}(s^t) \text{ for } i = j. \\ m_{i,j}(s^t) &= \hat{m}_{i,j}(s^t) - p_j(s^t)\, c_{i,j}(s^t) \text{ for } i \ne j. \end{aligned} \tag{4}$$

Given the money definition (4), the cash-in-advance constraints (3) can be rewritten as:

$$\begin{aligned} m_{i,j}(s^t) &\ge p_i(s^t)\,\mathbf{y}_i(s_t) \text{ for } i = j. \\ m_{i,j}(s^t) &\ge 0 \text{ for } i \ne j. \end{aligned} \tag{5}$$

Entering into date-events $(s^t, \sigma)_{\sigma\in\mathbf{S}}$, the nominal wealth available to country i households (in terms of the country i currency) is equal to the money holding plus the portfolio payout:

$$\omega_i(s^t, \sigma) = \frac{1}{\xi_i(s^t, \sigma)}\left(\sum_{j\in\mathbf{I}} \xi_j(s^t, \sigma)\left(m_{i,j}(s^t) + b_{i,j}(s^t)\right)\right). \tag{6}$$

Households are permitted to short-sell the bonds, so households in country i must satisfy the following implicit debt constraint:

$$\liminf_{t,s^t} \left(\sum_{j \in \mathbf{I}} q_j \left(s^t\right) b_{i,j} \left(s^t\right) \right) > -\infty. \tag{7}$$

The household optimization problem for households in country i is given by:

$$\begin{array}{ll} \underset{\{c_i(s^t), m_i(s^t), b_i(s^t)\}}{max} & E_0 \sum_{t=0}^{\infty} \beta^t u \left(c_i \left(s^t\right)\right) \\ subj.\ to & \text{budget constraint (2) with (4) and (6) } \forall t, s^t \\ & \text{cash-in-advance constraint (5) } \forall t, s^t \\ & \text{debt constraint (7)} \end{array} \tag{8}$$

2.2 Monetary authority constraints

Monetary authorities are responsible for choosing the domestic money supply, the price for the short-term domestic debt, and the exchange rates.[2]

The country i monetary authority chooses the portfolio of debt positions $B_i\left(s^t\right) = \left(B_{i,j}\left(s^t\right)\right)_{j \in \mathbf{I}} \in \mathbb{R}^N$ in each date-event s^t, where $B_{i,j}\left(s^t\right)$ is the amount of country j debt held. The domestic debt position must be nonnegative ($B_{i,i}\left(s^t\right) \geq 0$), while the foreign debt positions can either be positive or negative.[3]

The country i monetary authority issues the money supply $M_i\left(s^t\right)$ in date-event s^t. In the initial period s_0, the country i monetary authority has the nominal obligation $W_i\left(s_0\right)$.

The country i monetary authority constraint in the initial period s_0 is given by:

$$W_i\left(s_0\right) = M_i\left(s_0\right) + \frac{\sum_{j \in \mathbf{I}} \xi_j\left(s_0\right) q_j\left(s_0\right) B_{i,j}\left(s_0\right)}{\xi_i\left(s_0\right)}. \tag{9}$$

In any date-event s^t with $t > 0$, the country i monetary authority constraint is given by:

$$M_i\left(s^{t-1}\right) + \frac{\sum_{j \in \mathbf{I}} \xi_j\left(s^t\right) B_{i,j}\left(s^{t-1}\right)}{\xi_i\left(s^t\right)} = M_i\left(s^t\right) + \frac{\sum_{j \in \mathbf{I}} \xi_j\left(s^t\right) q_j\left(s^t\right) B_{i,j}\left(s^t\right)}{\xi_i\left(s^t\right)}. \tag{10}$$

[2] In reality, the fiscal authority issues the domestic debt and the monetary authority can buy or sell this debt. The fiscal authority can also hold foreign debt positions. Since the model does not contain fiscal policy choices of taxes or government spending, all choices of the fiscal authority are subsumed by the monetary authority. The assumption is that both authorities are acting in perfect concert.

[3] The monetary authority cannot hold more domestic debt than was initially issued by the fiscal authority.

As with households, implicit debt constraints are required:

$$\liminf_{t,s^t} \left(\sum_{j \in \mathbf{I}} q_j \left(s^t\right) B_{i,j}\left(s^t\right) \right) > -\infty. \tag{11}$$

After introducing an equilibrium, I will show that these monetary authority constraints are equivalent to constraints that equate the discounted expected revenues that the country i monetary authority can acquire from seigniorage, $s_i^M(s^t) = M_i(s^t) - M_i(s^{t-1})$, to the debt obligations.[4]

2.3 Sequential competitive equilibrium

Definition 1 *A sequential competitive equilibrium (SCE) is the vector of household variables* $\{c_i(s^t), m_i(s^t), b_i(s^t)\}_{i\in\mathbf{I}}$, *the monetary authority variables* $\{B_i(s^t), M_i(s^t)\}_{i\in\mathbf{I}}$, *and the price variables* $\{p_i(s^t), \xi_i(s^t), q_i(s^t)\}_{i\in\mathbf{I}}$ *such that:*

1. *Given* $\{p_i(s^t), \xi_i(s^t), q_i(s^t)\}_{i\in\mathbf{I}}$ *and* $\omega_i(s_0)$, *households in country* i *choose the sequence of variables* $\{c_i(s^t), m_i(s^t), b_i(s^t)\}$ *to solve the household problem (8).*

2. *Given* $W_i(s_0)$, *the monetary authority in country* i *chooses the variables* $\{B_i(s^t), M_i(s^t)\}$ *to satisfy (9), (10), and (11).*

3. *Markets clear:*

   (a) *Commodity markets clear:*

   $$\sum_{i\in\mathbf{I}} c_i\left(s^t\right) = \mathbf{Y}(s_t) \text{ for every } t, s^t. \tag{12}$$

   (b) *Total initial obligations of monetary authorities equal total initial household wealth:*

   $$\sum_{i\in\mathbf{I}} \xi_i(s_0)\,\omega_i(s_0) = \sum_{i\in\mathbf{I}} \xi_i(s_0)\,W_i(s_0). \tag{13}$$

   (c) *Total household money holdings equal money supply chosen by monetary authority:*

   $$\sum_{i\in\mathbf{I}} m_{i,j}\left(s^t\right) = M_j\left(s^t\right) \text{ for every } j \in \mathbf{I} \text{ and for every } t, s^t. \tag{14}$$

[4] In this model without fiscal policy, seigniorage is the only source of revenue for the government.

(d) Total net household savings equal total net monetary authority debt:

$$\sum_{i\in\mathbf{I}} b_{i,j}\left(s^t\right) = \sum_{i\in\mathbf{I}} B_{i,j}\left(s^t\right) \text{ for every } j\in\mathbf{I} \text{ and for every } t, s^t. \tag{15}$$

3 Equilibrium Properties

Standard existence results ensure that a SCE always exists.

Given the assumption that the commodities from both countries are perfect substitutes, the equilibrium exchange rates satisfy Purchasing Power Parity:

$$\xi_i\left(s^t\right) = \frac{p_1\left(s^t\right)}{p_i\left(s^t\right)} \quad \forall i\in\mathbf{I}.$$

The equilibrium asset prices $q_i\left(s^t\right) \leq 1 \ \forall i \in \mathbf{I}$.[5] If $q_i\left(s^t\right) < 1$, the cash-in-advance constraints (5) associated with country i currency will bind for all households. With binding cash-in-advance constraints (5), the market clearing condition for the money markets implies that the Quantity Theory of Money holds:

$$M_i\left(s^t\right) = p_i\left(s^t\right)\mathbf{y}_i\left(s_t\right). \tag{16}$$

The Friedman rule for country i in date-event s^t is such that $q_i\left(s^t\right) = 1$. Under the Friedman rule, money and the short-term bond are perfect substitutes. Market clearing for both implies that the sum of the two is pinned down for all households and the monetary authorities, but not the composition. The cash-in-advance constraints (5) need not bind under the Friedman rule. However, it is innocuous (i.e., without real effects) under the Friedman rule to set the household money holdings such that the cash-in-advance constraints (5) bind. This would allow the Quantity Theory of Money (16) to hold.

For simplicity, I define the new variables $\hat{b}_{i,j}\left(s^t\right)$ and $\hat{B}_{i,j}\left(s^t\right)$ such that

$$\begin{array}{ll} \hat{b}_{i,j}\left(s^t\right) = \mathbf{y}_i\left(s_t\right) + \frac{b_{i,j}\left(s^t\right)}{p_j\left(s^t\right)} \text{ for } i=j. & \hat{B}_{i,j}\left(s^t\right) = \mathbf{y}_i\left(s_t\right) + \frac{B_{i,j}\left(s^t\right)}{p_j\left(s^t\right)} \text{ for } i=j. \\ \hat{b}_{i,j}\left(s^t\right) = \frac{b_{i,j}\left(s^t\right)}{p_j\left(s^t\right)} \text{ for } i\neq j. & \hat{B}_{i,j}\left(s^t\right) = \frac{B_{i,j}\left(s^t\right)}{p_j\left(s^t\right)} \text{ for } i\neq j. \end{array}$$

The portfolios are denoted $\hat{b}_i\left(s^t\right) = \left(\hat{b}_{i,j}\left(s^t\right)\right)_{j\in\mathbf{I}}$ and $\hat{B}_i\left(s^t\right) = \left(\hat{B}_{i,j}\left(s^t\right)\right)_{j\in\mathbf{I}}$ for all $i\in\mathbf{I}$. If markets clear for the original variables, then markets clears for the new variables.

