Online Technical Appendix for "The Effects of Dependent Beliefs on Endogenous Leverage"

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B Online Technical Appendix

B.1 Statistics

Recall the following claim from the body of the paper:

Claim 1 *If $d_i > d_j$, then $h_i^* > h_j^*$.*

For simplicity, order the dividends such that $d_1 \geq ... \geq d_N$. From the previous claim, $h_1^* \geq ... \geq h_N^*$.

Consider an economy with N assets and focus on households with $(h_i, h_j) \geq \left(h_i^*, h_j^*\right)$ and $h_k < h_k^* \; \forall k \in \mathbf{I} \backslash \{i, j\}$, where $i < j$. Recall the following claim from the body of the paper:

Claim 2 *If $a_i^h > 0$, then $b_i^h = a_i^h$.*

The expected payout from only holding asset i equals $\frac{h_i}{h_i^*}$ and the expected payout from only holding asset j equals $\frac{h_j}{h_j^*}$. A household with $(h_i, h_j) \geq \left(h_i^*, h_j^*\right)$ therefore holds asset i only when $\frac{h_i}{h_i^*} > \frac{h_j}{h_j^*}$ and asset j only when $\frac{h_i}{h_i^*} < \frac{h_j}{h_j^*}$. The measure zero subset of households with $\frac{h_i}{h_i^*} = \frac{h_j}{h_j^*}$ that are indifferent between assets i and j are irrelevant when we add the budget constraints over all households and apply the market clearing conditions.

Define the random variable $Z_{ij} = \frac{h_j}{h_i}$. The cdf $F_{Z_{ij}}(x_{ij}) = P(Z_{ij} < x_{ij}) = P\left(\frac{h_j}{h_i} < x_{ij}\right)$. Remember that the marginal distribution for h_i is the uniform distribution between h_i^* and 1 and the marginal distribution for h_j is the uniform distribution between h_j^* and 1. Therefore,

$$P\left(\frac{h_j}{h_i} < x_{ij}\right) = \int_{h_i^*}^{1} P\left(\frac{h_j}{h_i} < x_{ij} | h_i = y\right) dy = \int_{h_i^*}^{1} P\left(h_j < x_{ij} y\right) \frac{dy}{1 - h_i^*}.$$

Observe that $P\left(h_j < x_{ij} y\right) = 1$ if $x_{ij} y > 1$. Without loss of generality, we focus on $x_{ij} \leq 1$, meaning that $P\left(h_j < x_{ij} y\right) = \frac{x_{ij} y - h_j^*}{1 - h_j^*}$. This means that for $x_{ij} \leq 1$:

$$P\left(\frac{h_j}{h_i} < x_{ij}\right) = \int_{h_i^*}^{1} \frac{x_{ij} y - h_j^*}{\left(1 - h_i^*\right)\left(1 - h_j^*\right)} dy = \frac{1}{1 - h_j^*}\left(\frac{x_{ij}}{2}\left(1 + h_i^*\right) - h_j^*\right).$$

Returning to the problem, the probability that households with $\left(h_i, h_j\right) \geq \left(h_i^*, h_j^*\right)$ only hold asset i is found by setting $x_{ij} = \frac{h_j^*}{h_i^*}$:

$$P\left(\frac{h_j}{h_i} < \frac{h_j^*}{h_i^*}\right) = \frac{1}{2} \frac{h_j^*}{h_i^*} \frac{\left(1 - h_i^*\right)}{\left(1 - h_j^*\right)} \leq \frac{1}{2}.$$

The probability that households with $\left(h_i, h_j\right) \geq \left(h_i^*, h_j^*\right)$ only hold asset j is equal to $1 - \frac{1}{2} \frac{h_j^*}{h_i^*} \frac{\left(1 - h_i^*\right)}{\left(1 - h_j^*\right)} \geq \frac{1}{2}$. Using the law of large numbers, these probabilities are equal to fractions of households.

For any $i < j$, define $\xi^{ij} = \frac{h_j^*}{h_i^*} \frac{\left(1 - h_i^*\right)}{\left(1 - h_j^*\right)}$. By definition, $\xi^{ij} \leq 1$. For the case where the choice is between only $m = 2$ assets, define $\left(\Delta_i^{ij}, \Delta_j^{ij}\right) \in \left\{\mathbb{R}^2 : \Delta_i^{ij} + \Delta_j^{ij} = 0\right\}$ such that the fraction of households with $\left(h_i, h_j\right) \geq \left(h_i^*, h_j^*\right)$ that only hold asset i equals $\frac{1}{2}\left(1 + \Delta_i^{ij}\right)$ and the fraction that only hold asset j equals $\frac{1}{2}\left(1 + \Delta_j^{ij}\right)$. The specific values are $\Delta_i^{ij} = \xi^{ij} - 1$ and $\Delta_j^{ij} = 1 - \xi^{ij}$.

Now focus on the choice between $m = 3$ assets, specifically households with $\left(h_i, h_j, h_k\right) \geq \left(h_i^*, h_j^*, h_k^*\right)$ and $h_l < h_l^* \ \forall l \in \mathbf{I} \backslash \{i, j, k\}$, where $i < j < k$. These households only hold asset i when $\frac{h_i}{h_i^*} > \max\left\{\frac{h_j}{h_j^*}, \frac{h_k}{h_k^*}\right\}$. Proceeding as above:

$$\begin{aligned} P\left(\frac{h_j}{h_i} < x_{ij} \ \& \ \frac{h_k}{h_i} < x_{ik}\right) &= \int_{h_i^*}^{1} P\left(\frac{h_j}{h_i} < x_{ij} \ \& \ \frac{h_k}{h_i} < x_{ik} | h_i = y\right) dy \\ &= \int_{h_i^*}^{1} P\left(h_j < x_{ij} y\right) P\left(h_k < x_{ik} y\right) \frac{dy}{1 - h_i^*}. \end{aligned}$$

We only consider $x_{ij} \leq 1$ and $x_{ik} \leq 1$. The calculus simplifies to:

$$P\left(\frac{h_j}{h_i} < x_{ij} \ \& \ \frac{h_k}{h_i} < x_{ik}\right) = \frac{\frac{x_{ij} x_{ik}}{3}\left(1 + h_i^* + \left(h_i^*\right)^2\right) - \frac{x_{ij} h_k^* + x_{ik} h_j^*}{2}\left(1 + h_i^*\right) + h_j^* h_k^*}{\left(1 - h_j^*\right)\left(1 - h_k^*\right)}.$$

The fraction of households that only hold asset i is found by setting $x_{ij} = \frac{h_j^*}{h_i^*}$ and $x_{ik} = \frac{h_k^*}{h_i^*}$:

$$P\left(\frac{h_j}{h_i} < \frac{h_j^*}{h_i^*} \ \& \ \frac{h_k}{h_i} < \frac{h_k^*}{h_i^*}\right) = \frac{1}{3}\left(\frac{h_j^*}{h_i^*} \frac{\left(1 - h_i^*\right)}{\left(1 - h_j^*\right)}\right)\left(\frac{h_k^*}{h_i^*} \frac{\left(1 - h_i^*\right)}{\left(1 - h_k^*\right)}\right).$$