[5] If not, the market clearing condition on the bond market is not satisfied as households strictly prefer to save using money holdings and not bond holdings.

Additionally, define the inflation rate

$$\pi_i\left(s^t\right)=\frac{p_i\left(s^t\right)}{p_i\left(s^{t-1}\right)} \text{ for } i\in\mathbf{I}.$$

3.1 Monetary authorities

The policy choices are the sequences of asset prices $\{q_i(s^t)\}_{i\in\mathbf{I}}$ and the sequence of debt positions $\left\{\hat{B}_i\left(s^t\right)\right\}$. These choices are mutually dependent as they must be consistent with the bond market clearing conditions. The policy choices $q_i\left(s^t\right)$ and $\hat{B}_{i,i}\left(s^t\right)$ serve the same purpose, namely to target the short-term nominal interest rate. The policy choices $\left(\hat{B}_{i,j}\left(s^t\right)\right)_{j\neq i}$ serve to target the exchange rates.

The monetary authority constraints (10) under the change of variables, after using the Quantity Theory of Money (16), are given by:

$$\sum_{j\in\mathbf{I}}\frac{\hat{B}_{i,j}\left(s^{t-1}\right)}{\pi_j\left(s^t\right)}=\mathbf{y}_i\left(s_t\right)\left(1-q_i\left(s^t\right)\right)+\sum_{j\in\mathbf{I}}q_j\left(s^t\right)\hat{B}_{i,j}\left(s^t\right)\quad\forall i\in\mathbf{I}. \tag{17}$$

Given $\{q_i(s^t)\}_{i\in\mathbf{I}}$ and $\left\{\hat{B}_i\left(s^t\right)\right\}$, the inflation rates are uniquely determined from (17).[6]

In any date-event s^t, the discounted expected revenues that the country i monetary authority can acquire from seigniorage (in real terms) are equal to

$$\sum_{k=0}^{\infty}\beta^k E_t\left[\left(\frac{c_i\left(s^{t+k}\right)}{c_i\left(s^t\right)}\right)^{-\rho}\frac{s_i^M\left(s^{t+k}\right)}{p_i\left(s^{t+k}\right)}\right]. \tag{18}$$

and the debt obligations (in real terms) are equal to

$$\sum_{j\in\mathbf{I}}\frac{B_{i,j}\left(s^{t-1}\right)}{p_j\left(s^t\right)}. \tag{19}$$

Claim 1 *In equilibrium, debt obligations (19) equal the discounted expected seigniorage revenues (18).*

Proof. See Section A.1. ■

[6] Define the $N\times N$ debt matrix $\hat{B}\left(s^t\right)=\left[\hat{B}_1\left(s^t\right)\ \ \dots\ \ \hat{B}_N\left(s^t\right)\right]$, where all vectors, including $\hat{B}_i\left(s^t\right)$, are considered column vectors unless otherwise specified. The monetary authority constraints (17), for all countries $i\in\mathbf{I}$ in date-event s^t, are written in matrix notation as $\left(\frac{1}{\pi_i(s^t)}\right)_{i\in\mathbf{I}}^T\left(\hat{B}\left(s^{t-1}\right)\right)=\left(\mathbf{y}_i\left(s_t\right)\left(1-q_i\left(s^t\right)\right)\right)_{i\in\mathbf{I}}^T+\left(q_i\left(s^t\right)\right)_{i\in\mathbf{I}}^T\left(\hat{B}\left(s^t\right)\right)$. Provided that the policies are such that $\hat{B}\left(s^{t-1}\right)$ is a full rank matrix, then $\left(\frac{1}{\pi_i(s^t)}\right)_{i\in\mathbf{I}}^T=\left[\hat{B}\left(s^{t-1}\right)\right]^{-1}\left[\left(\mathbf{y}_i\left(s_t\right)\left(1-q_i\left(s^t\right)\right)\right)_{i\in\mathbf{I}}^T+\left(q_i\left(s^t\right)\right)_{i\in\mathbf{I}}^T\left(\hat{B}\left(s^t\right)\right)\right]$.

3.2 Household problem

Define the real wealth for a country i household entering date-event s^t as $\hat{\omega}_i(s^t) = \frac{\omega_i(s^t)}{p_i(s^t)}$. By the definition of the scaled variables, $\hat{\omega}_i(s^t) = \sum_{j \in \mathbf{I}} \frac{\hat{b}_{i,j}(s^{t-1})}{\pi_j(s^t)}$. The equilibrium household budget constraints (2) (with (4) and (6)) under the change of variables are given by:

$$c_i\left(s^t\right) - q_i\left(s^t\right) \mathbf{y}_i\left(s_t\right) + \sum_{j \in \mathbf{I}} q_j\left(s^t\right) \hat{b}_{i,j}\left(s^t\right) = \hat{\omega}_i(s^t). \tag{20}$$

The first order conditions with respect to the bond holdings are given by:

$$q_j\left(s^t\right) = \beta \sum_{\sigma \in \mathbf{S}} \Gamma\left(s_t, \sigma\right) \left(\frac{c_i(s^t, \sigma)}{c_i(s^t)}\right)^{-\rho} \frac{1}{\pi_j\left(s^t, \sigma\right)} \ \forall (i,j) \in \mathbf{I}^2. \tag{21}$$

3.3 Stationary policy rules

This paper considers several different types of targeting rules, each of which can be supported with a stationary policy rule.

Definition 2 *The policy rule* $\left\{q(s^t), \hat{B}\left(s^t\right)\right\}$ *is stationary if there exists stationary vectors* $\left(\mathbf{q}_i\left(s\right)\right)_{(i,s)\in \mathbf{I}\times\mathbf{S}} \in \mathbb{R}_+^{SN}$ *and* $\left(\hat{\mathbf{B}}\left(s\right)\right)_{s \in \mathbf{S}} \in \mathbb{R}^{SN^2}$ *such that* $q_i\left(s^t\right) = \mathbf{q}_i\left(s_t\right)$ *for all* i *and all date-events* s^t *and* $\hat{B}\left(s^t\right) = \hat{\mathbf{B}}\left(s_t\right)$ *for all date-events* s^t.

Given that preferences are identical and homothetic (Assumption 3), a Pareto efficient allocation is characterized by the fractions $(\theta_i)_{i \in \mathbf{I}} \in \Delta^{N-1}$ such that $c_i\left(s^t\right) = \theta_i \mathbf{Y}\left(s_t\right)$ for all $i \in \mathbf{I}$ and for all date-events.

Claim 2 *Under stationary policy rules, if the equilibrium allocation is Pareto efficient, the real wealth vectors* $\hat{\omega}_i\left(s^t\right)$ *are stationary, meaning that there exists* $\left(\hat{\boldsymbol{\omega}}_i\left(s\right)\right)_{(i,s)\in \mathbf{I}\times\mathbf{S}}$ *such that the real wealth* $\hat{\omega}_i\left(s^t\right) = \hat{\boldsymbol{\omega}}_i\left(s_t\right)$ *for all* i *and all date-events.*

Proof. See Section A.2. ■

3.4 Discounted present value constraints

A discounted present value budget constraint is used to determine the initial period price levels $\left(p_i\left(s_0\right)\right)_{i \in \mathbf{I}}$ and to ensure that the policy rules are consistent with the initial conditions $\left(s_0, \left(\omega\left(s_0\right)\right)_{i \in \mathbf{I}}, \left(W\left(s_0\right)\right)_{i \in \mathbf{I}}\right)$.

Under stationary policy rules and Pareto efficiency, the initial period discounted present value constraints for households are given by:

$$\frac{\omega_i(s_0)}{p_i(s_0)} = \sum_{k=0}^{\infty} \beta^k E_0 \left[\left(\frac{\mathbf{Y}(s_k)}{\mathbf{Y}(s_0)} \right)^{-\rho} \left(\theta_i \mathbf{Y}(s_k) - \mathbf{q}_i(s_k) \mathbf{y}_i(s_k) \right) \right].^{7} \tag{22}$$

Under stationary policy rules and Pareto efficiency, the initial period discounted present value constraints for the monetary authorities are given by:

$$\frac{W_i(s_0)}{p_i(s_0)} = \sum_{k=0}^{\infty} \beta^k E_0 \left[\left(\frac{\mathbf{Y}(s_k)}{\mathbf{Y}(s_0)} \right)^{-\rho} \mathbf{y}_i(s_k) \left(1 - \mathbf{q}_i(s_k)\right) \right]. \tag{23}$$

The initial state s_0 and the initial nominal wealth values $(\omega_i(s_0), W_i(s_0))_{i \in \mathbf{I}}$ are parameters of the model. For each country $i \in \mathbf{I}$, the stationary bond prices $(\mathbf{q}_i(s))_{s \in \mathbf{S}}$ uniquely determine the initial price level $p_i(s_0)$ using equation (23). For $N-1$ countries $i \in \mathbf{I} \backslash \{N\}$, with $p_i(s_0)$ and $(\mathbf{q}_i(s))_{s \in \mathbf{S}}$ known, equation (22) uniquely determines the consumption fraction θ_i.[8]

4 Nominal GDP Targeting

Suppose that all countries adopt nominal GDP targeting rules. In the cash-in-advance model, nominal GDP targeting requires a constant growth rate for the money supply. This means $\pi_i(s^t, \sigma) = \frac{1}{\mu_i} \frac{\mathbf{y}_i(s)}{\mathbf{y}_i(\sigma)}$ for all i and all date-events, where the target variables are $(\mu_i)_{i \in \mathbf{I}}$. A zero monetary growth policy for country i refers to $\mu_i = 1$. Under Pareto efficiency, the Euler equations (21) provide equations for the stationary bond prices:

$$\mathbf{q}_i(s) = \beta \mu_i \sum_{\sigma \in \mathbf{S}} \Gamma(s, \sigma) \left(\frac{\mathbf{Y}(\sigma)}{\mathbf{Y}(s)} \right)^{-\rho} \frac{\mathbf{y}_i(\sigma)}{\mathbf{y}_i(s)} \ \forall (i, s) \in \mathbf{I} \times \mathbf{S}. \tag{24}$$

This restriction on the inflation rate implies that the bond holdings are constant. Define the constant bond holdings $(\mathbf{b}_{i,j})_{(i,j) \in \mathbf{I}^2}$ and constant debt holdings $(\mathbf{B}_{i,j})_{(i,j) \in \mathbf{I}^2}$ such that $\frac{\hat{b}_{i,j}(s^t)}{\mathbf{y}_j(s_t)} = \mathbf{b}_{i,j}$ and $\frac{\hat{B}_{i,j}(s^t)}{\mathbf{y}_j(s_t)} = \mathbf{B}_{i,j}$ for all (i,j) and for all date-events.