Using the definition for ξ^{ij} and ξ^{ik}, then $P\left(\frac{h_j}{h_i} < \frac{h_j^*}{h_i^*} \ \& \ \frac{h_k}{h_i} < \frac{h_k^*}{h_i^*}\right) = \frac{1}{3}\xi^{ij}\xi^{ik}$. For the case where the choice is between only $m = 3$ assets, define $\left(\Delta_i^{ijk}, \Delta_j^{ijk}, \Delta_k^{ijk}\right) \in \left\{\mathbb{R}^3 : \Delta_i^{ijk} + \Delta_j^{ijk} + \Delta_k^{ijk} = 0\right\}$ such that the fraction of households that only hold asset $p \in \{i, j, k\}$ equals $\frac{1}{3}\left(1 + \Delta_p^{ijk}\right)$. Since $P\left(\frac{h_j}{h_i} < \frac{h_j^*}{h_i^*} \ \& \ \frac{h_k}{h_i} < \frac{h_k^*}{h_i^*}\right) = \frac{1}{3}\xi^{ij}\xi^{ik}$, then $\Delta_i^{ijk} = \xi^{ij}\xi^{ik} - 1$. The remaining fraction $\left(1 - \frac{1}{3}\left(1 + \Delta_i^{ijk}\right)\right)$ of households either hold asset j or asset k. This reduces to the case with a choice between $m = 2$ assets, meaning that $\frac{1}{2}\left(1 + \Delta_j^{jk}\right)$ of the fraction $\left(1 - \frac{1}{3}\left(1 + \Delta_i^{ijk}\right)\right)$ only hold asset j. Setting $\frac{1}{3}\left(1 + \Delta_j^{ijk}\right)$ equal to the fraction that only hold asset j, I find $\Delta_j^{ijk} = \Delta_j^{jk} - \frac{1}{2}\Delta_i^{ijk} - \frac{1}{2}\Delta_j^{jk}\Delta_i^{ijk}$. The remaining households hold asset k, implying $\Delta_k^{ijk} = -\Delta_i^{ijk} - \Delta_j^{ijk}$.

The process continues by induction. Consider any $4 \le m \le N$ and the $m-$dimensional index $ijk....l$. Consider households such that $h_p \ge h_p^*$ for all assets p in the index $ijk....l$ and $h_p < h_p^*$ for all other assets. The variable $\Delta_p^{ijk....l}$ is defined such that the fraction of these households that only choose asset p equals $\frac{1}{m}\left(1 + \Delta_p^{ijk....l}\right)$. The construction of $\Delta_p^{ijk....l}$ proceeds by induction. The order in the index is such that $i < j < k < ... < l$. For the asset i, $\Delta_i^{ijk....l} = (\xi_{ij}) \cdots (\xi_{il}) - 1$. There now remain $(m-1)$ terms in the index $jk.....l$. Of the remaining fraction of households $\left(1 - \frac{1}{m}\left(1 + \Delta_i^{ijk....l}\right)\right)$, the fraction $\frac{1}{m-1}\left(1 + \Delta_j^{jk....l}\right)$ hold asset j, so $\Delta_j^{ijk....l}$ is defined such that:

$$\frac{1}{m}\left(1 + \Delta_j^{ijk....l}\right) = \left(1 - \frac{1}{m}\left(1 + \Delta_i^{ijk....l}\right)\right)\frac{1}{m-1}\left(1 + \Delta_j^{jk....l}\right).$$

There now remain $(m-2)$ terms in the index and the process continues until values for all variables $\Delta_p^{ijk....l}$ are specified. By construction, $-1 < \Delta_p^{ijk....l} < 1$ for all Δ terms.

B.2 Risk Aversion

One asset case

Consider risk-averse households. For simplicity, consider the single-asset $(N = 1)$ case. With risk-aversion, the cutoff household h^* has beliefs $h = h^*$ and is indifferent between buying and selling the asset. From the first order conditions for this cutoff household:

$$pu'\left(c^h(0)\right) = h^*u'\left(c^h(H)\right) + (1 - h^*)\,u'\left(c^h(L)\right)d, \tag{1}$$

where $\mathbf{S} = \{H, L\}$, the two dividend realizations for the single asset. With indifference, suppose h^* is a lender. The lender holds $a^h = 0$ units of the asset, implying that $c^h(H) = c^h(L)$. The lender uses a combination of storage and lending to transfer resources from the initial period $t = 0$ (with income $1+p$) into states H and L (with income 0). Both storage and lending take place such that consumption is perfectly smoothed: $c^h(0) = c^h(H) = c^h(L)$. This implies that equation (1) reduces to the original:

$$p = h^* + (1 - h^*)\, d.$$

With only a single asset, the only collateral is this asset. The possible borrowing contracts are $j \in \mathbb{R}_+$. The following result is identical to the Binomial No-Default Theorem of Fostel and Geanakoplos (2015) and general results by Araujo et al. (2012). The borrowing contract $j = d$ is termed the no-default borrowing contract as this is the highest promise repayment such that the borrower always repays its promised amount in all states. For all $j < d$, the payout vector is identical to the borrowing contract $j = d$. For all $j \geq 1$, the payout vector is identical to the asset itself (pays out 1 in state H and d in state L). For all $j \in (d, 1)$, the payout vector is equal to $\min\{j, 1\} = j$ in state H and $\min\{j, d\} = d$ in state L. This is simply a convex combination of the borrowing contract $j = d$ and the asset (the weight $\theta = \frac{1-j}{1-d}$ is placed on contract $j = d$ and the remaining weight $1 - \theta = \frac{j-d}{1-d}$ on the asset). For any households, including risk-averse ones, any contract $j \in \mathbb{R}_+ \backslash \{d\}$ can be traded in equilibrium, but it will be linearly dependent, meaning that the real equilibrium variables resulting from the trading of a contract $j \in \mathbb{R}_+ \backslash \{d\}$ are equivalent to the real equilibrium variables when only the no-default borrowing contract $j = d$ is traded.

Since it is innocuous (no real effects) to assume that the only contract traded is $j = d$, then I can use the budget constraints and first order conditions for $h \geq h^*$, together with market clearing conditions, to derive the remaining equilibrium equations relating p and h^*. Under risk-neutrality, the first order condition for households $h \geq h^*$ implies that they would set $c^h(0) = s^h = 0$ in order to purchase as many units of the asset as possible. This allowed me to pin down the asset holding a^h as all initial income $1 + p$ is spent for asset purchase. With risk-aversion, households $h \geq h^*$ optimally choose $c^h(0)$ and s^h to satisfy budget constraints and first order conditions.

The challenge with the equilibrium characterization under risk-aversion is two-fold: (i) more equilibrium equations (only 2 under risk-neutrality) and (ii) households $h \geq h^*$ no longer have symmetric asset holdings (as they did under risk-neutrality). With risk-neutrality, the asset holdings for $h \geq h^*$ were symmetric as they were constrained by the initial endowment. With risk-aversion, the asset holdings for $h \geq h^*$ are heterogeneous as they depend upon the belief parameter h.

Notice that this result does not extend outside the binomial economy. As shown in Araujo et al. (2012), with $S > 2$ possible dividend realizations, then there exists $S - 1$ borrowing contracts that are linearly independent. From this finite set of borrowing contracts, the optimal choice is a function of the degree of risk aversion.

Multiple asset case

With multiple assets ($N \geq 2$), we can still work with the binomial economy framework as each asset has binomial payouts and is an independent distribution. To do this, we have to impose the following additional assumption.

> Assumption A A borrowing contract is supported by the holding of one asset as collateral. It does not matter which of the N assets are specified by the contract, but combinations of different assets is not accepted as collateral.

Defining the unit vector $e_i = (0, ...0, 1, 0, ..., 0)$ with i^{th} element equal to 1, then the set of possible collateral is $\theta \in \{e_1, ..., e_N\}$. I will first show why Assumption A allows us to extend the results from the one asset case and then discuss why combinations of collateral $\theta \in \Delta^{N-1} \backslash \{e_1, ..., e_N\}$ are impermissible with risk aversion.

For any borrowing contract with collateral $\theta \in \{e_1, ..., e_N\}$, the results from the Binomial No-Default Theorem and the one asset case above are naturally extended and imply that either (i) in equilibrium, only the borrowing contracts $\{(e_1, d_1), ..., (e_N, d_N)\}$ are traded or (ii) any other equilibrium has identical real equilibrium variables as the equilibrium in which only $\{(e_1, d_1), ..., (e_N, d_N)\}$ are traded.

Suppose, in order to understand the sticking point in the argument, that Assumption A is not imposed. With N assets, each borrowing contract is secured with 1 total unit of asset (our normalization) and this 1 total unit can be any combination of the N available assets. Denote $\theta \in \Delta^{N-1}$ as one of the possible combinations. For each θ, the possible borrowing contracts are $j \in \mathbb{R}_+$.