[7] The derivation is contained in the proof of Claim 2.

[8] From equilibrium market clearing condition (13), there are only $N-1$ independent conditions in (22). The vector $(\theta_i)_{i \in \mathbf{I}}$ only has $N-1$ degrees of freedom as $\theta_N = 1 - \sum_{i \in \mathbf{I} \backslash \{N\}} \theta_i$.

From (20), the household budget constraints are given by:

$$\theta_i \mathbf{Y}(s) - \mathbf{q}_i(s)\mathbf{y}_i(s) + \sum_{j\in\mathbf{I}} \mathbf{q}_j(s)\,\mathbf{y}_j(s)\mathbf{b}_{i,j} = \sum_{j\in\mathbf{I}} \mu_j \mathbf{y}_j(s)\mathbf{b}_{i,j} \quad \forall (i,s) \in \mathbf{I}\times\mathbf{S}. \tag{25}$$

For bonds obtained in date-event s^t, the payout matrix for all date-events $(s^t,\sigma)_{\sigma\in\mathbf{S}}$ is given by:

$$\left[\Pi\left(s^t\right)^{-1}\right] = \begin{bmatrix} \frac{1}{\pi_1(s^t,1)} & .. & \frac{1}{\pi_N(s^t,1)} \\ : & .. & : \\ \frac{1}{\pi_1(s^t,S)} & .. & \frac{1}{\pi_N(s^t,S)} \end{bmatrix}.$$

The payout matrix $\left[\Pi\left(s^t\right)^{-1}\right]$ has the same rank and column space as the endowment matrix $\begin{bmatrix} \mathbf{y}_1(1) & .. & \mathbf{y}_N(1) \\ : & .. & : \\ \mathbf{y}_1(S) & .. & \mathbf{y}_N(S) \end{bmatrix}$. Under Assumption 2, $rank\left[\Pi\left(s^t\right)^{-1}\right] = N$.

4.1 Pareto inefficiency result

The set of sufficient conditions for (generic) Pareto efficiency requires that at least 2 countries adopt policies other than zero monetary growth policies.

Assumption 4 At least 2 countries adopt nonzero monetary growth policies.

Theorem 1 *Under Assumptions 1-4 and generically over the subset of household endowments* $(\mathbf{y}_i(s))_{(i,s)\in\mathbf{I}\times\mathbf{S}}$ *and initial period wealth* $(\omega_i(s_0))_{i\in\mathbf{I}}$, *the equilibrium allocation under nominal GDP targeting is not Pareto efficient.*

Proof. See Section A.3. ■

The theory in this subsection, however, cannot rule out the possibility that zero monetary growth policies ($\mu_i = 1$) for at least $N-1$ countries are consistent with Pareto efficiency and equilibrium. This justifies the inclusion of Assumption 4. The following subsection analyzes the role that Assumption 4 plays.

4.2 Necessary condition for Pareto efficiency under nominal GDP targeting

Define the bond returns as $R_i\left(s^t,\sigma\right)=\frac{1}{q_i(s^t)}\frac{p_i\left(s^t,\sigma\right)}{p_i(s^t)}$ for all $i\in\mathbf{I}$ and all date-events. Define the discounted present value of country i endowments (under Pareto efficiency) as

$$\Psi_i\left(s^t\right)=\sum_{k=0}^{\infty}\beta^k E_t\left[\left(\frac{\mathbf{Y}\left(s_{t+k}\right)}{\mathbf{Y}\left(s_t\right)}\right)^{-\rho}\mathbf{y}_i\left(s^{t+k}\right)\right].$$

By definition, $\Psi_i\left(s^t\right)$ is stationary, meaning there exists $\left(\boldsymbol{\Psi}_i\left(s\right)\right)_{s\in\mathbf{S}}$ such that $\Psi_i\left(s^t\right)=\boldsymbol{\Psi}_i\left(s_t\right)$ for all date-events.

Define the policies such that the bond returns for both countries $i\in\mathbf{I}$ satisfy:

$$R_i\left(s^t,\sigma\right)=\frac{\boldsymbol{\Psi}_i\left(\sigma\right)}{\boldsymbol{\Psi}_i\left(s_t\right)-\mathbf{y}_i\left(s_t\right)}. \tag{26}$$

The purpose of (26), as will be shown, is that such bond returns offer the spanning condition required for the household budget constraints and Euler equations to be satisfied. Policy choice satisfying (26) suffices for Pareto efficiency in a 2-period model and in a model with constant wealth distribution (as in Sheedy, 2014). Ultimately, it can be shown in the present model that debt positions consistent with both market clearing and the bond returns in (26) only exist over a closed and measure zero subset of economies.

By definition, $\boldsymbol{\Psi}_i\left(s_t\right)-\mathbf{y}_i\left(s_t\right)=\beta E_t\left[\left(\frac{\mathbf{Y}(\sigma)}{\mathbf{Y}(s_t)}\right)^{-\rho}\boldsymbol{\Psi}_i\left(\sigma\right)\right]$. By construction, the household Euler equations are satisfied as:

$$\beta E_t\left[\left(\frac{\mathbf{Y}(\sigma)}{\mathbf{Y}(s_t)}\right)^{-\rho}R_i\left(s^t,\sigma\right)\right]=\frac{\beta E_t\left[\left(\frac{\mathbf{Y}(\sigma)}{\mathbf{Y}(s_t)}\right)^{-\rho}\boldsymbol{\Psi}_i\left(\sigma\right)\right]}{\beta E_t\left[\left(\frac{\mathbf{Y}(\sigma)}{\mathbf{Y}(s_t)}\right)^{-\rho}\boldsymbol{\Psi}_i\left(\sigma\right)\right]}=1.$$

For the remainder of the argument, I will adopt a new change of variables. The household bond holdings $\tilde{b}_{i,j}\left(s^t\right)=\frac{q_j\left(s^t\right)b_{i,j}\left(s^t\right)}{p_j(s^t)}$ and the monetary authority debt positions satisfy $\tilde{B}_{i,j}\left(s^t\right)=\frac{q_j\left(s^t\right)B_{i,j}\left(s^t\right)}{p_j(s^t)}$ (for all (i,j) and in all date-events). Under Pareto efficiency ($c_i\left(s^t\right)=\theta_i\mathbf{Y}\left(s_t\right)$) and nominal GDP targeting ($\frac{\mathbf{y}_i(s_{t-1})}{\pi_i(s^t)}=\mu_i\mathbf{y}_i\left(s_t\right)$), the household budget constraints (2) (with (4) and (6)) are given by (for $i\in\mathbf{I}$):

$$\theta_i\mathbf{Y}\left(s_t\right)+\sum_{j\in\mathbf{I}}\tilde{b}_{i,j}\left(s^t\right)=\mu_i\mathbf{y}_i\left(s_t\right)+\sum_{j\in\mathbf{I}}R_j\left(s^t\right)\tilde{b}_{i,j}\left(s^{t-1}\right). \tag{27}$$

The monetary authority constraints (10) in date-event s^t under the change of variables are given by (for $i \in \mathbf{I}$):

$$\mu_i \mathbf{y}_i(s_t) + \sum_{j\in\mathbf{I}} R_j(s^t)\tilde{B}_{i,j}(s^{t-1}) = \mathbf{y}_i(s_t) + \sum_{j\in\mathbf{I}} \tilde{B}_{i,j}(s^t). \tag{28}$$

The bond holdings that satisfy (27) must be of the form:

$$\begin{array}{rl} \tilde{b}_{i,j}(s^t) = (\theta_i - \mu_i)(\mathbf{\Psi}_j(s_t) - \mathbf{y}_j(s_t)) & \text{if } j = i \\ \tilde{b}_{i,j}(s^t) = \theta_i(\mathbf{\Psi}_j(s_t) - \mathbf{y}_j(s_t)) & \text{if } j \neq i \end{array}.$$

From definition (26):

$$\begin{array}{rl} \tilde{b}_{i,j}(s^{t-1}) = (\theta_i - \mu_i)\left(\frac{\mathbf{\Psi}_j(s_t)}{R_j(s^t)}\right) & \text{if } j = i \\ \tilde{b}_{i,j}(s^{t-1}) = \theta_i\left(\frac{\mathbf{\Psi}_j(s_t)}{R_j(s^t)}\right) & \text{if } j \neq i \end{array}.$$