Define $\#\mathbf{I}^+$ as the number of elements in the set $\mathbf{I}^+ = \{i \in \mathbf{I} : \theta_i > 0\}$. Consider any $\theta \in \Delta^{N-1} \backslash \{e_1, ..., e_N\}$, meaning that $\#\mathbf{I}^+ > 1$ for the contract under consideration. The payout in any date-event $s \in \mathbf{S}$ is equal to $\min \left\{ \sum_{i \in \mathbf{I}} \theta_i d_i(s), k \right\}$. Generically on θ, there are a total of $2^{\#\mathbf{I}^+}$ date-events with distinct values for $\sum_{i \in \mathbf{I}} \theta_i d_i(s)$. Even allowing for redundancy, with $\#\mathbf{I}^+ > 1$, the minimum number of distinct values for $\sum_{i \in \mathbf{I}} \theta_i d_i(s)$ is equal to 3 (this includes $\sum_{i \in \mathbf{I}} \theta_i d_i$ and 1 and at least one value in between). With 3 or more date-events (defined in terms of distinct payouts), we are no longer in the binomial framework. As shown in Araujo et al. (2012), with more than 2 date-events, it is no longer the case that the no-default loan is the only contract traded. With S date-events, there are $S - 1$ linearly independent contracts that can be traded in equilibrium. This result holds for all monotonic household preferences (including risk-aversion) as the result has nothing to do with preferences, but only with the geometry of the payout vectors. In fuller detail, the equilibrium in which the traded contracts are optimally chosen from the set of $S-1$ linearly independent contracts has the same values for the real equilibrium variables as any other equilibrium.

This exposition identifies the key trade-off. With risk-averse household, Assumption A is required in order to guarantee that the equilibrium is equivalent to one in which only no-default loans are traded. With risk-averse households and without Assumption A, equilibrium real effects can arise from (i) default in equilibrium and (ii) multiple contracts traded with the same collateral requirement. Future research is required in such settings in order to fully analyze the equilibrium characterization and the belief dependence effects.

B.3 Comonotonic Beliefs

For simplicity, denote the household contract choices for (i, d_i) as b_i^h and the contract price as q_i. The payout of contract (i, d_i) is equal to d_i in all date-events in the final period (a no-default loan). For all households, the first order conditions require that:

$$q_i = d_i.$$

From the first order conditions for this cutoff household:

$$p_i = h^* + (1 - h^*)\, d_i. \tag{2}$$

Claim 3 *Under comonotonic beliefs, $h_i^* = h_1^*$ for all $i \in \mathbf{I}$.*

Proof. Suppose, in order to obtain a contradiction, that $h_i^* > h_k^*$ for some i, k. The down payment for asset i in the initial period is $p_i - d_i$. For each one unit of investment (which allows for $\frac{1}{p_i - d_i}$ units of the asset to be purchased), the difference between the expected payout and the price for asset i is:

$$\frac{(h_i + (1 - h_i)\, d_i) - p_i}{p_i - d_i} = \frac{h_i}{h_i^*} - 1,$$

after using the definition of the cutoff (2). In similar fashion, for each one unit of investment on asset k, the difference between the expected payout and the price is equal to:

$$\frac{(h_k + (1 - h_k)\, d_k) - p_k}{p_k - d_k} = \frac{h_k}{h_k^*} - 1.$$

Under comonotonic beliefs, $h = h_i = h_k$. For $h \geq h_i^*$, both differences are positive. Since $h_i^* > h_k^*$, then the difference for asset k is larger, meaning households $h \geq h_i^*$ only purchase asset k.

By definition, households $h < h_i^*$ will not purchase asset i. This violates the market clearing for asset i, since some households must be willing to purchase asset i in order to satisfy the market clearing condition $\int_{h \in \mathbf{H}} a_i^h = 1$. ■

For simplicity, define $h^* = h_i^*$ for all $i \in \mathbf{I}$.

In the initial period, households with $h \geq h^*$ are indifferent about which asset to buy. This feature is unique to comonotonic beliefs. Households $h \geq h^*$ set $c^h(0) = s^h = 0$ in order to purchase as many units of the assets as possible.

On the other side, households $h < h^*$ sell all assets to the point where the short-sale constraints bind: $a_i^h = 0$ for all $i \in \mathbf{I}$.