From market clearing, the monetary authority debt positions must take the form (for $i \in \mathbf{I}$ and $j \in \mathbf{I}$):

$$\tilde{B}_{i,j}(s^t) = \kappa_{i,j}(\mathbf{\Psi}_j(s_t) - \mathbf{y}_j(s_t)) \quad \tilde{B}_{i,j}(s^{t-1}) = \kappa_{i,j}\left(\frac{\mathbf{\Psi}_j(s_t)}{R_j(s^t)}\right)$$

for constants $(\kappa_{i,j})_{(i,j)\in\mathbf{I}^2}$ satisfying:

$$\sum_{i\in\mathbf{I}} \kappa_{i,j} = 1 - \mu_j \quad \forall j \in \mathbf{I}.$$

The monetary authority constraints (28) are satisfied iff

$$\begin{aligned} \mathbf{y}_1(s_t)(1-\mu_1) &= \sum_{j\in\mathbf{I}} \kappa_{1,j}\mathbf{y}_j(s_t) \quad \forall s_t \in \mathbf{S}. \\ &\vdots \\ \mathbf{y}_N(s_t)(1-\mu_N) &= \sum_{j\in\mathbf{I}} \kappa_{N,j}\mathbf{y}_j(s_t) \quad \forall s_t \in \mathbf{S}. \end{aligned} \tag{29}$$

Under Assumption 2, equations (29) are satisfied iff $(\kappa_{i,j})_{(i,j)\in\mathbf{I}^2} = \overrightarrow{0}$ and $(\mu_i)_{i\in\mathbf{I}} = \overrightarrow{1}$. The only equilibrium policies consistent with (26) are zero monetary growth policies.

Using the Euler equation (24) expression for the bond prices, the definition of returns implies that:

$$R_i(s^t, \sigma) = \frac{\mathbf{y}_i(\sigma)}{\beta E_t\left[\left(\frac{\mathbf{Y}(\sigma)}{\mathbf{Y}(s_t)}\right)^{-\rho}\mathbf{y}_i(\sigma)\right]}.$$

The returns required for optimal spanning, as introduced in (26), imply that the following equations must hold ($\forall\left(s_t,\sigma\right)\in\mathbf{S}^2$):

$$\frac{\mathbf{y}_i\left(\sigma\right)}{E_t\left[\left(\frac{\mathbf{Y}(\sigma)}{\mathbf{Y}(s_t)}\right)^{-\rho}\mathbf{y}_i\left(\sigma\right)\right]}=\frac{\mathbf{\Psi}_i\left(\sigma\right)}{E_t\left[\left(\frac{\mathbf{Y}(\sigma)}{\mathbf{Y}(s_t)}\right)^{-\rho}\mathbf{\Psi}_i\left(\sigma\right)\right]}.^{9} \tag{30}$$

Equation (30) is equivalent to:

$$\frac{\left(\mathbf{Y}(\sigma)\right)^{-\rho}\mathbf{y}_i\left(\sigma\right)}{\sum\limits_{\sigma\in\mathbf{S}}\Gamma\left(s_t,\sigma\right)\left(\mathbf{Y}(\sigma)\right)^{-\rho}\mathbf{y}_i\left(\sigma\right)}=\frac{\left(\mathbf{Y}(\sigma)\right)^{-\rho}\mathbf{\Psi}_i\left(\sigma\right)}{\sum\limits_{\sigma\in\mathbf{S}}\Gamma\left(s_t,\sigma\right)\left(\mathbf{Y}(\sigma)\right)^{-\rho}\mathbf{\Psi}_i\left(\sigma\right)}. \tag{31}$$

Even under zero monetary growth policies, Pareto efficiency requires a strong joint condition on the endowment processes and the Markov transition matrix. If equation (31) is violated for any $(i,s,\sigma)\in\mathbf{I}\times\mathbf{S}^2$, then it is not possible to implement policies as proposed in (26). Over a generic subset of household endowments, equation (31) is violated. This supports the (generic) Pareto inefficiency result even when all (or most) countries adopt zero monetary growth policies.

4.3 Comparison with inflation rate targeting

Suppose that all countries adopt inflation rate targeting rules. This implies that $\pi_i\left(s^t,\sigma\right)=\frac{1}{\phi_i}$ for all i and all date-events, where the target variables are $\left(\phi_i\right)_{i\in\mathbf{I}}$.

Under inflation rate targeting, $rank\left[\Pi\left(s^t\right)^{-1}\right]=1$ as

$$\left[\Pi\left(s^t\right)^{-1}\right]=\begin{bmatrix}\frac{1}{\pi_1(s^t,1)} & .. & \frac{1}{\pi_N(s^t,1)}\\ \vdots & .. & \vdots\\ \frac{1}{\pi_1(s^t,S)} & .. & \frac{1}{\pi_N(s^t,S)}\end{bmatrix}=\begin{bmatrix}\phi_1 & .. & \phi_N\\ \vdots & .. & \vdots\\ \phi_1 & .. & \phi_N\end{bmatrix}.$$

Define $\left\langle\Pi\left(s^t\right)^{-1}\right\rangle$ as the asset span

$$\left\langle\Pi\left(s^t\right)^{-1}\right\rangle=\left\{z\in\mathbb{R}^S:\exists b\in\mathbb{R}^J\text{ such that }z=\left[\Pi\left(s^t\right)^{-1}\right]b\right\}.$$

The asset span under inflation rate targeting is a $1-$dimensional linear subspace of $\mathbb{R}^S$.

In an economy without aggregate risk, the asset span under inflation rate targeting is a strict subset of the asset span under nominal GDP targeting. The asset span under nominal

[9] This condition is trivially satisfied in the final time period of a model, notably in the 2nd period of a 2-period model. In an infinite time horizon model, there is no final time period and the condition is only satisfied for atypical parameter values.

GDP targeting is a $J-$dimensional linear subspace of $\mathbb{R}^S$. This implies that nominal GDP targeting policies Pareto dominate inflation rate targeting policies as all households are better off under the former.[10] A similar conclusion holds (generically) for economies with aggregate risk, but the argument lies outside the scope of the paper.

A related policy is exchange rate stabilization. Exchange rate stabilization refers to policies such that $\xi_i(s^t, \sigma) = \xi_i(s^t)$ for all countries $i > 1$ and all realizations $\sigma \in \mathbf{S}$. The equation $\xi_i(s^t, \sigma) = \xi_i(s^t)$ implies $\pi_i(s^t, \sigma) = \pi_1(s^t, \sigma)$. Therefore, $rank \left[\Pi(s^t)^{-1}\right] = 1$, which is the same prediction as in the case of inflation rate targeting.

Suppose that country 1 adopts nominal GDP targeting and all other countries adopt exchange rate stabilization policies such that $\pi_i(s^t, \sigma) = \pi_1(s^t, \sigma)$ for all $i > 1$ and all date-events. The asset span under exchange rate stabilization is a strict subset of the asset span under nominal GDP targeting, implying that nominal GDP targeting policies Pareto dominate exchange rate stabilization policies.

5 Necesssary conditions for Pareto efficiency under stationary policies

5.1 Pareto efficient allocation

Select any country $i \in \mathbf{I}$. From the discounted present value constraint for the monetary authority (23):

$$\frac{W_i(s_0)}{p_i(s_0)} = \left(I_S - \hat{\Gamma}\right)^{-1}_{(s_0)} \left(\mathbf{y}_i(s)\left(1 - \mathbf{q}_i(s)\right)\right)_{s \in \mathbf{S}}, \tag{32}$$

where I_S is the $S \times S$ identity matrix, $\hat{\Gamma}$ is the $S \times S$ stochastic discount factor matrix with elements $\hat{\Gamma}(s, \sigma) = \beta\Gamma(s, \sigma)\left(\frac{\mathbf{Y}(\sigma)}{\mathbf{Y}(s)}\right)^{-\rho}$, and $\left(I_S - \hat{\Gamma}\right)^{-1}_{(s_0)}$ is row s_0 of the matrix $\left(I_S - \hat{\Pi}\right)^{-1}$. From the discounted present value constraint for households (22):

$$\frac{\omega_i(s_0)}{p_i(s_0)} = \left(I_S - \hat{\Gamma}\right)^{-1}_{(s_0)} \left(\theta_i \mathbf{Y}(s) - \mathbf{q}_i(s)\mathbf{y}_i(s)\right)_{s \in \mathbf{S}}. \tag{33}$$

[10] Under nominal GDP targeting, the equilibrium allocation is constrained Pareto efficient since there is only one commodity traded in each date-event. An analysis of multiple commodities per date-event and relative price effects goes beyond the scope of this paper.

Combining equations (32) and (33) allows me to solve for the consumption fraction θ_i in terms of the parameters $\omega_i(s_0)$ and $W_i(s_0)$ and the bond prices

$$\theta_i = \frac{\frac{\omega_i(s_0)}{W_i(s_0)}\left(I_S - \hat{\Gamma}\right)^{-1}_{(s_0)} (\mathbf{y}_i(s))_{s\in\mathbf{S}} + \left(1 - \frac{\omega_i(s_0)}{W_i(s_0)}\right)\left(I_S - \hat{\Gamma}\right)^{-1}_{(s_0)} (\mathbf{q}_i(s)\mathbf{y}_i(s))_{s\in\mathbf{S}}}{\left(I_S - \hat{\Gamma}\right)^{-1}_{(s_0)} (\mathbf{Y}(s))_{s\in\mathbf{S}}}. \quad (34)$$

Recall that the equilibrium definition included a general market clearing constraint (condition b) stating that the total initial obligations of monetary authorities must equal the total initial household wealth:

$$\sum_{i\in\mathbf{I}} \xi_i(s_0)\,\omega_i(s_0) = \sum_{i\in\mathbf{I}} \xi_i(s_0)\,W_i(s_0).$$

A more restrictive condition is that $\omega_i(s_0) = W_i(s_0)$ for all i, meaning that a monetary authority's initial nominal obligations are only owed to the domestic household. If $\omega_i(s_0) = W_i(s_0)$ for all i, the Pareto efficient allocation is independent of monetary policy:

$$\theta_i = \frac{\left(I_S - \hat{\Gamma}\right)^{-1}_{(s_0)} (\mathbf{y}_i(s))_{s\in\mathbf{S}}}{\left(I_S - \hat{\Gamma}\right)^{-1}_{(s_0)} (\mathbf{Y}(s))_{s\in\mathbf{S}}} \text{ for all } i \in \mathbf{I}.$$

5.2 Policies consistent with Pareto efficiency

Define $\tilde{N}$ as the number of countries that adopt stationary policies different from nominal GDP targeting. The stationary policies will be such that the inflation rates are given by $\pi_i(s^t,\sigma) = \frac{1}{\mu_i(\sigma)}\mathbf{y}_i(s)$ for all i and all date-events, where the stochastic targets are $(\mu_i(s))_{s\in\mathbf{S}}$. Define N^* as the number of countries that adopt nominal GDP targeting with nonzero monetary growth rate, meaning $\pi_i(s^t,\sigma) = \frac{1}{\mu_i}\frac{\mathbf{y}_i(s)}{\mathbf{y}_i(\sigma)}$ for all i and all date-events and $\mu_i \neq 1$. The remaining $N - \tilde{N} - N^*$ countries adopt nominal GDP targeting with zero monetary growth rate.

Without loss of generality, let countries $i = 1, ..., \tilde{N}$ be the ones to adopt policies different from nominal GDP targeting. The payout matrix $\left[\Pi(s^t)^{-1}\right]$ has the same rank and column space as the matrix

$$\begin{bmatrix} \mu_1(1) & .. & \mu_{\tilde{N}}(1) & \mathbf{y}_{\tilde{N}+1}(1) & .. & \mathbf{y}_N(1) \\ : & .. & : & : & .. & : \\ \mu_1(S) & .. & \mu_{\tilde{N}}(S) & \mathbf{y}_{\tilde{N}+1}(S) & .. & \mathbf{y}_N(S) \end{bmatrix}.$$

Define $\tilde{\mathbf{I}} = \left\{1, ..., \tilde{N}\right\}$. The policy choices for countries $i \in \tilde{\mathbf{I}}$ are $(\mu_i(s))_{(i,s)\in\tilde{\mathbf{I}}\times\mathbf{S}}$. Risk sharing is enhanced when $rank\left[\Pi(s^t)^{-1}\right] = N$. The following result will find conditions under which there exist policies $\left((\mu_i(s))_{(i,s)\in\tilde{\mathbf{I}}\times\mathbf{S}}, (\mu_i)_{i\in\mathbf{I}\backslash\tilde{\mathbf{I}}},\right)$ compatible with Pareto efficiency. The conditions are derived for policies $(\mu_i(s))_{(i,s)\in\tilde{\mathbf{I}}\times\mathbf{S}}$ such that $rank\left[\Pi(s^t)^{-1}\right] = N$.

Claim 3 *If $S > N + \frac{\tilde{N}(N-1)+N}{N+N^*-\tilde{N}-1}$, then generically over the subset of household endowments $(\mathbf{y}_i(s))_{(i,s)\in\mathbf{I}\times\mathbf{S}}$ and initial period wealth $(\omega_i(s_0))_{i\in\mathbf{I}}$, the equilibrium allocation is not Pareto efficient.*[11]

If $S \leq N + \frac{\tilde{N}(N-1)+N}{N+N^*-\tilde{N}-1}$, the number of equilibrium and Pareto efficiency equations is no greater than the number of equilibrium and policy variables. Generically over the subset of household endowments $(\mathbf{y}_i(s))_{(i,s)\in\mathbf{I}\times\mathbf{S}}$ and initial period wealth $(\omega_i(s_0))_{i\in\mathbf{I}}$, the equations are linearly independent. This means that a solution (and possibly multiple solutions) exists. If $S > N + \frac{\tilde{N}(N-1)+N}{N+N^*-\tilde{N}-1}$, a solution does not exist.

The proof of Theorem 1 provides all of the equations for the case with $\tilde{N} = 0$ (all countries adopt nominal GDP targeting).[12] The exact same argument applies in the general case, with the only difference being that the set of policy variables has now expanded to the dimension of the vector $\left((\mu_i(s))_{(i,s)\in\tilde{\mathbf{I}}\times\mathbf{S}}, (\mu_i)_{i\in\mathbf{I}\backslash\tilde{\mathbf{I}}},\right)$.

Suppose, for illustration, that no countries adopt zero monetary growth policies ($N^* = N - \tilde{N}$) and $S = 2N$ (a simple form of incomplete markets). Then Pareto efficiency requires that a minimum fraction of countries adopt policies other than nominal GDP targeting:

$$\frac{\tilde{N}}{N} \geq \frac{2N-2}{3N-1}.$$

The condition $S \leq N + \frac{\tilde{N}(N-1)+N}{N+N^*-\tilde{N}-1}$ is necessary, but not sufficient, for Pareto efficiency since equilibrium requires that the following inequalities are additonally satisfied:

1. Monetary authority net domestic positions are nonnegative:

$$\mathbf{B}_{i,i} \geq 0 \text{ for all } i \in \mathbf{I}. \tag{35}$$

[11] If $N + N^* - \tilde{N} \leq 1$, Pareto efficiency can be supported.

[12] For the $\tilde{N} = N$ case, Pareto inefficiency holds whenever $S > N + \frac{N}{N+N^*-1}$. Under Assumption 4 ($N^* > 1$), $\frac{N}{N+N^*-1} < 1$. Since incomplete markets implies $S \geq N+1$, then $S > N + \frac{N}{N+N^*-1}$ always holds and the allocation is Pareto inefficient.

2. The return on money cannot dominate the return on 1-period nominal bonds:

$$\mathbf{q}_i(s) \leq 1 \text{ for all } (i, s) \in \mathbf{I} \times \mathbf{S}. \tag{36}$$

3. Strictly positive consumption:

$$\theta_i \in (0, 1) \text{ for all } i \in \mathbf{I}. \tag{37}$$

6 Concluding Remarks

In a stochastic monetary model of a large open economy with incomplete markets and heterogeneous households, targeting rules are not Pareto efficient as additional policy tools must be provided to the central banks to generate the stochastic bond returns required for households to smooth consumption. The model in the present paper considered a pure exchange economy with cash-in-advance constraints. The nature of the monetary model is irrelevant to the broader claim. To carry out the complete analysis of the equilibrium equations, a choice needed to be made about how money is valued in the economy. The only notable implication unique to the cash-in-advance setting is that the Pareto efficient allocation varies with the short-term interest rates. This is due to the fact that the cash-in-advance constraints introduce a wedge between current consumption and income, whose size varies in proportion to the domestic interest rate.

The pure exchange economy is the purest setting to analyze the ability of bond markets to allocate risk efficiently (according to the Pareto criterion). Introducing an additional margin to the model, whether it is a production economy with a margin for household labor supply or a growth model with a margin for household savings (provided that the addition of the capital markets does not allow for perfect risk sharing by the households independent of policy), does not change the qualitative findings of this paper. Nominal GDP targeting remains incompatible with Pareto efficiency in these extensions of the model and additional policy tools in the form of more general monetary policies are required to support the asset spanning required by households for consumption smoothing.

The present paper already contains a margin due to the cash-in-advance distortion. If policy is concerned with the cash-in-advance friction (the marginal value of currency exceeds the marginal cost of printing it), then the Friedman rule should be implemented. The Friedman rule, a special case of interest rate targeting and closely related to inflation rate targeting, leads to a rank deficient payout matrix and is Pareto dominated by nominal GDP targeting policies. If policy is concerned with welfare (defined according to the Pareto criterion), this last argument carries the day.

The present model can be used to characterize the nature of monetary policy in a strategic seting of monetary policy competition. By understanding the incentives of monetary authorities, which may differ from the Pareto criterion and instead focus on domestic utility maximization, the model can evaluate the real effects of monetary policy and whether the monetary policy rules are simple enough to be implemented in reality. These avenues are left for future research to explore.