Recall the Equilibrium Assumption (EA) that is verified in the body of the paper:

> **Equilibrium Assumption (EA)** The only contracts traded in equilibrium are the N borrowing contracts $\{(1, d_1), ..., (N, d_N)\}$ and any linear combinations of these contracts.

The collateral constraint under EA is given by:

$$\max\left\{b_i^h, 0\right\} \leq a_i^h \quad \forall i \in \mathbf{I}.$$

Risk-neutral households that choose to borrow will borrow up until the point that the collateral constraint binds.

Claim 4 *If $h \geq h^*$, then $b_i^h = a_i^h$ for all $i \in \mathbf{I}$.*

Proof. For simplicity, define $D = \sum_{i \in \mathbf{I}} d_i$ and $P = \sum_{i \in \mathbf{I}} p_i$.

Households $h_i \geq h^*$ set $c^h(0) = s^h = 0$ in order to purchase as many units of asset i as possible. Their objective function is equal to expected consumption across all states $s \in \mathbf{S}$. The payout in state $s \in \mathbf{S}$ is given by

$$a_i^h d_i(s) - b_i^h d_i.$$

For s such that $d_i(s) = 1$, the payout equals $a_i^h - b_i^h d_i$. For s such that $d_i(s) = d_i$, the payout equals 0. The expected consumption for households $h : a_i^h > 0$ equals

$$h_i\left(a_i^h - b_i^h d_i\right).$$

Define the fraction ϕ_i^h such that $b_i^h = \phi_i^h a_i^h$. The collateral constraint requires that $\phi_i^h \leq 1$. The expected consumption is therefore expressed as

$$h_i a_i^h \left(1 - \phi_i^h d_i\right) = h_i a_i^h \left(1 - d_i\right) + h_i a_i^h \left(1 - \phi_i^h\right) d_i.$$

From the household budget constraint in the initial period:

$$a_i^h \left(p_i - d_i\right) + a_i^h \left(1 - \phi_i^h\right) d_i = 1 + P.$$

From (2),

$$p_i - d_i = h^* \left(1 - d_i\right).$$

The expected consumption is equal to:

$$\frac{h_i}{h^*} \left(1 + P\right) + \left(h_i - \frac{h_i}{h^*}\right) a_i^h \left(1 - \phi_i^h\right) d_i.$$

Since $h_i < \frac{h_i}{h^*}$, then expected consumption is maximized at $\phi_i^h = 1$. Expected consumption is simply $\frac{h_i}{h^*} \left(1 + P\right)$. Under comonotonic beliefs, $h = h_i = h_k$, so households $h \geq h^*$ must be such that $h_i \geq h^*$. They are indifferent between any asset i and it doesn't matter if they hold a portfolio with a mixture of assets or one asset exclusively. The expected consumption $\frac{h_i}{h^*} \left(1 + P\right)$ is higher than what is possible through storage alone. The optimal choice for households $h \geq h^*$ is to borrow so that the collateral constraints bind on all traded contracts.

■

All households $h \geq h^*$ will borrow until the collateral constraint binds. Using these facts, add up the initial period budget constraints for all households $h \geq h^*$:

$$\sum_{i \in \mathbf{I}} \left(\left(p_i - d_i\right) \int_{h^*}^{1} a_i^h dh \right) = \left(1 - h^*\right) \left(1 + P\right). \tag{3}$$

From the market clearing condition:

$$P - D = \left(1 - h^*\right) \left(1 + P\right). \tag{4}$$

The other equilibrium equations are from (2):

$$p_i = h^* + \left(1 - h^*\right) d_i \quad \forall i \in \mathbf{I}.$$

Claim 5 *The market for loans is slack, meaning that total available funds from lenders exceeds total borrowing.*

Proof. In equilibrium, each household $h \geq h^*$ borrows the amounts $\sum_{i \in \mathbf{I}} q_i b_i^h = \sum_{i \in \mathbf{I}} d_i a_i^h$. The total amount borrowed equals

$$\sum_{i \in \mathbf{I}} d_i \left(\int_{h \geq h^*} a_i^h dh \right).$$

By market clearing, the total amount borrowed by households $h \geq h^*$ equals $D = \sum_{i \in \mathbf{I}} d_i$. The total amount lent must also be D.