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A Appendix

A.1 Proof of Claim 1

Consider the constraint for the country i monetary authority in date-event s^t :

$$\frac{s_i^M\left(s^t\right)}{p_i\left(s^t\right)}+\sum_{j\in\mathbf{I}}q_j(s^t)\frac{B_{i,j}(s^t)}{p_j\left(s^t\right)}=\sum_{j\in\mathbf{I}}\frac{B_{i,j}(s^{t-1})}{p_j\left(s^t\right)}. \tag{38}$$

The exact same constraint can be written for date-event $\left(s^t,\sigma\right)_{\sigma\in\mathbf{S}}$:

$$\frac{s_i^M\left(s^t,\sigma\right)}{p_i\left(s^t,\sigma\right)}+\sum_{j\in\mathbf{I}}q_j(s^t,\sigma)\frac{B_{i,j}(s^t,\sigma)}{p_j\left(s^t,\sigma\right)}=\sum_{j\in\mathbf{I}}\frac{B_{i,j}(s^t)}{p_j\left(s^t,\sigma\right)}. \tag{39}$$

Multiply both sides of (39) by $\beta\left(\frac{c_i\left(s^t,\sigma\right)}{c_i\left(s^t\right)}\right)^{-\rho}$ and take the conditional expectation:

$$\begin{aligned}&E_t\left[\beta\left(\frac{c_i\left(s^t,\sigma\right)}{c_i\left(s^t\right)}\right)^{-\rho}\left(\frac{s_i^M\left(s^t,\sigma\right)}{p_i\left(s^t,\sigma\right)}+\sum_{j\in\mathbf{I}}q_j(s^t,\sigma)\frac{B_{i,j}(s^t,\sigma)}{p_j\left(s^t,\sigma\right)}\right)\right] \\ =&\sum_{j\in\mathbf{I}}B_{i,j}(s^t)\left\{\beta\sum_{\sigma}\Gamma\left(s_t,\sigma\right)\left(\frac{c_i\left(s^t,\sigma\right)}{c_i\left(s^t\right)}\right)^{-\rho}\frac{1}{p_j\left(s^t,\sigma\right)}\right\}.\end{aligned} \tag{40}$$

The Euler equations (21) imply that:

$$\beta\sum_{\sigma}\Gamma\left(s_t,\sigma\right)\left(\frac{c_i\left(s^t,\sigma\right)}{c_i\left(s^t\right)}\right)^{-\rho}\frac{1}{p_j\left(s^t,\sigma\right)}=\frac{q_j(s^t)}{p_j\left(s^t\right)}\quad\forall j\in\mathbf{I}.$$

This means that (40) is given by:

$$E_t\left[\beta\left(\frac{c_i\left(s^t,\sigma\right)}{c_i\left(s^t\right)}\right)^{-\rho}\left(\frac{s_i^M\left(s^t,\sigma\right)}{p_i\left(s^t,\sigma\right)}+\sum_{j\in\mathbf{I}}q_j(s^t,\sigma)\frac{B_{i,j}(s^t,\sigma)}{p_j\left(s^t,\sigma\right)}\right)\right]=\sum_{j\in\mathbf{I}}q_j(s^t)\frac{B_{i,j}(s^t)}{p_j\left(s^t\right)}. \tag{41}$$

Inserting this new expression (41) back into the date-event s^t budget constraint (38) and iterating forward yields:

$$\sum_{j\in\mathbf{I}}\frac{B_{i,j}(s^{t-1})}{p_j\left(s^t\right)}=\sum_{k=0}^{\infty}\beta^kE_t\left[\left(\frac{c_i\left(s^{t+k}\right)}{c_i\left(s^t\right)}\right)^{-\rho}\frac{s_i^M\left(s^{t+k}\right)}{p_i\left(s^{t+k}\right)}\right],$$

after citing the transversality condition.

A.2 Proof of Claim 2

Consider the budget constraint for country $i \in \mathbf{I}$ households in date-event s^t :

$$c_i\left(s^t\right) - \mathbf{q}_i(s_t)\mathbf{y}_i\left(s_t\right) + \sum_{j\in\mathbf{I}} \mathbf{q}_j(s_t)\hat{b}_{i,j}(s^t) = \sum_{j\in\mathbf{I}} \frac{\hat{b}_{i,j}(s^{t-1})}{\pi_j\left(s^t\right)}. \tag{42}$$

The exact same constraint can be written for date-event $\left(s^t,\sigma\right)_{\sigma\in\mathbf{S}}$:

$$c_i\left(s^t,\sigma\right) - \mathbf{q}_i\left(\sigma\right)\mathbf{y}_i\left(\sigma\right) + \sum_{j\in\mathbf{I}} \mathbf{q}_j(s_t,\sigma)\hat{b}_{i,j}(s^t,\sigma) = \sum_{j\in\mathbf{I}} \frac{\hat{b}_{i,j}(s^t)}{\pi_j\left(s^t,\sigma\right)}. \tag{43}$$

Multiply both sides of (43) by $\beta\left(\frac{c_i\left(s^t,\sigma\right)}{c_i\left(s^t\right)}\right)^{-\rho}$ and take the conditional expectation:

$$\begin{aligned} & E_t\left[\beta\left(\frac{c_i\left(s^t,\sigma\right)}{c_i\left(s^t\right)}\right)^{-\rho}\left(c_i\left(s^t,\sigma\right) - \mathbf{q}_i\left(\sigma\right)\mathbf{y}_i\left(\sigma\right) + \sum_{j\in\mathbf{I}}\mathbf{q}_j(s_t,\sigma)\hat{b}_{i,j}(s^t,\sigma)\right)\right] \\ = & \sum_{j\in\mathbf{I}}\hat{b}_{i,j}(s^t)\left\{\beta\sum_{\sigma}\Gamma\left(s_t,\sigma\right)\left(\frac{c_i\left(s^t,\sigma\right)}{c_i\left(s^t\right)}\right)^{-\rho}\frac{1}{\pi_j\left(s^t,\sigma\right)}\right\}. \end{aligned} \tag{44}$$

The Euler equations (21) imply that:

$$\beta\sum_{\sigma}\Gamma\left(s_t,\sigma\right)\left(\frac{c_i\left(s^t,\sigma\right)}{c_i\left(s^t\right)}\right)^{-\rho}\frac{1}{\pi_j\left(s^t,\sigma\right)} = \mathbf{q}_j(s_t) \quad \forall j\in\mathbf{I}.$$

This means that (44) is given by:

$$E_t\left[\beta\left(\frac{c_i\left(s^t,\sigma\right)}{c_i\left(s^t\right)}\right)^{-\rho}\left(c_i\left(s^t,\sigma\right) - \mathbf{q}_i\left(\sigma\right)\mathbf{y}_i\left(\sigma\right) + \sum_{j\in\mathbf{I}}\mathbf{q}_j\left(\sigma\right)\hat{b}_{i,j}(s^t,\sigma)\right)\right] = \sum_{j\in\mathbf{I}}\mathbf{q}_j(s_t)\hat{b}_{i,j}(s^t). \tag{45}$$

Inserting this new expression (45) back into the date-event s^t budget constraint (42) and iterating forward yields:

$$\sum_{j\in\mathbf{I}}\frac{\hat{b}_{i,j}(s^{t-1})}{\pi_j\left(s^t\right)} = \sum_{k=0}^{\infty}\beta^k E_t\left[\left(\frac{c_i\left(s^{t+k}\right)}{c_i\left(s^t\right)}\right)^{-\rho}\left(c_i\left(s^{t+k}\right) - \mathbf{q}_i(s_{t+k})\mathbf{y}_i\left(s_{t+k}\right)\right)\right], \tag{46}$$

after citing the transversality condition. By definition, $\hat{\omega}_i(s^t) = \sum_{j\in\mathbf{I}}\frac{\hat{b}_{i,j}(s^{t-1})}{\pi_j(s^t)}$. The equilibrium equation (46) must hold in all date-events s^t.

If the equilibrium allocation is Pareto efficient, there exists θ_i such that $c_i\left(s^{t+k}\right) = \theta_i \mathbf{Y}\left(s_{t+k}\right)$ for all date-events. Equation (46) is updated as:

$$\hat{\omega}_i(s^t) = \sum_{k=0}^{\infty} \beta^k E_t \left[\left(\frac{\mathbf{Y}\left(s_{t+k}\right)}{\mathbf{Y}\left(s_t\right)} \right)^{-\rho} \left(\theta_i \mathbf{Y}\left(s_{t+k}\right) - \mathbf{q}_i(s_{t+k}) \mathbf{y}_i\left(s_{t+k}\right)\right) \right]. \tag{47}$$

The right-hand side of (47), by definition, only depends upon the current state s_t. It does not depend upon any other realizations from the history s^{t-1}.

A.3 Proof of Theorem 1

It is straightforward for any country $i \in \mathbf{I}$ to adopt a nominal GDP target $\mu_i = 1$. This is accomplished by setting $\hat{B}_{i,i}\left(s^t\right) = \mathbf{y}_i\left(s_t\right)$ and $\left(\hat{B}_{i,j}\left(s^t\right)\right)_{j \neq i} = 0$ (equivalently, the constant debt holdings are $\mathbf{B}_{i,i} = 1$ and $\mathbf{B}_{i,j} = 0$ for $j \neq i$). This trivially implies that $\pi_i\left(s^t, \sigma\right) = \frac{\mathbf{y}_i(s)}{\mathbf{y}_i(\sigma)}$ from (17) for all date-events.

Define $\mathbf{I}^* = \{1, ..., N^*\}$ as the set of countries that adopt a nominal GDP target $\mu_i \neq 1$. The monetary authority constraints (17) for countries $i \in \mathbf{I}^*$ are given by:

$$\mathbf{y}_i(s)\left(1 - \mathbf{q}_i(s)\right) + \sum_{j \in \mathbf{I}} \mathbf{q}_j\left(s\right) \mathbf{y}_j(s) \mathbf{B}_{i,j} = \sum_{j \in \mathbf{I}} \mu_j \mathbf{y}_j(s) \mathbf{B}_{i,j} \quad \forall\left(i, s\right) \in \mathbf{I} \times \mathbf{S}. \tag{48}$$

For country $i \in \mathbf{I} \backslash \mathbf{I}^*$, the monetary authority constraints (17) reduce to:

$$\begin{aligned} \mathbf{B}_{i,i} &= 1. \\ \mathbf{B}_{i,j} &= 0 \text{ for } j \neq i. \end{aligned} \tag{49}$$

Walras' Law implies that the budget constraint for the household in country N is redundant given the constraints for all other households and all monetary authorities.

The total number of equations is $2N - 1 + S\left(2N - 1\right) + SN^* + N(N - N^*)$ and consists of initial period discounted present value equations (22) and (23), Euler equations (24), and budget constraints (25), (48), and (49).

The equilibrium variables are:

initial prices	$(p_i(s_0))_{i\in\mathbf{I}}$	N variables
consumption fractions	$(\theta_i)_{i\in\mathbf{I}}$	$N-1$ variables
asset prices	$(\mathbf{q}_i(s))_{(i,s)\in\mathbf{I}\times\mathbf{S}}$	SN variables
nominal GDP targets	$(\mu_i)_{i\in\mathbf{I}}$	N variables
bond holdings	$(\mathbf{b}_{i,j})_{(i,j)\in\mathbf{I}\setminus\{N\}\times\mathbf{I}}$	$N(N-1)$ variables
debt positions	$(\mathbf{B}_{i,j})_{(i,j)\in\mathbf{I}\times\mathbf{I}}$	N^2 variables

.

Table 4: Equilibrium variables for nominal GDP targeting

Lemma 1 *Under Assumption 4, if $S \leq N + \frac{N}{N+N^*-1}$, then generically over the subset of household endowments $(\mathbf{y}_i(s))_{(i,s)\in\mathbf{I}\times\mathbf{S}}$ and initial period wealth $(\omega_i(s_0))_{i\in\mathbf{I}}$, the $2N-1+S(2N-1)+SN^*+N(N-N^*)$ equations (22), (23), (24), (25), (48), and (49) are linearly independent.*

Proof. See Section A.3.1. ■

Lemma 1 provides a (generically) necessary condition for Pareto efficiency, namely $S \leq N + \frac{N}{N+N^*-1}$. Under Assumptions 1 and 6, $N^* > 1$ and $N < S$. Taken together, the inequality $S \leq N + \frac{N}{N+N^*-1}$ is violated.

A.3.1 Proof of Lemma 1

The variables $\xi \in \mathbb{R}^{2N-1+S(N+1)+N(2N-1)}$ are

$$\xi = \left((p_i(s_0))_{i\in\mathbf{I}}, (\theta_i)_{i\in\mathbf{I}}, (\mathbf{q}_i(s))_{(i,s)\in\mathbf{I}\times\mathbf{S}}, (\mu_i)_{i\in\mathbf{I}}, (\mathbf{b}_{i,j})_{(i,j)\in\mathbf{I}\setminus\{N\}\times\mathbf{I}}, (\mathbf{B}_{i,j})_{(i,j)\in\mathbf{I}\times\mathbf{I}}\right)$$

and the parameters $\theta \in \mathbb{R}^{N(S+1)}$ are

$$\theta = \left((\mathbf{y}_i(s))_{(i,s)\in\mathbf{I}\times\mathbf{S}}, (\omega_i(s_0))_{i\in\mathbf{I}}\right).$$

Define the system of equations as

$$\Phi : \mathbb{R}^{2N-1+S(N+1)+N(2N-1)+S(N+1)} \longrightarrow \mathbb{R}^{2N-1+S(2N-1)+SN^*+N(N-N^*)}$$

where $\Phi(\xi,\theta) = 0$ iff the following equations are satisfied: (i) initial period discounted present value equations (22) and (23), (ii) Euler equations (24), and (iii) budget constraints (25), (48), and (49).

Define the projection $\phi : \mathbb{R}^{2N-1+S(N+1)+N(2N-1)+S(N+1)} \longrightarrow \mathbb{R}^{N(S+1)}$ as the mapping $(\xi,\theta) \mapsto \theta$ such that $\Phi(\xi,\theta) = 0$. The mapping ϕ is proper iff for any compact subset of the range Y', the inverse image $\phi^{-1}(Y')$ is also compact. The payout matrix has full rank since the only variables considered are those with linearly independent payouts. This implies that the projection ϕ is proper.

To complete the argument, it suffices to prove that $D_{\xi,\theta}\Phi(\xi,\theta)$ has full row rank $2N - 1 + S(2N-1) + SN^* + N(N-N^*)$ and apply the parametric transversality theorem.

Consider the columns for derivatives with respect to the price variables $(p_i(s_0))_{i\in\mathbf{I}}$ and the parameters $(\omega_i(s_0))_{i\in\mathbf{I}}$. The submatrix in $D_{\xi,\theta}\Phi(\xi,\theta)$ for the initial period discounted present value equations (22) and (23) and the derivatives with respect to $(p_i(s_0))_{i\in\mathbf{I}}$ and $(\omega_i(s_0))_{i\in\mathbf{I}}$ has full row rank. Moreover, the variables $(p_i(s_0))_{i\in\mathbf{I}}$ and parameters $(\omega_i(s_0))_{i\in\mathbf{I}}$ do not appear in any other equations. The initial period discounted present value equations are therefore linearly independent of all other equations.

Consider the columns for derivatives with respect to the variables $(\theta_i)_{i\in\mathbf{I}}$, $(\mathbf{b}_{i,j})_{(i,j)\in\mathbf{I}\setminus\{N\}\times\mathbf{I}}$, and $(\mathbf{B}_{i,j})_{(i,j)\in\mathbf{I}\times\mathbf{I}}$ and the parameters $(\mathbf{y}_i(s))_{(i,s)\in\mathbf{I}\times\mathbf{S}}$. The remaining equations are the Euler equations (24) and the budget constraints (25), (48), and (49). The following lemma shows that the budget constraints are linearly independent of all other equations.

Lemma 2 *The submatrix in $D_{\xi,\theta}\Phi(\xi,\theta)$ for the budget constraints (25), (48), and (49) and the derivatives with respect to $(\theta_i)_{i\in\mathbf{I}}$, $(\mathbf{b}_{i,j})_{(i,j)\in\mathbf{I}\setminus\{N\}\times\mathbf{I}}$, $(\mathbf{B}_{i,j})_{(i,j)\in\mathbf{I}\times\mathbf{I}}$, and $(\mathbf{y}_i(s))_{(i,s)\in\mathbf{I}\times\mathbf{S}}$ has full row rank.*

Proof. See Subsection A.3.2 below. ■

Consider the columns for derivatives with respect to the asset prices $(\mathbf{q}_i(s))_{(i,s)\in\mathbf{I}\times\mathbf{S}}$. The submatrix in $D_{\xi,\theta}\Phi(\xi,\theta)$ for the Euler equations (24) and the derivatives with respect to $(\mathbf{q}_i(s))_{(i,s)\in\mathbf{I}\times\mathbf{S}}$ has full row rank. Given the previous two steps, the Euler equations are linearly independent of all other equations.

This completes the argument.

A.3.2 Proof of Lemma 2

A sufficient condition for full row rank is that for any

$$\alpha^T = \left(\left(\Delta b_i^T\right)_{i\in\mathbf{I}\setminus\{N\}}, \left(\Delta B_i^T\right)_{i\in\mathbf{I}} \right) \in \mathbb{R}^{S(N-1)+SN^*+N(N-N^*)},$$

the product

$$\alpha^T D_{\theta,\mathbf{b},\mathbf{B},e}\Phi'(\xi,\theta) = 0 \tag{50}$$

implies $\alpha^T = 0$. Here, the subsystem of equations $\Phi' : \mathbb{R}^{2N-1+S(N+1)+N(2N-1)+S(N+1)} \to \mathbb{R}^{S(N-1)+SN^*+N(N-N^*)}$ is defined by:

$$\Phi'(\xi,\theta)=\begin{pmatrix} \left(\theta_i\mathbf{Y}(s)-\mathbf{q}_i(s)\,\mathbf{y}_i(s)+\sum_{j\in\mathbf{I}}\mathbf{q}_j(s)\,\mathbf{y}_j(s)\mathbf{b}_{i,j}-\sum_{j\in\mathbf{I}}\mu_j\mathbf{y}_j(s)\mathbf{b}_{i,j}\right)_{(i,s)\in\mathbf{I}\backslash\{N\}\times\mathbf{S}} \\ \left(\mathbf{y}_i(s)\,(1-\mathbf{q}_i(s))+\sum_{j\in\mathbf{I}}\mathbf{q}_j(s)\,\mathbf{y}_j(s)\mathbf{B}_{i,j}-\sum_{j\in\mathbf{I}}\mu_j\mathbf{y}_j(s)\mathbf{B}_{i,j}\right)_{(i,s)\in\mathbf{I}^*\times\mathbf{S}} \\ \begin{pmatrix} \mathbf{B}_{i,i}=1 \\ \mathbf{B}_{i,j}=0 \text{ for } j\neq i \end{pmatrix}_{i\in\mathbf{I}\backslash\mathbf{I}^*} \end{pmatrix}.$$

The submatrix $D_{\theta,\mathbf{b},\mathbf{B},e}\Phi'(\xi,\theta)$ includes the derivatives of the budget constraints (25), (48), and (49) with respect to $(\theta_i)_{i\in\mathbf{I}}$, $(\mathbf{b}_{i,j})_{(i,j)\in\mathbf{I}\backslash\{N\}\times\mathbf{I}}$, and $(\mathbf{B}_{i,j})_{(i,j)\in\mathbf{I}\times\mathbf{I}}$ and the parameters $(\mathbf{y}_i(s))_{(i,s)\in\mathbf{I}\times\mathbf{S}}$.

Consider $S=N+1$ states, which is the largest number of states satisfying the upper bound $S\leq N+\frac{N}{N+N^*-1}$ and Assumption 4.