Recall that households with beliefs $h \leq h^*$ sell all assets and become lenders. This means that the loan size for each lender is equal to $\frac{D}{F_{H_1,\ldots,H_N}(h^*,\ldots,h^*)} = \frac{D}{h^*}$. The income that lenders have available to lend has value $1 + P$. Lenders have sufficient resources to lend provided that

$$(1 + P) > \frac{D}{h^*}. \tag{5}$$

From (4)

$$P - D = (1 - h^*)(1 + P). \tag{6}$$

Equation (6) can be rewritten:

$$(1 + P) h^* = 1 + D. \tag{7}$$

Equation (7) implies that inequality (5) is satisfied. ■

Plug the price equations into the first equilibrium equation to obtain a single equation in terms of the variable h^*. The single equation can be written as a quadratic equation in the form:

$$(h^*)^2 (N - D) + h^* (1 + D) - (1 + D) = 0.$$

The equilibrium equation indicates that h^* does not depend on the dividend distribution, only on N and total dividends D.

Claim 6 $h^* \in (0, 1)$.

Proof. Since $d_i < 1$ for all $i \in \mathbf{I}$, then $N > D$. Using this fact, there exists a unique strictly positive solution to the quadratic equation:

$$h^* = \frac{-(1 + D) + \sqrt{(1 + D)^2 + 4(N - D)(1 + D)}}{2(N - D)}.$$

Since $N > D$, it is straightforward to show that $h^* \in (0, 1)$. ■

Denote lev_i as the leverage ratio for asset i. The price of asset i equals p_i and the loan size is equal to the low dividend payout d_i. By definition,

$$lev_i = \frac{p_i}{p_i - d_i}. \tag{8}$$

Using the equilibrium price equations, the leverage ratios are equivalently expressed as:

$$lev_i = \frac{h^* (1 - d_i) + d_i}{h^* (1 - d_i)}.$$

Leverage lev_i is strictly decreasing in h^* and strictly increasing in d_i. This is verified via the derivatives:

$$\frac{\partial lev_i}{\partial h^*} = \frac{-d_i (1 - d_i)}{[h^* (1 - d_i)]^2} < 0.$$
$$\frac{\partial lev_i}{\partial d_i} = \frac{h^*}{[h^* (1 - d_i)]^2} > 0.$$

Theorem 1 *With comonotonic beliefs, the leverage ratios for all assets are a strictly increasing function of m, the number of replications of the financial side of the economy.*

Proof. For the replica economies, the quadratic equation is given by

$$(h^*)^2 (mN - mD) + h^* (1 + mD) - (1 + mD) = 0.$$

Notice that if the endowment was scaled up proportionately (by m), then the wealth per asset market would not change and the equilibrium (h^*, p) would not change. There exists a unique strictly positive solution to the quadratic equation:

$$h^* = \frac{-(1 + mD) + \sqrt{(1 + mD)^2 + 4m(N - D)(1 + mD)}}{2m(N - D)}.$$

As before, $h^* \in (0, 1)$. Using the quotient rule:

$$\frac{\partial h^*}{\partial m} = \frac{2(N - D)\Psi}{[2m(N - D)]^2},$$

where

$$\Psi = 1 + \frac{(1 + mD)^2 - (1 + mD) + 4m(N - D)(1 + mD) - 2m(N - D)}{\sqrt{(1 + mD)^2 + 4m(N - D)(1 + mD)}}$$
$$- \sqrt{(1 + mD)^2 + 4m(N - D)(1 + mD)}.$$

Algebraically $\Psi < 0$ iff

$$\sqrt{(1+mD)^2 + 4m(N-D)(1+mD)} < (1+mD) + 2m(N-D),$$

where the latter strict inequality holds using the same algebra used in the proof of Claim 6. Therefore, $\frac{\partial h^*}{\partial m} < 0$.

In each asset market, the dividends remain unchanged. From above, $\frac{\partial lev_i}{\partial h^*} < 0$. Therefore, $\frac{\partial lev_i}{\partial m} > 0$ for all asset markets. ■

References

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