Choose country $i\in\mathbf{I}\backslash\{N\}$. Consider the columns for the derivatives with respect to $\left((\mathbf{b}_{i,j})_{j\in\{1,\ldots,N\}},\theta_i\right)$. The equations $\alpha^T D_{\theta,\mathbf{b},\mathbf{B},e}\Phi'(\xi,\theta)=0$ for these columns are:

$$\Delta b_i^T\begin{bmatrix} \mathbf{y}_1(1)\,(\mathbf{q}_1(1)-\mu_1) & .. & \mathbf{y}_N(1)\,(\mathbf{q}_N(1)-\mu_N) & \mathbf{Y}(1) \\ : & .. & : & : \\ \mathbf{y}_1(S)\,(\mathbf{q}_1(S)-\mu_1) & .. & \mathbf{y}_N(S)\,(\mathbf{q}_N(S)-\mu_N) & \mathbf{Y}(S) \end{bmatrix}=0. \tag{51}$$

I claim that $\begin{bmatrix} \mathbf{y}_1(1)\,(\mathbf{q}_1(1)-\mu_1) & .. & \mathbf{y}_N(1)\,(\mathbf{q}_N(1)-\mu_N) & \mathbf{Y}(1) \\ : & .. & : & : \\ \mathbf{y}_1(S)\,(\mathbf{q}_1(S)-\mu_1) & .. & \mathbf{y}_N(S)\,(\mathbf{q}_N(S)-\mu_N) & \mathbf{Y}(S) \end{bmatrix}$ is a full rank matrix.

By definition, $\mathbf{q}_i(s)=\beta\mu_i\sum_{\sigma\in\mathbf{S}}\Gamma(s,\sigma)\left(\frac{\mathbf{Y}(\sigma)}{\mathbf{Y}(s)}\right)^{-\rho}\frac{\mathbf{y}_i(\sigma)}{\mathbf{y}_i(s)}$. This can be rewritten as:

$$(\mathbf{q}_i(s)\,\mathbf{y}_i(s))_{s\in\mathbf{S}}=\mu_i\hat{\Gamma}\,(\mathbf{y}_i(\sigma))_{\sigma\in\mathbf{S}}.$$

The first N columns of the matrix in (51) are equivalently expressed as:

$$\left(\hat{\Gamma}-I_S\right)\begin{bmatrix} \mathbf{y}_1(1) & .. & \mathbf{y}_N(1) \\ : & .. & : \\ \mathbf{y}_1(S) & .. & \mathbf{y}_N(S) \end{bmatrix} diag\left((\mu_i)_{i\in\mathbf{I}}\right). \tag{52}$$

The submatrix from (52) has full column rank, since $\begin{bmatrix} \mathbf{y}_1(1) & .. & \mathbf{y}_N(1) \\ : & .. & : \\ \mathbf{y}_1(S) & .. & \mathbf{y}_N(S) \end{bmatrix}$ has full rank and the matrix $\left(\hat{\Gamma} - I_S\right)$ has full rank, the latter owing to the facts that $\beta \in (0,1)$ and Γ is a transition matrix. The final column in the matrix in (51) is linearly independent from any of the first N columns, provided that $\mathbf{q}_j(s) \neq \mathbf{q}_j(s')$ for some (j, s, s').

Lemma 3 $\mathbf{q}_j(s) \neq \mathbf{q}_j(s')$ *for some* (j, s, s').

Proof. See Subsection A.3.3 below. ■

Since the matrix in (51) is a full rank matrix, $\Delta b_i^T = 0$.

Choose country $i \in \mathbf{I} \backslash \mathbf{I}^*$. The monetary authority constraints (48) reduce to:

$$\begin{aligned} \mathbf{B}_{i,i} &= 1. \\ \mathbf{B}_{i,j} &= 0 \text{ for } j \neq i. \end{aligned}$$

From the derivatives with respect to $(\mathbf{B}_{i,j})_{j \in \{1,\ldots,N\}}$, the equations $\alpha D_{\theta,\mathbf{b},\mathbf{B},e} \Phi'(\xi, \theta) = 0$ imply that $\Delta B_i^T = 0$.

Choose country $i \in \mathbf{I}^*$. Consider the columns for the derivatives with respect to $(\mathbf{B}_{i,j})_{j \in \{1,\ldots,N\}}$. The equations $\alpha^T D_{\theta,\mathbf{b},\mathbf{B},e} \Phi'(\xi, \theta) = 0$ for these columns are:

$$\Delta B_i^T \begin{bmatrix} \mathbf{y}_1(1)\left(\mathbf{q}_1(1) - \mu_1\right) & \mathbf{y}_N(1)\left(\mathbf{q}_N(1) - \mu_N\right) \\ : & : \\ \mathbf{y}_1(S)\left(\mathbf{q}_1(S) - \mu_1\right) & \mathbf{y}_N(S)\left(\mathbf{q}_N(S) - \mu_N\right) \end{bmatrix} = 0. \tag{53}$$

The matrix $\begin{bmatrix} \mathbf{y}_1(1)\left(\mathbf{q}_1(1) - \mu_1\right) & \mathbf{y}_N(1)\left(\mathbf{q}_N(1) - \mu_N\right) \\ : & : \\ \mathbf{y}_1(S)\left(\mathbf{q}_1(S) - \mu_1\right) & \mathbf{y}_N(S)\left(\mathbf{q}_N(S) - \mu_N\right) \end{bmatrix}$, as previously shown, has full column rank $S-1$. The vector ΔB_i^T belongs to a 1-dimensional linear subspace of $\mathbb{R}^S$. This implies that $\exists \left(\delta_s\right)_{s \in \mathbf{S}} \in \mathbb{R}^S$ such that $\Delta B_i(s) = \delta_s \Delta B_i(1)$ for all states $s \in \mathbf{S}$.

For any country $j \in \mathbf{I}^*$, consider the columns for the derivatives with respect to $\left(\mathbf{y}_j(s)\right)_{s \in \mathbf{S}}$. The equations (50) imply that:

$$\left(\mathbf{q}_j(s) - \mu_j\right) \sum_{i \in \mathbf{S}} \Delta B_i(s) \mathbf{B}_{i,j} + \Delta B_j(s)\left(1 - \mathbf{q}_j(s)\right) = 0 \;\; \forall (j,s) \in \mathbf{I}^* \times \mathbf{S}. \tag{54}$$

Using the result that $\Delta B_j(s) = \delta_s \Delta B_j(1)$ for all countries $j \in \mathbf{I}^*$ and for all states $s \in \mathbf{S}$, then (54) is equivalently given by (as the terms δ_s cancel):

$$\left(\mathbf{q}_j(s)-\mu_j\right)\sum_{i\in\mathbf{S}}\Delta B_i(1)\mathbf{B}_{i,j}+\Delta B_j(1)\left(1-\mathbf{q}_j(s)\right)=0 \quad \forall (j,s)\in\mathbf{I}^*\times\mathbf{S}. \tag{55}$$

This implies that $\forall s,s'\in\mathbf{S}$,

$$\mathbf{q}_j(s)\left(\sum_{i\in\mathbf{S}}\Delta B_i(1)\mathbf{B}_{i,j}-\Delta B_j(1)\right)=\mathbf{q}_j(s')\left(\sum_{i\in\mathbf{S}}\Delta B_i(1)\mathbf{B}_{i,j}-\Delta B_j(1)\right),$$

as the asset prices $\mathbf{q}_j(s)$ are the only terms that depend on s.

From Lemma 3, $\forall j$, $\exists s,s'$ such that $\mathbf{q}_j(s)\neq\mathbf{q}_j(s')$. This implies that $\sum_{i\in\mathbf{S}}\Delta B_i(1)\mathbf{B}_{i,j}=\Delta B_j(1)$. From (55), this implies $\Delta B_j(1)\left(1-\mu_j\right)=0$. Since $\mu_j\neq 1$ for $j\in\mathbf{I}^*$, then $\Delta B_j(1)=0$. Since $\Delta B_j(s)=\delta_s\Delta B_j(1)$ for all states $s\in\mathbf{S}$, then $\Delta B_j^T=0$.

Since $\alpha^T D_{\theta,\mathbf{b},\mathbf{B},e}\Phi'(\xi,\theta)=0$ implies $\alpha^T=0$, the matrix $D_{\theta,\mathbf{b},\mathbf{B},e}\Phi'(\xi,\theta)$ has full row rank.

A.3.3 Proof of Lemma 3

Suppose, in order to obtain a contradiction, that $\exists j\in\mathbf{I}$ such that $\mathbf{q}_j(s)=\mathbf{q}_j(1)$ for all $s\in\mathbf{S}$. Given the nominal GDP targeting rules, the Euler equations (24) in matrix form are given by:

$$\left(\mathbf{Y}(\sigma)\,\mathbf{y}_j(\sigma)\right)_{\sigma\in\mathbf{S}}=\frac{\beta\mu_j}{\mathbf{q}_j(1)}\Gamma\left(\mathbf{Y}(\sigma)\,\mathbf{y}_j(\sigma)\right)_{\sigma\in\mathbf{S}}.$$

This implies that:

$$\left[I_S-\frac{\beta\mu_j}{\mathbf{q}_j(1)}\Gamma\right]\left(\mathbf{Y}(\sigma)\,\mathbf{y}_j(\sigma)\right)_{\sigma\in\mathbf{S}}=0. \tag{56}$$

Under incomplete markets, $S>2$. The system of equations (56) contains S equations and only two policy variables $\left(\mathbf{q}_j(1),\mu_j\right)$. Generically over the set of household endowments $\left(\mathbf{y}_j(s)\right)_{s\in\mathbf{S}}$, there does not exist a solution to the system of equations (56).